ACMS 40390: Test I

February 20, 2019

By signing you confirm that you are following the honor code for this test.

Name: ....

To receive credit you must show your work.

Given the total points T that you score for the test problems, the points that you receive for the test will be $\min\{100, T\}$

Problem Number	*Maximum Points*	*Points attained*
1	*10*	
2	*18*	
3	*16*	
4	*14*	
5	*12*	
6	*20*	
7	*14*	
TOTAL	*104*	

Some Useful Results

The following result may be of use. You may assume it in any argument you need to give.

Theorem 0.1 (Interpolation Error Formula) *Let* $x_0, \ldots, x_n$ *be distinct real numbers in the interval,* $[a, b]$*. Let* f *be an* $n+1$ *times continuously differentiable function on* $[a, b]$*. Let* $p(x)$ *denote the the unique polynomial of degree* $\leq n$ *such that for* $0 \leq i \leq n$, $p(x_i) = f(x_i)$*. Then for each* $x \in [a, b]$*, there exists a point* $\xi \in (a, b)$ *such that*

$$f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0) \cdots (x - x_n).$$

Problems

In the following you must show your work.

Problem 1 (10 points total) *Perform the following computations in a) 3 digit rounding arithmetic; and compute b) the absolute error, and c) the relative error in each case:*

1. $(100 + 0.45) + 0.45$;
2. $100 + (0.45 + 0.45)$.

Put your answers in the table below.

expression to be evaluated	*3 digit rounding answer* (3 points each)	*absolute error* (1 point each)	*relative error* (1 point each)
$(100 + 0.45) + 0.45$	100	0.9	$\frac{0.9}{100.9}$
$100 + (0.45 + 0.45)$	101	0.1	$\frac{0.1}{100.9}$

Problem 2 (18 points total) *A person wishes to find a zero of $f(x) = x^3+x-1$ for $0 \leq x \leq 1$. That person decides to use the bisection method to accomplish this.* Assume for simplicity that you are using exact arithmetic, i.e., that the errors of computer arithmetic play no role here.

6 pts: *State a theorem and show that it applies to guarantee that $f(x)$ has a zero on $[0, 1]$.*

The intermediate value theorem states that on the interval ($[0, 1]$ in this case), all values between $f(0)$ and $f(1)$ are taken on. So in this case $f(0) > 0$ and $f(1) < 0$ implies that there is at least one x in $[0, 1]$ with $f(x) = 0$.

6 pts: *The first approximation to a solution by the bisection method is 0.5 with $f(0.5) = -0.375$. What is the second approximation to a solution by the bisection method?*

Since $f(1) > 0$ and $f(0.5) < 0$, the second approximation is 0.75.

6 pts: *You would like to find an approximation to a zero of* $f(x)$ *on* $[0, 1]$ *with an absolute error of no more than* 0.02. *Using the bisection method as outlined in the previous part of this problem, and the error estimate for the bisection method, which is the smallest integer* n *for which you know that on the* n*-th approximation you will be within* 0.02 *of the correct answer. An answer without justification by the error estimate for the bisection method will receive no credit.*

We need the smallest n such that $\frac{1}{2^n} < \frac{1}{50}$. Thus we need the smallest integer n such that $2^n > 50$, i.e., $n = 6$.

Problem 3 (16 points total) *A person wishes to find a zero of* $f(x) = x^3 + x - 1$ *for* $0 \le x \le 1$*. That person decides to use Newton's method (also known as the Newton-Raphson method) for finding a solution of the equation* $f(x) = 0$ *on the interval,* $[0, 1]$.

8 points: *Write down iteration formula that Newton's method gives for solving* $f(x) = 0$*; and*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x^3 + x - 1}{3x^2 + 1}.$$

8 points: *using* 1.0 *as a starting guess find the first approximation to a solution of* $f(x) = 0$ *given by this formula.*

$$x_1 = 1 - \frac{1}{4} = 0.75.$$

Problem 4 (14 points total) *Write down the Lagrange form* (NOT THE NEWTON FORM) *of the interpolating polynomial of degree* ≤ 1 *with the value* 1 *at* $x_0 = 0$ *and the value* 2 *at* $x_1 = 1$.

$$1 \cdot \frac{x-1}{0-1} + 2 \cdot \frac{x-0}{1-0} = -(x-1) + 2x.$$

Problem 5 (11 points total) *Let* $f(x) = x^2 + 1$. *Assume that you have computed the degree* ≤ 1, *interpolation polynomial,* $p(x)$, *which satisfies* $p(0) = f(0)$ *and* $p(1) = f(1)$. *Compute an error bound for the approximation of* $p(x)$ *to* $f(x)$.

Since $f''(x) = 2$, *we have*

$$|f(x) - p(x)| = \left|\frac{2}{2!}(x-0)(x-1)\right| \leq \frac{1}{4}.$$

Problem 6 (20 points total) *Given the function $f(x) = x^2$:*

10 pts: *Compute the divided differences,* $f[-1], f[-1,0], f[-1,0,1], f[-1,0,1,2]$;

The divided difference table is

$$\begin{array}{cccccc}
2 & 4 & & & \\
 & & 3 & & \\
1 & 1 & & 1 & \\
 & & 1 & & 0 \\
0 & 0 & & 1 & \\
 & & -1 & & \\
-1 & 1 & & &
\end{array}$$

Therefore

$$\begin{aligned}
f[-1] &= 1 \\
f[-1,0] &= -1 \\
f[-1,0,1] &= 1 \\
f[-1,0,1,2] &= 0.
\end{aligned}$$

10 pts: *Write down the interpolating polynomial,* $p_3(x)$, *of degree* ≤ 3 *for* $f(x)$ *with the node points* $x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$ *using Newton's interpolatory divided-difference formula, i.e., the Newton form of the interpolation polynomial built out of divided differences* (NOT THE LAGRANGE FORM).

$$1 - (x+1) + (x+1)x.$$

Problem 7 (12 points total) *Assume that aproximations* $N_1(h)$ *to an integral*

$$M = \int_1^3 f(x)\mathrm{d}x$$

satisfy

$$M = N_1(h) + K_1 h^2 + O(h^4),$$

where K_1 *is a constant (i.e.,* K_1 *is the same for any* $h > 0$*). Assume that you have two estimates* $N_1(h)$ *and* $N_1\left(\frac{h}{2}\right)$*. What is the Richardson extrapolation* $N_2(h)$ *of these two estimates, that satisfies*

$$M = N_2(h) + O(h^4) \quad ?$$

We have

$$M = N_1(h) + K_1 h^2 + O(h^4),$$

and

$$M = N_1\left(\frac{h}{2}\right) + K_1\left(\frac{h}{2}\right)^2 + O(h^4).$$

Therefore

$$4M - M = 4N_1\left(\frac{h}{2}\right) - N_1(h) + O(h^4).$$

So the Richardson extrapolation $N_2(h)$ *equals*

$$\frac{4N_1\left(\frac{h}{2}\right) - N_1(h)}{3}.$$