Homework 8



1b $\| \|_\infty = \max\{2, 1, |-3|, 4\} = 4$

$\| \|_2 = \sqrt{2^2 + 1^2 + (-3)^2 + 4^2} = \sqrt{30}$

1d) Computing the first few

| $k$ | $\frac{4}{k+1}$ | $2/k^2$ | $\frac{k^2}{e^k}$ |
|---|---|---|---|
| 1 | 2 | 2 | $\frac{1}{e}$ |
| 2 | $\frac{4}{3}$ | $\frac{1}{2}$ | $\frac{4}{e^2}$ |
| 3 | 1 | $2/9$ | $9/e^3$ |
| 4 | $\frac{4}{5}$ | $\frac{2}{16}$ | $16/e^4$ |

leads to the guess that $\frac{4}{k+1}$ is the larger

Let's check

$\frac{4}{k+1} \geq 2/k^2$ is the same as

$4k^2 \geq 2k+2$ or $2k(2k-1) \geq 2$

or $k(2k-1) \geq 1$

since $2k-1 \geq 1$, this is clear.

$\frac{4}{k+1} \geq \frac{k^2}{e^k}$ is the same as $\underset{\text{VI}}{4e^k} \geq k^3 + k^2$

$4 + 4k + 2k^2 + \frac{2}{3}k^3 + \frac{k^4}{6}$

so $\| \|_\infty = \frac{4}{k+1}$ when $k \geq 2$ $\geq k^2 + \frac{2}{3}k^3 + \frac{1}{3}k^3 \frac{k}{2} \geq k^3 + k^2$

$$\| \quad \|_2 = \sqrt{\left(\frac{4}{k+1}\right)^2 + \left(\frac{2}{k^2}\right)^2 + \left(\frac{k^2}{e^k}\right)^2}$$

4b elementary calculus!

$$\left.\begin{array}{l} \dfrac{2+k}{k} \to 2 \\[2ex] \dfrac{k}{2+k} \to 1 \\[2ex] \dfrac{2k+1}{k} \to 2 \end{array}\right\} \text{ all by l'hopital's rule}$$

4d $\dfrac{|\cos(k)|}{k} \le \dfrac{1}{k} \to 0$

$\left|\dfrac{\sin(k)}{k}\right| \le \dfrac{1}{k} \to 0$

$\dfrac{1-k}{k^2+1} \to 0$ by l'hopital

$\dfrac{3k-2}{4k+1} \to \dfrac{3}{4}$ ''

5) a 15
b) 16
d 12

$\Big\}$ max of $\| \ \|_1$ ~~[illegible]~~ of the row vectors

6 d) 2 ''

8 bd) See maple worksheet at the ~~[illegible]~~ bottom of these notes.

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9 a) $\|x\|_1$ sum of nonnegative numbers $\Rightarrow \|x\|_1 \geq 0$

$\|x\|_1 = 0 \Rightarrow \sum |x_i| = 0 \Rightarrow$ each $x_i = 0 \Rightarrow x = 0$

$|x_i + y_i| \leq |x_i| + |y_i| \Rightarrow \sum_i |x_i + y_i| \leq \sum_i |x_i| + \sum_i |y_i|$; i.e. $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$

$\|cx\|_1 = \sum_i |cx_i| = \sum_i |c||x_i| = |c| \sum |x_i| = |c| \, \|x\|_1$

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$3 + 4 + 3/2$, $2+1+3+4$, $|\sin(k)| + |\cos(k)| + 2^k$, $\frac{4}{k+1} + \frac{2}{k^2} + \frac{k^2}{e^k}$

9c $\quad \|X\|_1 \geq \|X\|_2$

is the same as

$$|x_1| + \cdots + |x_m| \geq \sqrt{x_1^2 + \cdots + x_m^2}$$

$$\iff \left(|x_1| + \cdots + |x_m|\right)^2 \geq x_1^2 + \cdots + x_m^2$$

$$\shortparallel$$

$x_1^2 + \cdots + x_m^2$

$+ \left(2|x_1||x_2| + \text{the rest of the non negative cross terms}\right)$

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1 a $\quad (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 \qquad \lambda = 1, 3$

$\lambda = 1 \qquad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = b$ so multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 3 \qquad \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = -b$ so multiples of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

1d)

$$\det\begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & (3-\lambda) \end{pmatrix}$$

$$= (3-\lambda)\left((2-\lambda)^2 - 1\right) = (3-\lambda)(\lambda-1)(\lambda-3)$$

$\lambda = 1$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow$$

$c = 0 \quad a = -a \quad$ so $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and its multiples

$c = 3$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$a = b$, $c$ arbitrary

$\begin{pmatrix} a \\ a \\ c \end{pmatrix}$ two dimensional eigenspace

Note letting

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$T^{-1} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Note $T$ is orthogonal (i.e., $T^{-1} = T^t$

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2 a)

$$\det \begin{pmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{pmatrix} = 0$$

$$\shortparallel$$

$$\lambda^2 + \lambda$$

So eigenvalues are $0, -1$

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} a \\ -a \end{pmatrix}$$ are the eigenvectors for $\lambda = 0$

$$\begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow b = -2a$$

$\begin{pmatrix} a \\ -2a \end{pmatrix}$ are the eigenvectors for $\lambda = -1$

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2 d)

$$\det \begin{pmatrix} 3-\lambda & 2 & -1 \\ 1 & -2-\lambda & 3 \\ 2 & 0 & 4-\lambda \end{pmatrix} = -(3-\lambda)(4-\lambda)(2+\lambda) + 12 + 0 - (4+2\lambda) - (8-2\lambda)$$

$$= \cancel{-}(3-\lambda)(4-\lambda)(2+\lambda)$$

so eigenvalues are $\lambda = -2, 4, 3$

$\lambda = 3$

$$\begin{pmatrix} 0 & 2 & -1 \\ 1 & -5 & 3 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$$\left.\begin{aligned} c &= -2a \\ c &= 2b \\ b &= -a \end{aligned}\right\} \Rightarrow a \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \text{ are the eigenvectors}$$

$\lambda = 4$

$$\begin{pmatrix} 3-4 & 2 & -1 \\ 1 & -2-4 & 3 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -6 & 3 \\ 2 & 0 & 0 \end{pmatrix}$$

so $a = 0$

$c = 2b$

$$b \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \text{ are the eigenvectors}$$

$\lambda = -2$

$$\begin{pmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 0 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \implies$$

$$-3c = a$$

$$-4b = a$$

so $\begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \cdot a$

give

$\lambda = -2$

$$\begin{pmatrix} 5 & 2 & -1 \\ 1 & 0 & 3 \\ 2 & 0 & 6 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow$$

$$\begin{aligned} a &= -3c \\ b &= 8c \end{aligned} \Rightarrow c\begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix}$$

are the eigenvectors

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5 a) $\max\{1, 3\} = 3$

d) $\max\{3, 1, 3\} = 3$

6 a) $\max\{0, |-1|\} = 1$

d) $\max\{3, 4, |-2|\} = 4$

7 a d) Neither since spectral radii $\geq 1$

8 a d) " " " " "

9 a d) Since both are symmetric,

3, 3

12) assume $Av = \lambda v$ $\quad \lambda \neq 0, v \neq 0$

then $A^m v = \lambda^m v$ $\quad$ a contradiction

$$\begin{array}{c} A^m v \\ \shortparallel \\ 0 \cdot v \\ \shortparallel \\ 0 \end{array}$$

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2d) 2, 3, 2 are the eigenvalues

(since matrix is upper triangular

$$\begin{pmatrix} \overset{0}{2-2} & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow b = -c$$

$$\begin{pmatrix} a \\ b \\ -b \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$ are linearly independent eigenvectors for $2 = \lambda$

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \Rightarrow \begin{array}{l} c = 0 \\ b = a \end{array} \quad \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

so indeed there are 3 linearly ind. eigenvectors

3d) $4.75 \pm 2.5$

$4.75 \pm 3.5$

$4.75 \pm 1.5$

So

$$1.25 \le |\lambda| \le 8.25$$

4a) $-4 \pm 4$

$-4 \pm 3$

$-2 \pm 3$

$-4 \pm 4$

all eigenvalues are real since matrix is selfadjoint

$$-8 \le \lambda \le 1$$

4d) $6 \le \lambda \le 12$

$1 \le \lambda \le 5$

again all real

7ab see Maple worksheet below

~~5ab~~ $\cancel{\det\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = 3 \Rightarrow}$ ~~lin. ind.~~

$\cancel{\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 1\cdot(-1) + 1\cdot 1 \neq -2 \Rightarrow}$ ~~lin. ind.~~

```
> restart;
```

Homework 8

Problem 8b pg 447

```
> x1:=1;
  x2:=1;
  x1Tilde := 1.02;
  x2Tilde := 0.98;
  L1:= (x,y) -> x+2*y-3.0;
  L2:= (x,y) ->1.001*x - y - 0.001;
```

$$x1 := 1$$
$$x2 := 1$$
$$x1Tilde := 1.02$$
$$x2Tilde := 0.98$$
$$L1 := (x, y) \rightarrow x + 2\,y - 3.0$$
$$L2 := (x, y) \rightarrow 1.001\,x - y - 0.001 \quad (1)$$

```
> max(abs(x1-x1Tilde),abs(x2-x2Tilde));
  max(abs(L1(x1,x2)),abs(L2(x1,x2)));
  max(abs(L1(x1Tilde,x2Tilde)),abs(L2(x1Tilde,x2Tilde)));
```

$$0.02$$
$$0.$$
$$0.04002 \quad (2)$$

Problem 8d pg 447

```
> x1:=1.81;
  x2:=-1.81;
  x3:=0.65;
  x1Tilde := 2.0;
  x2Tilde := -2;
  x3Tilde := 1.0;
  L1 := (x,y,z) -> 0.04*x+0.01*y-0.01*z-0.0478;
  L2 := (x,y,z) -> 0.4*x +0.1*y - 0.2*z-0.413;
  L3 := (x,y,z) -> x+2*y+3*z-0.14;
```

$$x1 := 1.81$$
$$x2 := -1.81$$
$$x3 := 0.65$$
$$x1Tilde := 2.0$$
$$x2Tilde := -2$$
$$x3Tilde := 1.0$$
$$L1 := (x, y, z) \rightarrow 0.04\,x + 0.01\,y - 0.01\,z - 0.0478$$
$$L2 := (x, y, z) \rightarrow 0.4\,x + 0.1\,y - 0.2\,z - 0.413$$
$$L3 := (x, y, z) \rightarrow x + 2\,y + 3\,z - 0.14 \quad (3)$$

```
> max(abs(x1-x1Tilde),abs(x2-x2Tilde),abs(x3-x3Tilde));
  max(abs(L1(x1,x2,x3)),abs(L2(x1,x2,x3)),abs(L3(x1,x2,x3)));
  max(abs(L1(x1Tilde,x2Tilde,x3Tilde)),abs(L2(x1Tilde,x2Tilde,
  x3Tilde)),abs(L3(x1Tilde,x2Tilde,x3Tilde)));
```

$$0.35$$
$$0.$$
$$0.86 \quad (4)$$

Problem 7ab page 576

```
> with(LinearAlgebra):
> v1:= Vector([1.0,1.0]);
  v2:=Vector([-2,1.0]);
```

$$v1 := \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}$$

$$v2 := \begin{bmatrix} -2 \\ 1.0 \end{bmatrix} \tag{5}$$

```
> e1:=v1/Norm(v1,2);
```

$$e1 := \begin{bmatrix} 0.707106781400000 \\ 0.707106781400000 \end{bmatrix} \tag{6}$$

```
> f2:= v2-DotProduct(e1,v2)*e1:
  e2:= f2/Norm(f2,2);
```

$$e2 := \begin{bmatrix} -0.707106780907698 \\ 0.707106781192302 \end{bmatrix} \tag{7}$$

Just to check orthonormality

```
> DotProduct(e1,e1);
  DotProduct(e2,e2);
  DotProduct(e1,e2);
```

$$1.00000000060373484$$

$$0.99999999961378550$$

$$2.01244965136737619\ 10^{-10} \tag{8}$$

```
> v1:= Vector([1,1,0]);
  v2:= Vector([1,0,1]);
  v3:= Vector([0,1,1]);
```

$$v1 := \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$v2 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$v3 := \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{9}$$

```
> e1:= v1/Norm(v1,2);
```

(10)

$$e1 := \begin{bmatrix} \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} \\ 0 \end{bmatrix} \quad (10)$$

```
> f2:= v2-DotProduct(e1,v2)*e1:
  e2:= f2/Norm(f2,2);
```

$$e2 := \begin{bmatrix} \frac{1}{6}\sqrt{6} \\ -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{6} \end{bmatrix} \quad (11)$$

```
> f3:= v3-DotProduct(e1,v3)*e1-DotProduct(e2,v3)*e2:
  e3:=f3/Norm(f3,2);
```

$$e3 := \begin{bmatrix} -\frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \\ \frac{1}{3}\sqrt{3} \end{bmatrix} \quad (12)$$

Let's confirm orthonormality

```
> for j from 1 to 3 do for k from 1 to 3 do
  print(DotProduct(e||j,e||k));
  od;
  od;
```

$$1$$
$$0$$
$$0$$
$$0$$
$$1$$
$$0$$
$$0$$
$$0$$
$$1 \quad (13)$$