```
> restart;
> with(LinearAlgebra);
```

[*&x*, *Add*, *Adjoint*, *BackwardSubstitute*, *BandMatrix*, *Basis*, *BezoutMatrix*, *BidiagonalForm*, *BilinearForm*, *CARE*, *CharacteristicMatrix*, *CharacteristicPolynomial*, *Column*, *ColumnDimension*, *ColumnOperation*, *ColumnSpace*, *CompanionMatrix*, *CompressedSparseForm*, *ConditionNumber*, *ConstantMatrix*, *ConstantVector*, *Copy*, *CreatePermutation*, *CrossProduct*, *DARE*, *DeleteColumn*, *DeleteRow*, *Determinant*, *Diagonal*, *DiagonalMatrix*, *Dimension*, *Dimensions*, *DotProduct*, *EigenConditionNumbers*, *Eigenvalues*, *Eigenvectors*, *Equal*, *ForwardSubstitute*, *FrobeniusForm*, *FromCompressedSparseForm*, *FromSplitForm*, *GaussianElimination*, *GenerateEquations*, *GenerateMatrix*, *Generic*, *GetResultDataType*, *GetResultShape*, *GivensRotationMatrix*, *GramSchmidt*, *HankelMatrix*, *HermiteForm*, *HermitianTranspose*, *HessenbergForm*, *HilbertMatrix*, *HouseholderMatrix*, *IdentityMatrix*, *IntersectionBasis*, *IsDefinite*, *IsOrthogonal*, *IsSimilar*, *IsUnitary*, *JordanBlockMatrix*, *JordanForm*, *KroneckerProduct*, *LA_Main*, *LUDecomposition*, *LeastSquares*, *LinearSolve*, *LyapunovSolve*, *Map*, *Map2*, *MatrixAdd*, *MatrixExponential*, *MatrixFunction*, *MatrixInverse*, *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *ProjectionMatrix*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SplitForm*, *StronglyConnectedBlocks*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *SylvesterSolve*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*] **(1)**

First the Markov Chain for the dice game craps

```
> P:=Matrix(9):
  P(1,2) := 8/36:
  P(1,3) := 4/36:
  P(1,4) :=3/36:
  P(1,5) := 4/36:
  P(1,6):= 5/36:
  P(1,7) :=3/36:
  P(1,8) := 4/36:
  P(1,9):= 5/36:
  P(2,2):=1:
  P(3,3):=1:
> P(4,2):=3/36:
  P(4,3):=1/6:
  P(4,4):= 27/36:
  P(7,2):=3/36:
  P(7,3):=1/6:
  P(7,7):= 27/36:
  P(5,2):=4/36:
  P(5,3):=1/6:
  P(5,5):= 26/36:
  P(8,2):=4/36:
  P(8,3):=1/6:
  P(8,8):= 26/36:
  P(6,2):=5/36:
  P(6,3):=1/6:
  P(6,6):= 25/36:
```

```
P(9,2):=5/36:
P(9,3):=1/6:
P(9,9):= 25/36:
```

```
> P;
#check to make sure the rows add up to 1
for i from 1 to 9 do
x:=0;
for j from 1 to 9 do
x:=x+P(i,j);
od;
print(x);
od:
```

$$\begin{bmatrix} 0 & \frac{2}{9} & \frac{1}{9} & \frac{1}{12} & \frac{1}{9} & \frac{5}{36} & \frac{1}{12} & \frac{1}{9} & \frac{5}{36} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{6} & \frac{3}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{9} & \frac{1}{6} & 0 & \frac{13}{18} & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{36} & \frac{1}{6} & 0 & 0 & \frac{25}{36} & 0 & 0 & 0 \\ 0 & \frac{1}{12} & \frac{1}{6} & 0 & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{9} & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{13}{18} & 0 \\ 0 & \frac{5}{36} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{25}{36} \end{bmatrix}$$

$$\begin{matrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{matrix} \tag{2}$$

```
> A:=MatrixInverse(IdentityMatrix(9)-s*P);
  sP:= unapply(A,s):
```

$$A := \left[\left[1, -\frac{4}{9}\ \frac{s\left(250\,s^3-1373\,s^2+2358\,s-1296\right)}{(-1+s)\,(-4+3\,s)\,(-18+13\,s)\,(-36+25\,s)},\right.\right. \tag{3}$$

$$\left.\left.\frac{1}{9}\ \frac{s\left(385\,s^3+298\,s^2-3024\,s+2592\right)}{(-1+s)\,(-4+3\,s)\,(-18+13\,s)\,(-36+25\,s)}, -\frac{1}{3}\ \frac{s}{-4+3\,s}, -\frac{2\,s}{-18+13\,s}, -\frac{5\,s}{-36+25\,s},\right.\right.$$

$$\left[-\frac{1}{3}\ \frac{s}{-4+3s}, -\frac{2s}{-18+13s}, -\frac{5s}{-36+25s}\right],$$

$$\left[0, -\frac{1}{-1+s}, 0, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, 0, -\frac{1}{-1+s}, 0, 0, 0, 0, 0, 0\right],$$

$$\left[0, \frac{1}{3}\ \frac{s}{(-1+s)\ (-4+3s)}, \frac{2}{3}\ \frac{s}{(-1+s)\ (-4+3s)}, -\frac{4}{-4+3s}, 0, 0, 0, 0, 0\right],$$

$$\left[0, \frac{2s}{(-1+s)\ (-18+13s)}, \frac{3s}{(-1+s)\ (-18+13s)}, 0, -\frac{18}{-18+13s}, 0, 0, 0, 0\right],$$

$$\left[0, \frac{5s}{(-1+s)\ (-36+25s)}, \frac{6s}{(-1+s)\ (-36+25s)}, 0, 0, -\frac{36}{-36+25s}, 0, 0, 0\right],$$

$$\left[0, \frac{1}{3}\ \frac{s}{(-1+s)\ (-4+3s)}, \frac{2}{3}\ \frac{s}{(-1+s)\ (-4+3s)}, 0, 0, 0, -\frac{4}{-4+3s}, 0, 0\right],$$

$$\left[0, \frac{2s}{(-1+s)\ (-18+13s)}, \frac{3s}{(-1+s)\ (-18+13s)}, 0, 0, 0, 0, -\frac{18}{-18+13s}, 0\right],$$

$$\left[0, \frac{5s}{(-1+s)\ (-36+25s)}, \frac{6s}{(-1+s)\ (-36+25s)}, 0, 0, 0, 0, 0, -\frac{36}{-36+25s}\right]\Bigg]$$

```
> M:=Matrix(9,9):
  for i from 1 to 9 do
  for j from 1 to 9 do
  M(i,j):=limit((1-s)*sP(s)(i,j),s=1);
  od:
  od:
```

```
> M;
```

$$\begin{bmatrix} 0 & \frac{244}{495} & \frac{251}{495} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{11} & \frac{6}{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5}{11} & \frac{6}{11} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \textbf{(4)}$$

Now a five state Markov chin with two transient states and three intercommunicating persistent states

```
> P:=Matrix(5,5):
  P(1,1):=2/3:
  P(1,2):=1/3:
  P(2,1):=3/5:
  P(2,2):=1/5:
  P(2,3):=1/5:
  P(3,4):=2/3:
  P(3,3):=1/6:
  P(3,5):=1/6:
  P(4,3):=2/5:
  P(4,4):=1/5:
  P(4,5):=2/5:
  P(5,3):=2/7:
  P(5,4):=1/7:
  P(5,5):=4/7:
```

```
> for i from 1 to 5 do
  x:=0;
  for j from 1 to 5 do
  x:=x+P(i,j);
  od;
  print(x);
  od:
```

$$1,\ 1,\ 1,\ 1,\ 1 \tag{5}$$

```
> P;
```

$$\begin{bmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{3}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & 0 & \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{2}{7} & \frac{1}{7} & \frac{4}{7} \end{bmatrix} \tag{6}$$

```
> A:=MatrixInverse(IdentityMatrix(5)-s*P);
  sP:= unapply(A,s):
```

$$A := \left[\left[\frac{3\,(s-5)}{s^2+13\,s-15}, -\frac{5\,s}{s^2+13\,s-15}, -\frac{6\,(2\,s^2-27\,s+35)\,s^2}{(s^2+13\,s-15)\,(14\,s^3-27\,s^2-197\,s+210)}, \right.\right. \tag{7}$$

$$\left.\left.\frac{5\,(15\,s-28)\,s^3}{(s^2+13\,s-15)\,(14\,s^3-27\,s^2-197\,s+210)}, -\frac{7\,(7\,s+5)\,s^3}{(s^2+13\,s-15)\,(14\,s^3-27\,s^2-197\,s+210)}\right],\right.$$

$$\left[-\frac{9s}{s^2+13s-15}, \frac{5(-3+2s)}{s^2+13s-15}, \frac{6s(2s^2-27s+35)(-3+2s)}{(14s^3-27s^2-197s+210)(s^2+13s-15)},\right.$$
$$\left.-\frac{5s^2(15s-28)(-3+2s)}{(14s^3-27s^2-197s+210)(s^2+13s-15)}, \frac{7s^2(7s+5)(-3+2s)}{(14s^3-27s^2-197s+210)(s^2+13s-15)}\right],$$
$$\left[0, 0, \frac{6(2s^2-27s+35)}{14s^3-27s^2-197s+210}, -\frac{5(15s-28)s}{14s^3-27s^2-197s+210}, \frac{7(7s+5)s}{14s^3-27s^2-197s+210}\right],$$
$$\left[0, 0, -\frac{12(2s-7)s}{14s^3-27s^2-197s+210}, \frac{5(2s^2-31s+42)}{14s^3-27s^2-197s+210}, \frac{84s}{14s^3-27s^2-197s+210}\right],$$
$$\left[0, 0, \frac{60s}{14s^3-27s^2-197s+210}, \frac{5s(6+7s)}{14s^3-27s^2-197s+210}, -\frac{7(7s^2+11s-30)}{14s^3-27s^2-197s+210}\right]\Bigg]$$

```
> M:=Matrix(5,5):
  for i from 1 to 5 do
  for j from 1 to 5 do
  M(i,j):=limit((1-s)*sP(s)(i,j),s=1);
  od:
  od:
```

```
> M;
```

$$\begin{bmatrix} 0 & 0 & \frac{60}{209} & \frac{65}{209} & \frac{84}{209} \\ 0 & 0 & \frac{60}{209} & \frac{65}{209} & \frac{84}{209} \\ 0 & 0 & \frac{60}{209} & \frac{65}{209} & \frac{84}{209} \\ 0 & 0 & \frac{60}{209} & \frac{65}{209} & \frac{84}{209} \\ 0 & 0 & \frac{60}{209} & \frac{65}{209} & \frac{84}{209} \end{bmatrix} \quad (8)$$

```
> B:=Matrix(5,5):
  B:=evalf(P);
```

$$B := \begin{bmatrix} 0.6666666667 & 0.3333333333 & 0. & 0. & 0. \\ 0.6000000000 & 0.2000000000 & 0.2000000000 & 0. & 0. \\ 0. & 0. & 0.1666666667 & 0.6666666667 & 0.1666666667 \\ 0. & 0. & 0.4000000000 & 0.2000000000 & 0.4000000000 \\ 0. & 0. & 0.2857142857 & 0.1428571429 & 0.5714285714 \end{bmatrix} \quad (9)$$

```
> for j from 1 to 10 do
  B:=MatrixMatrixMultiply(B,B):
  od:
  B;
```

$$[[1.92813790877284\ 10^{-29}, 8.71169764875651\ 10^{-30}, 0.287081348028416, 0.311004793717524, 0.401913887227130],] \quad (10)$$

$$\left[1.56810557693298\ 10^{-29},\ 7.08500237737844\ 10^{-30},\ 0.287081348053141,\ 0.311004793744309,\ 0.401913887261744\right],$$
$$\left[0.,\ 0.,\ 0.287081348168523,\ 0.311004793869306,\ 0.401913887423279\right],$$
$$\left[0.,\ 0.,\ 0.287081348144623,\ 0.311004793843414,\ 0.401913887389818\right],$$
$$\left[0.,\ 0.,\ 0.287081348141326,\ 0.311004793839843,\ 0.401913887385203\right]\Big]$$

```
> evalf(M);
```

$$\begin{bmatrix} 0. & 0. & 0.2870813397 & 0.3110047847 & 0.4019138756 \\ 0. & 0. & 0.2870813397 & 0.3110047847 & 0.4019138756 \\ 0. & 0. & 0.2870813397 & 0.3110047847 & 0.4019138756 \\ 0. & 0. & 0.2870813397 & 0.3110047847 & 0.4019138756 \\ 0. & 0. & 0.2870813397 & 0.3110047847 & 0.4019138756 \end{bmatrix} \tag{11}$$

a few mean times

```
> F13:=unapply(sP(s)(1,3)/sP(s)(3,3),s);
  dF13:=unapply(diff(F13(s),s),s);
  F23:=unapply(sP(s)(2,3)/sP(s)(3,3),s);
  dF23:=unapply(diff(F23(s),s),s);
  F33:=unapply(1-1/sP(s)(3,3),s);
  dF33:=unapply(diff(F33(s),s),s);
```

$$F13 := s \to -\frac{s^2}{s^2+13\,s-15}$$

$$dF13 := s \to -\frac{2\,s}{s^2+13\,s-15} + \frac{s^2\,(2\,s+13)}{\left(s^2+13\,s-15\right)^2}$$

$$F23 := s \to \frac{s\,(-3+2\,s)}{s^2+13\,s-15}$$

$$dF23 := s \to \frac{-3+2\,s}{s^2+13\,s-15} - \frac{s\,(-3+2\,s)\,(2\,s+13)}{\left(s^2+13\,s-15\right)^2} + \frac{2\,s}{s^2+13\,s-15}$$

$$F33 := s \to 1 - \frac{1}{6}\,\frac{14\,s^3-27\,s^2-197\,s+210}{2\,s^2-27\,s+35}$$

$$dF33 := s \to \frac{1}{6}\,\frac{\left(14\,s^3-27\,s^2-197\,s+210\right)\,(4\,s-27)}{\left(2\,s^2-27\,s+35\right)^2} - \frac{1}{6}\,\frac{42\,s^2-54\,s-197}{2\,s^2-27\,s+35} \tag{12}$$

```
> dF13(1);
  dF23(1);
  dF33(1);
```

$$17$$
$$14$$
$$\frac{209}{60} \tag{13}$$

```
> F13(1);
```

$$1 \tag{14}$$