Mathematics 40485: Test II

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March 26, 2015

Name: ....

This test is due at the start of class on Tuesday, March 31, 2015.

This test is being conducted under the honor code. From the time you receive this exam *until the time it is collected you may not communicate about course material with anyone (in the class or not)*, except me. This includes comments on which problems are easy or hard. *You may not use anyone else's notes or programs on the test.* You are allowed and encouraged to use your notes, Maple, any books, or material on websites (at your own risk of course) while doing the tests.

If you find some problem on the test which you think is unclear (or wrong), please contact me immediately—my e-mail address is `sommese@nd.edu`.

If necessary I will send a correction or clarification to the class by e-mail. It is your responsibility to check your e-mail regularly between now and Tuesday, March 31, 2015.

There are a total of 111 points below. Letting T denote the total number of points you get, your test grade will be $\min\{T, 100\}$.

In the following you must show your work and include your reasoning. You should also include your Maple work sheets used on the problems. You may use the differentiation and algebraic simplification commands (including `taylor`, which computes Taylor series), but you must show all work as discussed in class.

Please write neatly!

Problems

In all problems involving inverse Laplace transforms, you must use the Bromwich contour.

Problem 1 (12 points) *Compute*

$$\frac{1}{2\pi i}\int_C \sinh\left(\frac{1}{z}\right) \mathrm{d}z,$$

where C is the unit circle traversed counterclockwise.

Problem 2 (16 points) *Evaluate*

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{5 - 4\cos(\theta)}$$

using residue integration.

Problem 3 (16 points) *To compute $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ by using residues:*

1. *show for $N \geq 1$*

$$\frac{1}{2\pi i}\int_{C_N} \frac{\pi\cot(\pi z)}{z^2+1}\mathrm{d}z = \sum_{n=-N}^{N} \frac{1}{n^2+1} + \mathrm{Res}_i\left(\frac{\pi\cot(\pi z)}{z^2+1}\right) + \mathrm{Res}_{-i}\left(\frac{\pi\cot(\pi z)}{z^2+1}\right)$$

where C_N denotes the square with vertices

$$\pm\left(N+\frac{1}{2}\right) + \pm\left(N+\frac{1}{2}\right) i; \text{and}$$

2. *show*

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2}\coth(\pi) - \frac{1}{2}.$$

You may assume that

$$\lim_{N\to\infty}\left[\frac{1}{2\pi i}\int_{C_N} \frac{\pi\cot(\pi z)}{z^2+1}\mathrm{d}z\right] = 0.$$

Problem 4 (16 points) *Let* $g(z) = z^4 + 1$ *and let* $f(z) = e^z$.

1. *Show that the roots of* $g(z) = 0$ *have absolute value* 1.
2. *Show that* $|f(z)| < 9$ *when* $|z| = 2$.
3. *Show that* $|g(z)| \geq 15$ *when* $|z| = 2$.
4. *Explain how Rouché's Theorem lets you conclude how many roots (counting multiplicities)* $e^z + z^4 + 1$ *has 4 zeroes when* $|z| \leq 2$.

Problem 5 (15 points) *Do Problem 10 on page 256, i.e., compute the inverse Laplace transform of* $\frac{1}{\sqrt{z}}$ *using a keyhole contour.*

Problem 6 (15 points) *Show that*

$$\int_{-\infty}^{+\infty} \frac{(\cos(x) - 1)}{x^2(x^2+1)} \mathrm{d}x = -\frac{\pi}{e}.$$

Problem 7 (16 points) *Recall that the Riemann sphere consists of the complex plane* C *plus the point at* ∞

1. *Write down a function* $f(z)$ *such that*
   (a) $z \to f(z)$ *is a one-to-one onto mapping of the Riemann sphere to itself; and*
   (b) $f(0) = \infty$, $f(\infty) = 0$, *and* $f(1) = 2$.
2. *Write down the inverse function of* $f(z)$, *i.e., the function* $g(z)$ *with* $f(g(z)) = z$ *and* $g(f(z)) = z$.

Problem 8 (5 points) *What is the cross ratio of* $0, 1, \infty, 3$?

Test 2 problems

1 $\sinh\left(\frac{1}{z}\right) = \frac{1}{z} + \frac{1}{3!z^3} + \cdots$

so $\sinh\left(\frac{1}{z}\right)$ is analytic except at $z = 0$ where it has a residue of 1, i.e..

$$\frac{1}{2\pi i}\int_C \sinh\left(\frac{1}{z}\right)dz = \operatorname{Res}_0\left(\sinh\left(\frac{1}{z}\right)\right) = 1$$

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2 $C = \{z \mid |z| = 1\}$ parameterize $z = e^{i\theta}$

$\cos(\theta) = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right),\ \dfrac{dz}{z} = i\,d\theta$

So $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 - 4\cos(\theta)}\,d\theta = i^{-1}\int_C \frac{1}{5 - \frac{4}{2}\left(z + \frac{1}{z}\right)}\frac{dz}{z}$

$= \displaystyle\frac{1}{i}\int_0^{2\pi} \frac{dz}{-2z^2 + 5z - 2}$

$-2z^2 + 5z - 2 = -(z-2)(2z-1)$

$\|$ only $z = \frac{1}{2}$ is within the circle. So

$\displaystyle\frac{2\pi i}{i}\operatorname{Res}_{\frac{1}{2}}\frac{1}{-(z-2)(2z-1)} = -2\pi \cdot \frac{1}{-3/2 \cdot 2} = \frac{2}{3}\pi$

3 $$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} = 2\sum_{n=1}^{\infty} \frac{1}{n^2+1} + 1$$

$$= -\pi \left(\mathrm{Res}_{i} \frac{\cot(\pi z)}{z^2+1} + \mathrm{Res}_{-i}\left(\frac{\cot(\pi z)}{z^2+1}\right)\right)$$

$$= -\left(\frac{\pi}{2i}\cot(\pi i) + \frac{\pi \cot(-\pi i)}{-2i} \right)$$

$$= -\pi \frac{\frac{e^{i(i\pi)}+e^{-i(i\pi)}}{2}}{e^{i(i\pi)}-e^{-i(i\pi)}} + \pi \left(\frac{e^{-ii\pi}+e^{ii\pi}}{e^{-ii\pi}-e^{ii\pi}} \right)\frac{1}{2}$$

$$= \pi \frac{\cosh(\pi)}{\sinh(\pi)}.$$ So

$$\sum_{1}^{\infty} \frac{1}{n^2+1} = \frac{1}{2}\pi \coth(\pi) - \frac{1}{2}.$$

4) 1) $z^4 = -1 \Rightarrow |z|^4 = 1$

$\Rightarrow |z| = 1$

2) $|e^z|$ on $|z|=2$

$$\| \quad |e^{2\cos\theta + i2\sin\theta}| = e^{2\cos\theta} \le e^2 < 9$$

(since e^x is increasing for x real
as $2\cos(\theta) \le 2$)

3) $|g|$ on $|z|=2$

$$\| \quad |z^4+1| \ge |z|^4 - 1 = 15$$

4) $|f(z)| < |g(z)|$ on $|z|=2$ by 2 and 3

$g(z)$ has all zeros with $|z|<2$
since a zero of $g(z)$ satisfies $|z|=1$
by 1)

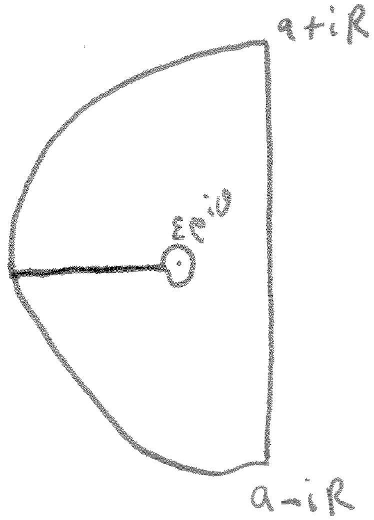
Thus by Rouché z^4+1+e^z has
the same # of zeros (counting multiplicities)
as z^4+1, i.e., 4, in $|z|<2$

5 Compute

singularities at 0, so
$a > 0$ is ok
$t > 0$ is ok

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{zt}\,dz}{\sqrt{z}}$$

$\sqrt{z}$ defined by $\sqrt{1}=1$ negative x axis not crossed



$$\left|\int_{|z|=\varepsilon} \frac{e^{\varepsilon e^{i\theta}\cdot t}\,\varepsilon i e^{i\theta}\,d\theta}{\sqrt{\varepsilon}\, e^{i\theta/2}}\right|$$

$|z|=\varepsilon$
clockwise
from π to $-\pi$

$$\le \int_{-\pi}^{\pi} \left|e^{\varepsilon e^{i\theta}t}\right| \frac{\varepsilon}{\sqrt{\varepsilon}}\,d\theta$$

Since $|e^{\varepsilon e^{i\theta}t}|$ is near 1 for small ε
we have $\lim\limits_{\varepsilon\to 0}\int_{-\pi}^{\pi} \le 2\pi$

$$\lim_{\varepsilon\to 0} 2\sqrt{\varepsilon}\, 2\pi = 0$$

$$\int_{-R}^{-\varepsilon} \frac{e^{xt}}{i\sqrt{|x|}}\,dx = \int_{\varepsilon}^{R} \frac{e^{-xt}}{i\sqrt{x}}\,dx$$

$$\int_{-\varepsilon}^{-R} \frac{e^{xt}}{-i\sqrt{|x|}}\,dx = -\int_{\varepsilon}^{R} \frac{e^{-xt}}{-i\sqrt{x}}\,dx$$

both exist but are equal

$$\int_{\frac{\pi}{2}}^{\pi} \frac{R\, e^{Re^{i\theta}t}\, i e^{i\theta}\, d\theta}{\sqrt{R}\; e^{i\theta/2}} \longrightarrow 0 \quad \text{as } R \to \infty$$

(by the ~~reasoning~~ reasoning in Jordan's lemma.

~~Jordead~~

$$\left| \int_{\frac{\pi}{2}}^{\pi} \right| \leq \sqrt{R} \int_{\frac{\pi}{2}}^{\pi} e^{Rt\cos(\theta)}\, d\theta$$

$$= \sqrt{R} \int_0^{\frac{\pi}{2}} e^{-Rt\sin(\theta)}\, d\theta$$

so is

Now on $\left(0, \frac{\pi}{2}\right]$ $\sin(\theta) \geq \frac{2}{\pi}\theta$

$$\leq \sqrt{R} \int_0^{\frac{\pi}{2}} e^{-Rt\frac{2}{\pi}\theta}\, d\theta = \frac{\sqrt{R}}{Rt\,2/\pi}\left(e^{-Rt} - 1\right)$$

$$\to 0$$

Similarly for $\int_{\mathcal{C}}$ (6)

So we have

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} = \left(-\frac{1}{2\pi i}\right)\frac{2}{i}\int_0^\infty \frac{e^{-xt}}{\sqrt{x}}\,dx$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{-xt}}{\sqrt{x}}\,dx \qquad \text{set } u = xt$$

$$\parallel \quad \frac{1}{\pi}\int_0^\infty e^{-u}\frac{du}{\sqrt{u}\sqrt{t}} \qquad \text{set } u = y^2$$

$$= \frac{1}{\pi\sqrt{t}}\int_0^\infty e^{-y^2}\,2\,du = \frac{2}{\pi\sqrt{t}}\int_0^\infty e^{-u^2}\,du$$

$$= \frac{2}{\pi\sqrt{t}}\cdot\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{\pi t}}$$

why?

6 Note $\cos(x) - 1 = -\frac{x^2}{2} + \cdots$

so singularities are at $\pm i$

$$\text{So } \operatorname{Re} PV \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x^2(x^2+1)}\, dx = \text{answer}$$

$$PV \int_{-\infty}^{\infty} \frac{e^{ix} - 1}{x^2(x^2+1)}\, dx + -\frac{1}{2}\, 2\pi i \operatorname{Res}_0 \frac{e^{iz} - 1}{z^2(z^2+1)}$$

$$= 2\pi i \operatorname{Res}_i \frac{e^{iz} - 1}{z^2(z^2+1)} \qquad \text{since } \int_{\frown} \to 0$$

by Jord. lemma

So answer $=$

$$\pi i \cdot i + \frac{e^{-1} - 1}{-1 \cdot 2i}\, 2\pi i = -\pi + \pi\left(1 - \frac{1}{e}\right)$$

$$= \frac{-\pi}{e}$$

(8)

(7) $f(z) = \dfrac{az+b}{cz+d}$ with $ad-bc \neq 0$

So $\dfrac{a \cdot 0 + b}{c \cdot 0 + d} = \dfrac{b}{d} = \infty \Longrightarrow d = 0$

$f(\infty) = \dfrac{a}{c} = 0 \Longrightarrow a = 0$

$f(z) = \dfrac{b}{cz}$ $\quad \dfrac{b}{c \cdot 1} = 2 \Rightarrow \dfrac{b}{c} = 2$

$2z^{-1}$ is the function

$w = \dfrac{2}{z} \Rightarrow$ inverse is given by $\dfrac{2}{z}$

8 cross ratio of z_1, z_2, z_3, z_4

is $\dfrac{(z_1 - z_4)(z_3 - z_2)}{(z_1 - z_2)(z_3 - z_4)}$

$$= \frac{0 - 3}{0 - 1} = 3$$