Mathematics 40485: Test I

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February 10, 2015

Name: ....

This test is due at the start of class on Thursday, February 12, 2015.

This test is being conducted under the honor code. From the time you receive this exam *until the time it is collected you may not communicate about course material with anyone (in the class or not)*, except me. This includes comments on which problems are easy or hard. *You may not use anyone else's notes or programs on the test.* You are allowed and encouraged to use your notes, Maple, any books, or material on websites (at your own risk of course) while doing the test.

If you find some problem on the test which you think is unclear (or wrong), please contact me immediately—my email address is

`sommese@nd.edu`.

If necessary I will send a correction or clarification to the class by e-mail. It is your responsibility to check your e-mail regularly between now and Thursday, February 12, 2015.

There are a total of 111 points below. Letting T denote the total number of points you get, your test grade will be $\min\{T, 100\}$.

In the following you must show your work, e.g., in Problem 1, a number without the reason for it would receive no credit! You should also include your Maple, Mathematica, Matlab ... worksheets used on the problems. You may use the differentiation, partial fraction, and algebraic simplification commands (including `taylor`, which computes Taylor series), but you must show all work as discussed in class. You cannot use the integral commands or a command that directly computes residues.

Problems

Problem 1 (11 points) *The function*

$$u(x,y) = \frac{x}{x^2+y^2}$$

is the real part of an analytic function $f(z)$. *What is the function?*

Problem 2 (12 points) *The polynomial*

$$p(z) = z^6 - 1$$

has six complex roots. What are they?

Problem 3 (5 points) *If you expanded*

$$e^{\frac{1}{z}}$$

in a power series around $1+i$, *i.e., in a series*

$$\sum_{n=0}^{\infty} a_n(z-1-i)^n,$$

what would the radius of convergence be?

Problem 4 (12 points) *Let* C *denote the counterclockwise path made by joining the eight points*

$$e^{\frac{ki\pi}{4}}$$

with $k = 0, \ldots, 7$. *Compute*

$$\int_C \frac{\mathrm{d}z}{z}.$$

Problem 5 (12 points) *Given*

$$f(z) = \frac{z}{(1-z^2)},$$

expand $f(z)$ *in a Laurent series in powers of* z *in the regions*

1. $|z| < 1$; *and*
2. $1 < |z|$.

Problem 6 (12 points) *Letting* C *denote the unit circle traversed counter-clockwise, compute*

$$\frac{1}{2\pi i}\int_C e^{\left(\frac{8z}{4z^2-1}\right)}\mathrm{d}z.$$

Problem 7 (11 points) *Letting* C *denote the unit circle traversed counter-clockwise, compute*

$$\frac{1}{2\pi i}\int_C \frac{e^{z^3}}{z^7}\mathrm{d}z.$$

Problem 8 (12 points) *Compute*

$$\frac{1}{2\pi i}\int_C \frac{\cosh(z)}{z}\mathrm{d}z,$$

where C *is the boundary of the square with diagonal opposite corners* $1+i$ *and* $-(1+i)$.

Problem 9 (12 points) *Prove the zeroes of* $\sin(z)$ *are*

$$\pi n,$$

where n *runs over the integers.*

Problem 10 (12 points) *Compute*

$$\frac{1}{2\pi i}\int_C \frac{\mathrm{d}z}{\sin(z)},$$

where C *is the boundary of the disk* $\Delta_4(0)$.

Answers to Test I

(1) $\frac{x}{x^2+y^2}$ is the real part of

$\frac{1}{z} + ic$ where c is any real number

$u+iv$ Find v

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = \frac{2xy}{(x^2+y^2)^2}$$

C.R. equat.

So by the second equation

$$v = \frac{-y}{x^2+y^2} + C(y)$$

By 1st eq. $\frac{\partial v}{\partial y} = C'(y) + \frac{2y^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2}$

$$\frac{y^2-x^2}{x^2+y^2} = C'(y) + \frac{y^2-x^2}{(x^2+y^2)^2}$$

or $C =$ real constant

S $u+iv = \frac{x-iy}{x^2+y^2} + ic = \frac{1}{x+iy} + ic$

$$= \frac{1}{z} + ic$$

(2)

$z^6 - 1 = 0$ $\qquad z = e^{i\theta}$

since $|z|^6 = 1$

$\Rightarrow \quad e^{6i\theta} = 1$

$\Rightarrow \quad 6\theta = 2\pi N$

or $\quad \theta = \dfrac{\pi N}{3}$

$\theta = \frac{\pi}{3}, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi, \frac{5}{3}\pi, 2\pi$

give distinct roots and then the repeat

So $\quad 1, w, w^2, w^3, w^4, w^5$

where $w = e^{i\pi/3}$

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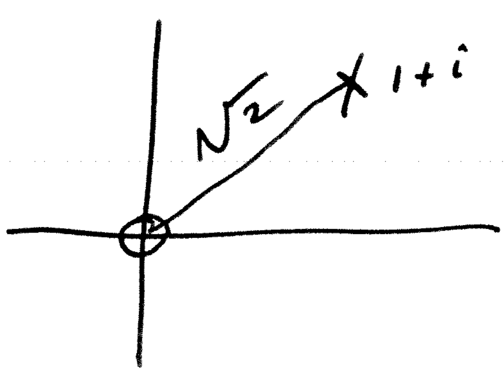
(3) $\quad e^{\frac{1}{z}}$ has a singularity at 0 and nowhere else. So.



$R = \sqrt{2}$

Since the power series at a point converges up to the ~~[illegible]~~ nearest sing. to the point we are ~~[illegible]~~ expanding around

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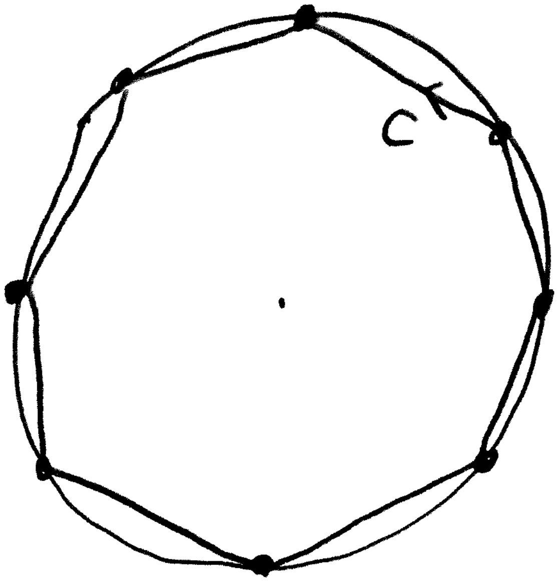

By Cauchy's THM

$$\int_C \frac{dz}{z} = 2\pi i$$

5 1)

$$\frac{z}{1-z^2} = z + z^3 + \cdots = \sum_{n=0}^{\infty} z^{2n+1}$$

2)

$$\frac{z}{1-z^2} = \frac{z}{-z^2}\left(\frac{1}{1-\frac{1}{z^2}}\right) =$$

$$-\frac{1}{z}\,\frac{1}{1-\frac{1}{z^2}} = -\frac{1}{z} - \frac{1}{z^3} - \frac{1}{z^5} \cdots = -\sum_{n=0}^{\infty} \frac{1}{z^{2n+1}}$$

(6)

$e^{\frac{8z}{4z^2-1}}$ has singularities at $z=\pm\frac{1}{2}$

Computing the residues at $z=\pm\frac{1}{2}$ is very involved! So we let $z=\frac{1}{w}$

Then

$$\frac{1}{2\pi i}\int_{\substack{C \\ \| \\ \partial\Delta_{\underset{z=0}{\uparrow 0}}(1)}} e^{\frac{8z}{4z^2-1}}\,dz = \frac{1}{2\pi i}\int_{\substack{C \\ \| \\ \partial\Delta_{\underset{w=1}{\uparrow 0}}(1)}} \frac{e^{\frac{8w}{4-w^2}}\,dw}{w^2}$$

$\operatorname{Res}_{w=0} \dfrac{e^{\frac{8w}{4-w^2}}}{w^2}$ = coefficient of w in the expansion of $e^{\frac{8w}{4-w^2}}$

$$e^{\frac{8w}{4-w^2}} = e^{2\frac{w}{1-\frac{w^2}{4}}} = 1+2\left(1+\frac{w^2}{4}+\cdots\right)w + \text{H.O.T.}$$

So $\operatorname{Res}_{w=0}(*) = 2$. and $\int = 2$

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$\frac{e^{z^3}}{z^7}$ has singularities only at 0. Thus

$$\frac{1}{2\pi i}\int_C \frac{e^{z^3}}{z^7}\,dz = \mathrm{Res}_0 \frac{e^{z^3}}{z^7}$$

The Laurent expansion of

$$\frac{e^{z^3}}{z^7} \text{ is } \frac{1}{z^7}+\frac{z^3}{z^7}+\frac{z^6}{2!\,z^7}+\frac{z^9}{3!\,z^7}+\cdots$$

the coefficient of the $\frac{1}{z}$ term is $\frac{1}{2}$. So the integral

$$= \frac{1}{2}$$

(8) $$\frac{\cosh(z)}{z} = \frac{e^z + e^{-z}}{2z}$$

Thus $\frac{\cosh(z)}{z}$ only has a singularity at $z = 0$, which is inside the square.

So $$\frac{1}{2\pi i} \int_C \frac{\cosh(z)}{z}\, dz$$

$$= \operatorname{Res}_0 \frac{\cosh z}{z}$$

$\frac{\cosh(z)}{z}$ has the Laurent expansion at 0

$$\frac{\cosh z}{z} = \frac{1}{z} + z + \text{higher order terms}$$

So answer is 1

9 $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = 0$

iff $\quad e^{iz} = e^{-iz}$

iff $\quad e^{2iz} = 1$

iff $\quad 2z = N \cdot 2\pi \qquad$ N arb. integer

iff $\quad z = N\pi \qquad$ "

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10 $\quad \sin(z)$ has zeroes at $N\pi$

So these are the singularities of $\frac{1}{\sin(z)}$

inside the disk of radius 4

$|N\pi| \le 4$ gives $N = 0, 1, -1$.

Thus $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{dz}{\sin(z)} = \mathrm{Res}_{-1} \frac{1}{\sin(z)}$

$+ \mathrm{Res}_0 \frac{1}{\sin(z)} + \cancel{\mathrm{Res}}$

$\mathrm{Res}_1 \frac{1}{\sin(z)}$

$\dfrac{1}{\cos(0)} = 1$

$\dfrac{1}{\cos(\pi)} = -1 \qquad$ So

$\dfrac{1}{\cos(-\pi)} = -1$

$-1 + 1 + -1 = -1$