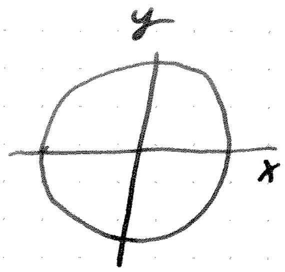
More on curves



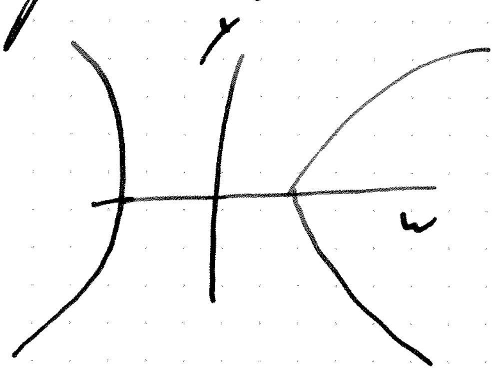
Consider $x^2+y^2-1=0$.

Homogenize $\left(\left(\frac{x}{w}\right)^2+\left(\frac{y}{w}\right)^2-1\right)w^2$

$$= x^2+y^2-w^2$$

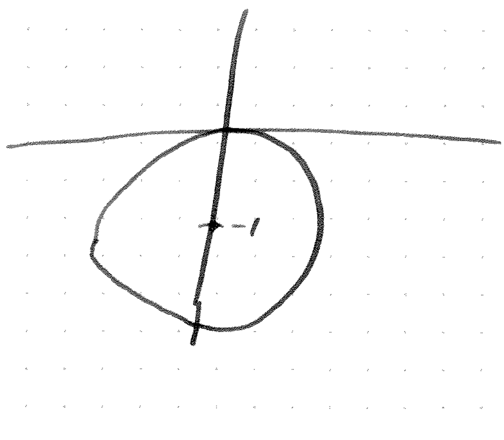
Dehomogenize with respect to $x=0$

$$\Rightarrow \quad 1+y^2-w^2=0$$



Move the circle over

$$x^2+(y+1)^2=1$$

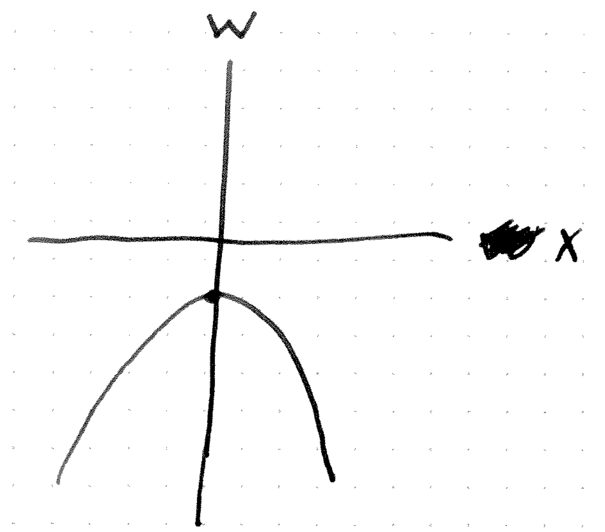


Homogenize

$$x^2+(y+w)^2=w^2$$

Dehomogenize with respect to $y=0$

$$x^2+(1+w)^2=w^2 \Leftrightarrow x^2+1+2w=0$$



So the conic sections are more or less all the same.

Finding singular points

$$x^4 + y^4 - w^4 = 0$$

Hard way $\quad x^4 + y^4 - 1$ $\quad$ (the patch $\mathbb{C}^2 \longrightarrow \mathbb{P}^2$, $(x,y) \longrightarrow [1,x,y]$)

$$\begin{aligned} 4x^3 &= 0 \\ 4y^3 &= 0 \end{aligned} \implies \begin{aligned} x &= 0 \\ y &= 0 \end{aligned}$$

but they do not satisfy $x^4 + y^4 - 1 = 0$

$$x^4 + 1 - w^4 = 0$$

(the patch $\mathbb{C}^2 \longrightarrow \mathbb{P}^2$, $(x,w) \mapsto [w,x,1]$)

Same calculation

$$1 + y^4 - w^4 = 0$$

Same calc.

(the patch $\mathbb{C}^2 \to \mathbb{P}^2$, $(y,w) \longrightarrow [w,1,y]$)

Easy way

singular points of $p(x,y,w)=0$ gotten by going through patches, the same as the solutions of

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = \frac{\partial p}{\partial z} = 0$$

We don't even need to check these solutions back into $p(x,y,z)=0$.

<u>Example</u> $\quad x^4-2x^2y^2+y^4=0$

$$\frac{\partial(x^4-2x^2y^2+y^4)}{\partial x} = 4x^3-4xy^2=0$$

$$\frac{\partial}{\partial y}(\qquad) = 4y^3-4x^2y=0$$

Solutions are the solutions of

$x(x-y)(x+y)=0$ $\quad$ $[0,0]$ not a point of $\mathbb{P}$

$y(x-y)(x+y)=0$ $\quad$ $[1,1],[1,-1]$ are solutions

Euler's Theorem Let p be a degree d homogeneous polynomial on $\mathbb{P}^n$ (with homogeneous coordinates $z_0, \dots, z_n$)

Then

$$z_0 \frac{\partial}{\partial z_0} p + \quad \cdots \quad + z_n \frac{\partial}{\partial z_n} p$$

$$= d\, p$$

Ex $\quad z_0^d + z_1^d + z_2^d$

$$z_0 \left(d \cdot z_0^{d-1} \right) + z_1 \left(d z_1^{d-1} \right) + z_2 \left(d \cdot z_2^{d-1} \right)$$

$$= d \left(z_0^d + z_1^d + z_2^d \right)$$

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pf $\quad p$ is a function on $\mathbb{C}^{N+1}$

Consider $p(z_0 t, \dots, z_n t)$

$$= t^d \, p(z_0, \quad , z_n)$$

Differentiate with respect to t
and evaluate at $t=1$

$$\frac{\partial p}{\partial t}(t z) = \frac{\partial p}{\partial z_0}\Big|_{t=1} \frac{\partial (t z_0)}{\partial t} + \cdots$$

$$= \sum_{i=0}^{n} z_i \frac{\partial p}{\partial z_i}$$

LHS

RHS $= \frac{\partial t^d}{\partial t}\Big|_{t=1} p(z_0, \quad z_n)$

$= d\, p(z_0, \quad , z_n)$

Automorphisms of curves

$$z_0^3 + z_1^3 + z_2^3 = 0$$

$$\left.\begin{aligned}
[z_0, z_1, z_2] \longrightarrow\ & [z_1, z_0, z_2] \\
\longrightarrow\ & [z_1, z_2, z_0] \\
\longrightarrow\ & [z_2, z_0, z_1] \\
\longrightarrow\ & [z_2, z_1, z_0] \\
\longrightarrow\ & [z_0, z_1, z_2] \\
\longrightarrow\ & [z_0, z_2, z_1]
\end{aligned}\right\} \text{6 symmetries on 3 letters}$$

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what about multiplying

$$[z_0, z_1, z_2] \longrightarrow [z_0, \alpha z_1, \beta z_2]$$

if $\alpha^3 = \beta^3 = 1$ the new point satisfies

$z_0^3 + (\alpha z_1)^3 + (\beta z_2)^3 = 0$ if original point

satisfies $z_0^3 + z_1^3 + z_2^3 = 0$

ω = 3rd root of unity

$$z_0^3 z_1 + z_1^3 z_2 + z_2^3 z_0 = 0$$

Klein quartic

$$z_0, z_1, z_2 \to z_1, z_2, z_0$$

$$\to z_2, z_0, z_1$$

$$z_0, z_1, z_2 \to z_0, \alpha z_1, \beta z_2$$

$$\alpha z_0^3 z_1 + \alpha^3 \beta z_1^3 z_2 + \beta^3 z_2^3 z_0$$

$\alpha^2 \beta = 1$ $\quad \beta^7 = 1$ $\quad \beta$ a 7th root of unity

$\beta^3 = \alpha$

ω

$\alpha = \omega^3$

$7 \cdot 3 = 21$

Group is actually 168 (Famous simple group)

$g = 3$

$2 \cdot 3 - 2 = 4$ $\quad \boxed{84(g-1)}$ bound for automorphisms

$x^4 + y^4 - z^4 = 0$

1) smooth (no singular points)

2) at ∞, 4 solutions

$[1, \omega, 0]$ $\qquad$ ω a 4th root of -1

$$\begin{array}{ccc} [x,y,z] & & \mathbb{P}^2 - \{[0,0,1]\} \\ \downarrow & & \downarrow \\ [x,y] & & \mathbb{P}^1 \end{array}$$

well defined on $V(x^4+y^4-z^4)$ since $[0,0,1] \notin$

4 to 1 $\qquad z^4 = x^4+y^4$

except when $x^4+y^4=0$

Then it is 1-1

So triangulating $\mathbb{P}^1$ with solutions of $x^4+y^4=0$ among the vertices then we have

$$v - e + f = 2 \quad \text{on } \mathbb{P}^1$$

$\underline{\underline{d=4}}$

$$d \cdot v - \underbrace{4(d-1)}_{\substack{\text{overcount} \\ \text{at 4 points of} \\ V(x^4+y^4)}} - de + df = 2-2g$$

where g is the genus of $V(z^4 - x^4 - y^4)$

or $d(v - e + f) - 4(d-1) = 2 - 2g$

$$\parallel$$

$$2d - 4d + 4$$

$$\parallel$$

$$4 - 2d = -4$$

$$\Rightarrow g = 3$$

More generally $x^d + y^d - z^d = 0$

$$g = \frac{d^2 - 3d + 2}{2}$$

d	g
1	0
2	0
3	1
4	3
5	6
6	10

All smooth curves of degree d have the same genus.

Other automorphism?

$$PL(2) \qquad [z_0, z_1, z_2]$$

$$\mapsto \quad A \cdot \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix}$$

A an invertible 3×3 matrix

note $A = \begin{pmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \lambda \end{pmatrix}$ acts as

the identity. There may be other automorphisms besides these.