Homework 8 (due Tuesday, March 24)

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(2) let $g(z) = f(z) - f_0$.

$$\frac{1}{2\pi i}\oint_C \frac{g'}{g}\,dz = \# \text{ zeroes of } g \text{ within } C.$$
counting mult.

zeroes of g = points where $f(z) = f_0$

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(5)
$$f = -4z^2$$
$$g = e^z - 1$$
on $\partial\Delta_1(0)$ $\quad |f| = 4$
$$|g| \leq |e^z| + 1 = |e^{R\cos(\theta)}| + 1$$
$$\leq e + 1 < 4 = |f|$$

So by $e^z - 4z^2 - 1$ has the same # of zeroes in $\Delta_1(0)$ as $-4z^2$, i.e., 2 zeroes.

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$$\int_0^\infty f''(x) e^{-sx} dx =$$

$$f'(x) e^{-sx}\Big|_0^\infty + s\int f'(x) e^{-sx} dx$$

$$= -f'(0) + s f(x) e^{-sx}\Big|_0^\infty + s^2 \int_0^\infty f(x) e^{-sx} dx$$

$$= -f'(0) - s f(0) + s^2 \mathcal{L}(f)(s)$$

$\hat{F}(s)$ book's notation for

13 (a) $$\mathcal{L}\left(e^{ax} f(x)\right) = \int_0^\infty e^{-sx} e^{ax} f(x)\, dx$$

$$= \int_0^\infty e^{-(s-a)x} f(x)\, dx = \hat{F}(s-a)$$

(b) $a>0$ $$\mathcal{L}\left(f(x-a) H(x-a)\right) = \int_0^\infty e^{-sx} f(x-a) H(x-a) dx$$

$$= \int_a^\infty e^{-sx} f(x-a)\, dx = \int_0^\infty e^{-su} e^{-as} f(u)\, du$$

$u = x-a$
$du = dx$

$$= e^{-as} \int_0^\infty e^{-su} f(u) du$$

$$= e^{-as} \hat{F}(s)$$

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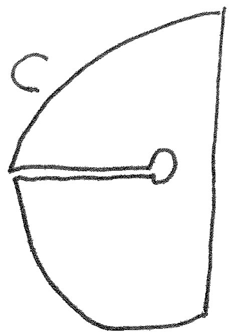
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{sx}\frac{\ln(s)}{s^2+w^2}\,ds \qquad w>0.$$

$c>0$



C as in figure 4.5.2

$$\frac{1}{2\pi i}\int_C \longrightarrow ds = \operatorname{Res}_{iw}\frac{e^{sx}\ln s}{s^2+w^2} + \operatorname{Res}_{-iw}\frac{e^{sx}\ln s}{s^2+w^2}$$

on the ε circle the integrand is

$$\frac{e^{\varepsilon e^{i\theta}x}(\ln\varepsilon+i\theta)}{\varepsilon^2e^{2i\theta}+w^2}$$

So $\left|\int_{\varepsilon\text{ circle}}\right| \le$

$$\varepsilon\int_{-\pi}^{\pi}\frac{e^{\varepsilon x\cos(\theta)}(|\ln\varepsilon|+\theta)}{1-\varepsilon^2}\,d\theta$$

$$\le \frac{\varepsilon}{1-\varepsilon^2}\left(|\ln\varepsilon|+2\pi\right)\int_{-\pi}^{\pi}e^{\varepsilon x\cos\theta}\,d\theta$$

$$\le \frac{2\pi\varepsilon}{1-\varepsilon^2}\left(|\ln\varepsilon|+2\pi\right)e^{\varepsilon x}\xrightarrow[\varepsilon\to 0]{\text{as}} 0$$

on $z = c + Re^{i\theta}$ $\quad \frac{\pi}{2} \le \theta \le \pi$

$$\left| \frac{e^{sx} \ln(s)}{s^2 + w^2} \right| = \left| \frac{e^{c + Re^{i\theta}} (\ln R + i\theta)}{R^2 e^{2i\theta} + w^2} \right|$$

$$\le \frac{e^c e^{R\cos\theta}}{R^2 - w^2} (\ln R + \pi)$$

$$\le \frac{e^c (\ln R + \pi)}{R^2 - w^2}$$

since $\cos\theta \le 0$

when $\frac{\pi}{2} \le \theta \le \pi$

So $$\left| \int \quad \right| \le \frac{R(\ln R + \pi)}{R^2 - w^2} e^c 2\pi \longrightarrow 0$$

on $z = c + Re^{i\theta}$ $\quad \pi \le \theta \le \frac{3}{2}\pi$

the same argument

shows $\int \to 0$

Thus $\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{e^{sx} \ln s}{s^2 + w^2} \, ds$ ~~[illegible]~~ $=$

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residue $-\frac{1}{4\pi i}\int_{\infty}^{0} \frac{e^{-ux}(\ln u + i\pi)}{u^2+w^2}\, d(-u)$

$$-\frac{1}{2\pi i}\int_{0}^{\infty} \frac{e^{-ux}(\ln u - i\pi)}{u^2+w^2}\, d(-u)$$

$= \text{residue} + -\int_{0}^{\infty} \frac{e^{-ux}}{u^2+w^2}\, du$

$$\text{residue} = \frac{e^{iwx}\left(\ln w + i\frac{\pi}{2}\right)}{2(iw)} + \frac{e^{-iwx}\left(\ln w - i\frac{\pi}{2}\right)}{(-2iw)}$$

$$= \frac{\left(e^{iwx} - e^{-iwx}\right)}{2i}\frac{\ln w}{w} + \frac{i\pi}{2iw}\frac{\left(e^{iwx} + e^{-iwx}\right)}{2}$$

$$= \frac{\sin(wx)}{w}\ln w + \frac{\pi}{2w}\cos(wx)$$

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~~Show~~

$w = \sqrt{1-z^2}$

$w^2 = 1 - z^2$

$w = u + iv$. Show $2x^2 - 2y^2 = 1$ goes to $2u^2 - 2v^2 = 1$

$u^2 - v^2 = 1 - x^2 + y^2$

$2uv = -2xy$

$2x^2 - 2y^2 - 1 = 2v^2 - 2u^2$
-1

so $2x^2 - 2y^2 - 1 = 2v^2 - 2u^2 + 1$

if $2x^2 - 2y^2 = 1$ then $2v^2 - 2u^2 + 1 = 0$
or $2u^2 - 2v^2 = 1$