Homework 7, March 5, 2015

By evenness of cos replacing k, m by $|k|, |m|$ does not change the answer

pg 253

1 a)

$$\lim_{R\to\infty} \int \frac{e^{ikx}-e^{imx}}{x^2}\,dx = ⨍_{-\infty}^{\infty} \frac{e^{ikx}-e^{imx}}{x^2}\,dx$$

$$-\pi i \operatorname{Res}_0 \frac{e^{ikx}-e^{imx}}{x^2}$$

$$= -\pi i\big((ik-im)\big)$$

$$= +\pi(|k|-|m|)$$

$$2\int_0^{\infty} \frac{\cos(kx)-\cos(mx)}{x^2}\,dx$$ is the real part of

So Answer is $\frac{\pi}{2}(|k|-|m|)$

(b) $\sin^2(x) + \cos^2(x) = 1$

$-\sin^2(x) + \cos^2(x) = \cos(2x)$

$$\Rightarrow \sin^2(x) = \frac{1-\cos(2x)}{2}$$

Taking $k=0$, $m=2$

we get $$\int_0^{\infty} \frac{\sin^2(x)}{x^2}\,dx = \frac{\pi}{4}\cdot 2 = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin(x)}{x(x^2+1)}\,dx$$

$$= \frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{e^{ix}}{x\,(x^2+1)}\,dx$$

note $\int \to 0$ (why?).

So $\int_{-\infty}^{\infty} + -\pi i\,\mathrm{Res}_0\,\frac{e^{ix}}{x(x^2+1)} =$

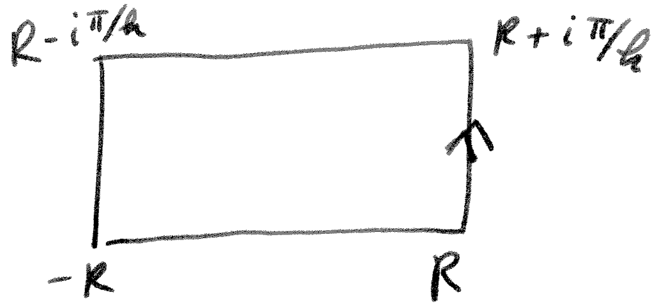
$2\pi i\,\mathrm{Res}_i\,\frac{e^{ix}}{x(x^2+1)}$

or $\int_0^\infty \frac{\sin(x)}{x(x^2+1)}\,dx =$

$$\mathrm{Im}\left(\frac{\pi i}{2} + \pi i\,\frac{e^{-1}}{i\cdot 2i}\right)$$

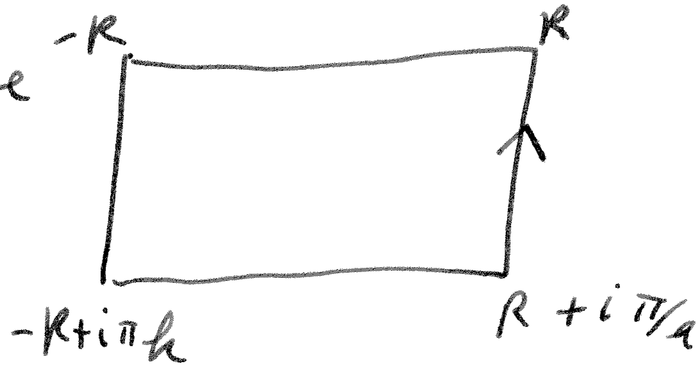
$$= \frac{\pi}{2}\,\mathrm{Im}\; i\left(1-\frac{1}{e}\right) = \frac{\pi}{2}\left(1-\frac{1}{e}\right)$$

(4) we let k be > 0.



is the contour of the book before we worry about indentations

If k was < 0, the contour would be



and would be traversing in opposite directions for the integrals we used in the case $k>0$ leading to the minus sign in the book answer

$$\frac{\pi^2}{4k|k|}$$

The zeroes of $\sinh(kx)$ satisfy $e^{kx} = e^{-kx}$ or

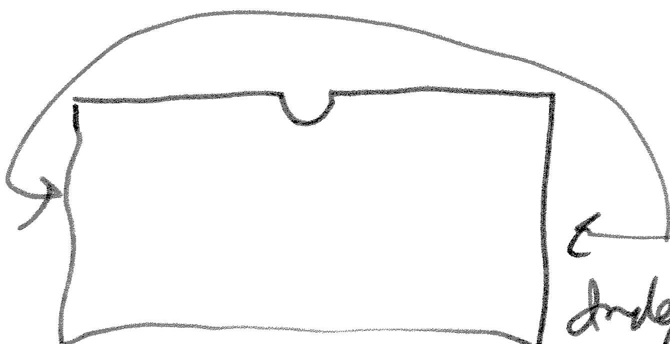
$$x = \pi i \frac{\nu}{k}$$ where ν is an integer.

$\frac{x}{\sinh(kx)}$ has no singularity

at 0 since $\frac{x}{\sinh(kx)} = \frac{2}{k + \frac{k^3}{3!}x^2 + \cdots}$

but it does ~~have~~ have a first order pole at $\frac{\pi i}{k}$. So we use the

contour



Note on the vertical parts of the contour $\int \to 0$.

Indeed note $|\sinh(kR+iy)|$

$$\geq \left|\frac{e^{kR} - e^{-kR}}{2}\right|$$

$$\geq \frac{e^{kR}}{4}$$

Same for $|\sinh(-kR+iy)|$

(5)

So
$$\int_{-\infty}^{\infty} \frac{x\,dx}{\sinh(kx)} + \int_{\infty}^{-\infty} \frac{x + i\pi/k}{\sinh(kx + i\pi)}\,dx$$

$$+ \pi i \operatorname{Res}_{i\pi/k}\left(\frac{z}{\sinh(kz)}\right) \underset{\substack{\uparrow \\ \text{why?}}}{=} 0$$

So since $\sinh(kx)$ is odd

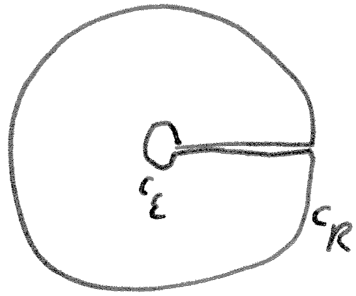
$$\int_{-\infty}^{\infty} \frac{x\,dx}{\sinh(kx)} = 2\int_{0}^{\infty} \frac{x\,dx}{\sinh(kx)}$$

$$\int_{\infty}^{-\infty} \frac{x + i\pi/k}{\sinh(kx+i\pi)}\,dx = \int_{\infty}^{-\infty} \frac{x + i\pi/k}{-\sinh(kx)}\,dx$$

$$= \underbrace{\int_{\infty}^{-\infty} \frac{x\,dx}{-\sinh(kx)}}_{\overset{\text{why}}{=} 2\int_0^\infty \frac{x\,d}{\sinh(kx)}} + \underbrace{\left(-\frac{i\pi}{k}\right)\int_{\infty}^{-\infty} \frac{dx}{\sinh(kx)}}_{= 0 \text{ since } \sinh(kx) \text{ is odd}}$$

Thus
$$4\int_0^\infty \frac{x\,dx}{\sinh(kx)} = \pi i \cdot \frac{\pi i/k}{k\cosh(\pi i)} = \frac{\pi^2}{k^2}$$

7a)



$0 < k < 1$

$a > 0$

$R > a$

$$\left| \int_{c_R} \frac{z^{k-1}}{z+a} dz \right| = \left| \int_0^{2\pi} \frac{e^{(k-1)\theta i} R^{k-1} R i e^{i\theta} d\theta}{Re^{i\theta} + a} \right|$$

$$\leq \frac{R^k}{R-a} 2\pi \longrightarrow 0$$

Tak $\varepsilon < a$

$$\left| \int_{c_\varepsilon} \right| \leq \frac{\varepsilon^k}{a-\varepsilon} 2\pi \to 0$$

So $$\int_0^\infty \frac{x^{k-1}}{x+a} dx + \int_\infty^0 \frac{x^{k-1} e^{i(k-1)2\pi}}{x+a} dx$$

$$= 2\pi i \operatorname{Res}_{-a} \frac{z^{k-1}}{z+a} = 2\pi i \, e^{\pi i (k-1)} \cdot a^{k-1}$$

So $$\int_0^\infty \frac{x^{k-1}}{x+a} dx = \frac{2\pi i \, a^{k-1}}{1 - e^{i(k-1)2\pi}} e^{\pi i (k-1)}$$

$$= \frac{-2\pi i \, a^{k-1}}{e^{-\pi i k} - e^{\pi i k}} = \frac{-\pi a^{k-1}}{\sin(-\pi k)} = \frac{\pi a^{k-1}}{\sin(\pi k)}$$

7b

Same argument

$$\int_0^\infty \frac{x^{k-1}}{(x+1)^2}\,dx = \frac{2\pi i}{1-e^{(k-1)2\pi i}}\,\mathrm{Res}_{-1}\,\frac{x^{k-1}}{(x+1)^2}$$

$\mathrm{Res}_{-1}\dfrac{x^{k-1}}{(x+1)^2}$ is the coef of the $(x+1)$ term of the power series expansion of x^{k-1} =

$(k-1)\,x^{k-2}$ evaluated at $e^{i\pi}$

$$= (k-1)\,e^{(k-2)\pi i}$$

$$\text{So } \int_0^\infty \frac{x^{k-1}}{(x+1)^2}\,dx = \frac{i2\pi(k-1)e^{k\pi i}}{1-e^{k2\pi i}}$$

$$= \frac{\pi(k-1)}{\sin(k\pi)}$$

Note the derivative with respect to a of the first integral is after setting a to 1 gives the second (Check!)
This only works for $0<k<1$ though. Why?

11 Done in class

13

$$\left| \int_{C_R \cap c_\varepsilon} \frac{z^{\frac{1}{2}} \ln z}{1+z^2}\, dz \right| \to 0$$

Why? (Same as before, but check it).

$$\text{so } \int_0^\infty \frac{x^{\frac{1}{2}} \ln x}{1+x^2}\, dx + \int_\infty^0 \frac{-x^{\frac{1}{2}}(\ln(x) + 2\pi i)}{1+x^2}\, dx$$

$$= 2\pi i \; \mathrm{Res}_i \left(\frac{z^{\frac{1}{2}} \ln(z)}{1+z^2} + \mathrm{Res}_{-i} \frac{z^{\frac{1}{2}} \ln(z)}{1+z^2} \right)$$

$$= 2\pi i \left(\frac{e^{i\pi/4} \frac{\pi}{2} i}{2i} + \frac{e^{3/4 \pi i} \frac{3}{2} \pi i}{2(-i)} \right)$$

$$= \frac{\pi^2}{2} \left(i e^{i\pi/4} \cdot - i e^{3/4 \pi i} 3 \right)$$

$$= \cancel{\pi}\frac{\pi^2}{2} i \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i - 3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \right)\right)$$

$$= \frac{\pi^2}{2} i \, 2\sqrt{2} \quad \cdot \quad - \sqrt{2}\, i \frac{\pi^2 i}{2}$$

$$= \frac{\pi^2 \sqrt{2}}{2} + \pi^2 \sqrt{2}\, i$$

So

$$2\int_0^\infty \frac{x^{\frac{1}{2}} \ln x}{1+x^2}\, dx + 2\pi i \int_0^\infty \frac{x^{\frac{1}{2}}}{1+x^2}\, dx$$

$$= \frac{\pi^2 \sqrt{2}}{2} + \pi^2 \sqrt{2}\, i$$

which equating real and imaginary parts gives the answers.

consider $\boxed{\text{almost prob 11}}$ 10

$$\frac{1}{2\pi i}\int_{C_R}\frac{dz}{\sqrt{z^2-1}} \qquad R>1$$

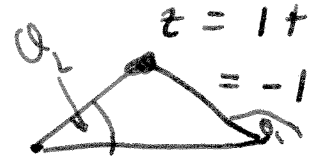
$$\|$$

let $z=\frac{1}{w}$

$$\frac{1}{2\pi i}\int_{C_{\frac{1}{R}}}\frac{dw}{w^2\sqrt{\frac{1}{w^2}-1}} = \frac{1}{2\pi i}\int_{C'_{\frac{1}{R}}}\frac{dw}{w\sqrt{1-w^2}}$$

$$= 1$$

$$\int_{C_R} = \int_{-1}^{1} + \int_{1}^{-1}$$



$$z = 1+R_1e^{i\theta_1}$$
$$= -1+R_2e^{i\theta_2}$$

$$\sqrt{z^2-1}$$

from above $\theta_2 \to 0$

$$\sqrt{z^2-1} = i\sqrt{1-x^2}$$

$$= \sqrt{R_1e^{i\theta_1}}\sqrt{R_2e^{i\theta_2}}$$

$$= \sqrt{R_1R_2}\cdot e^{i\frac{\theta_1+\theta_2}{2}}$$

from below $\theta_1 \to \pi$, $\theta_2 \to 2\pi$

$$= -i\sqrt{1-x^2}$$

$$\sqrt{R_1R_2}\,e^{i\frac{3\pi}{2}}$$

So

$$2\pi i = \int_{1}^{-1} \frac{-i\,dx}{\sqrt{1-x^2}} + \int_{-1}^{1} \frac{+i\,dx}{\sqrt{1-x^2}}$$

$$= +2i \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$$

So $$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = +\pi$$