Homework 6 (Due Thursday, February 26)

Pg. 235: 3, 4, 5, 6, 7, 8a, 9.

$\cos\theta = \left(z + \frac{1}{z}\right)\frac{1}{2}$ on path $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$

3) $$\int_0^{2\pi} \cos^{2m}(\theta)\,d\theta =$$

$$\int_{\partial\Delta_1(0)} \frac{\left(z+\frac{1}{z}\right)^{2m}}{2^{2m}} \frac{dz}{iz} = \frac{2\pi i}{2^{2m}\, i} \operatorname{Res}_0 \frac{(z^2+1)^{2m}}{z^{2m+1}}$$

$$= \frac{2\pi}{2^{2m}} \text{ coef of } z^{2m} \text{ in } (z^2+1)^{2m}$$

$$= \frac{2\pi}{2^{2m}} \text{ coef of } z^{m} \text{ in } (z+1)^{2m} = \frac{2\pi}{2^{2m}} \binom{2m}{m}$$ it seems perverse to write

the answer as the book does!

$$\binom{2m}{m} = \frac{(2m)!}{m!\,m!} = \frac{2^m m! \cdot (2m-1)\cdots 1}{m!\,m!} = \frac{2^m (2m-1)\cdots 1}{m \cdots 1}$$

$$= 2^{2m} \cdot \frac{(2m-1) \quad \cdots \quad 1}{(2m)\ (2m-2) \quad \cdot 2}$$

$$\text{Answer} = 2\pi \cdot \frac{2m-1}{2m} \cdots \frac{1}{2} < 2\pi$$

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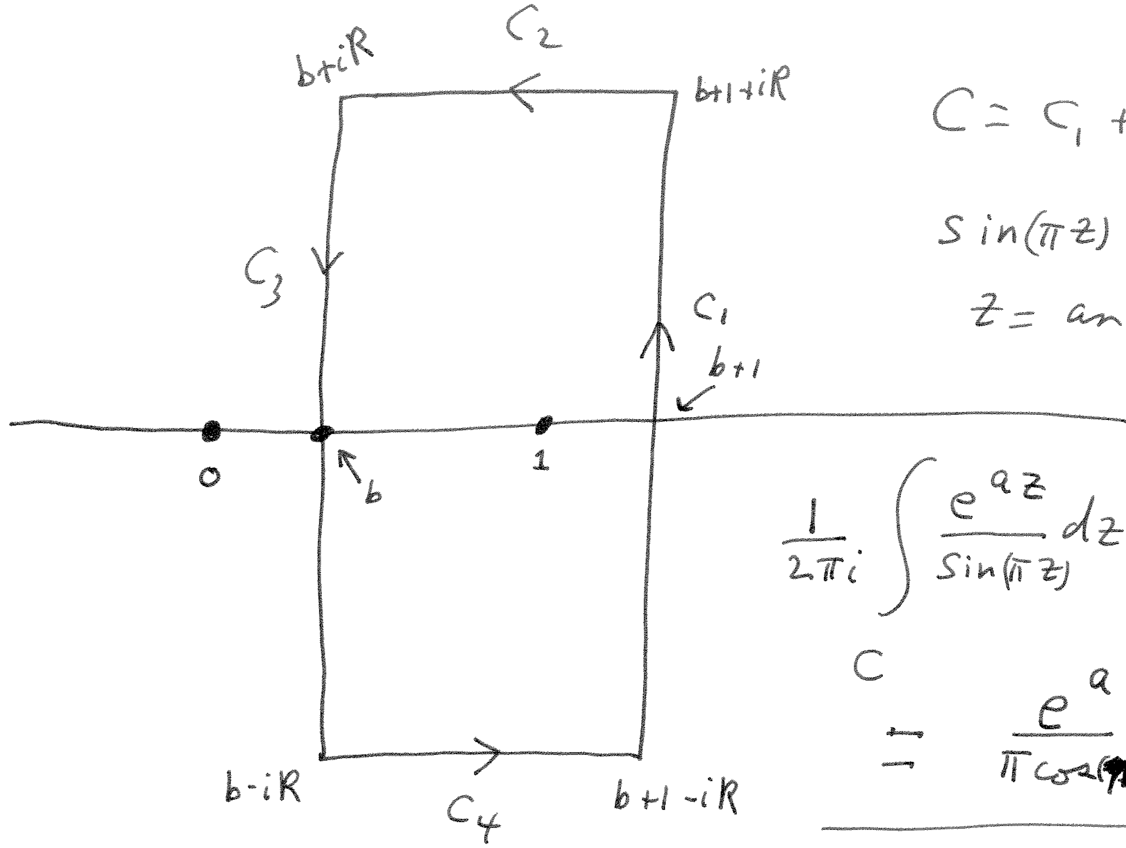
note $\cos\left(\theta - \frac{\pi}{2}\right) = \sin(\theta)$ (*)

$$\int_0^{2\pi} \cos^{2m}(\theta)\,d\theta = \int_{\pi/2}^{5/2\,\pi} \cos^{2m}\left(\varphi - \frac{\pi}{2}\right) d\varphi \overset{\text{periodicity}}{=} \int_0^{2\pi} \cos^{2m}\left(\varphi - \frac{\pi}{2}\right) d\theta$$

change variable $\theta = \varphi - \frac{\pi}{2}$ $\quad d\theta = d\varphi$

$$\overset{(*)}{=} \int_0^{2\pi} \sin^{2m}(\varphi)\,d\varphi$$



$C = C_1 + C_2 + C_3 + C_4$

$\sin(\pi z) = 0$ at

$z =$ an integer $\Rightarrow$

$$\frac{1}{2\pi i}\int_C \frac{e^{az}}{\sin(\pi z)}\,dz = \operatorname{Res}_1\left(\frac{e^{az}}{\sin(\pi z)}\right)$$

$$= \frac{e^a}{\pi\cos(\pi)} = -\frac{e^a}{\pi}$$

$a = \alpha + iA$ with $|A| < \pi$

$$\int_{C_1} \frac{e^{az}}{\sin(\pi z)}\,dz = \int_{b-iR+1}^{b+iR+1} \frac{e^{az}}{\sin(\pi z)}\,dz$$

shorthand for

$$i\int_{-R}^{R} \frac{e^{a(b+1)+ayi}}{\sin(\pi(b+1)+i\pi y)}\,dy$$

where we use $z = b+1+iy$ $\quad -R \le y \le R$ to parameterize the path

Note $\sin(z+\pi) = \dfrac{e^{iz+i\pi} - e^{-iz-i\pi}}{2i}$

$$= \frac{-1\cdot e^{iz} - (-1)e^{-iz}}{2i} = -\sin(z)$$

$\Rightarrow \sin(\pi(b+1)+i\pi y) = -\sin(\pi b + i\pi y)$

So the last integral equals

$$-i\int_{-R}^{R} \frac{e^{a(b+1)+ayi}}{\sin(\pi b + i\pi y)}\,dy =$$

$$-i\,e^a\int_{-R}^{R} \frac{e^{a(b+yi)}}{\sin(\pi(b+iy))}\,dy = e^a\int_{C_3}\frac{e^{az}}{\sin(\pi z)}\,dz$$

So if we show

$\lim_{R\to\infty}\int_{C_2} = 0$, $\lim_{R\to\infty}\int_{C_4} = 0$

we will have $-(1+e^a)\lim_{R\to\infty}\dfrac{1}{2\pi i}\displaystyle\int_{b-iR}^{b+iR}\frac{e^{az}}{\sin(\pi z)}\,dz = -\frac{e^a}{\pi}$, which gives $$\frac{1}{2\pi i} \lim_{R\to\infty} \int_{b-iR}^{b+iR} \frac{e^{az}}{\sin(\pi z)}\,dz = \frac{e^a}{\pi(1+e^a)}$$ ③

$$= \frac{1}{\pi(e^{-a}+1)}$$

So let's look at $\int_{C_2}$.

$$\left|\int_{C_2} \frac{e^{az}}{\sin(\pi z)}\,dz\right| \underset{\uparrow}{=} \left| -\int_0^1 \frac{e^{ab+aiR+ax}\,dx}{\sin(\pi b+\pi Ri+\pi x)}\right| = \quad \text{use } a=\alpha+iA$$

$z = b+iR+x$
$0 \le x \le 1$

$$\stackrel{?}{=} |e^{ab}| \left|\int_0^1 \frac{e^{\alpha Ri - AR + \alpha x + i a A}}{\frac{e^{i\pi b - \pi R + \pi x i} - e^{-(i\pi b - \pi R + i\pi x)}}{2i}}\,dx\right|$$

It is convenient to note that

$|\sin(w)| \ge |\sinh(\operatorname{Im} w)|$

Indeed $\sin(w) = \frac{e^{iw}-e^{-iw}}{2i}$

let $w = u+iv$ u, v real

$|\sin(w)| = \left|\frac{e^{iu-v} - e^{-iu+v}}{2i}\right|$

$\ge \left|\frac{e^{-v}-e^{v}}{2}\right| = |\sinh(v)|$

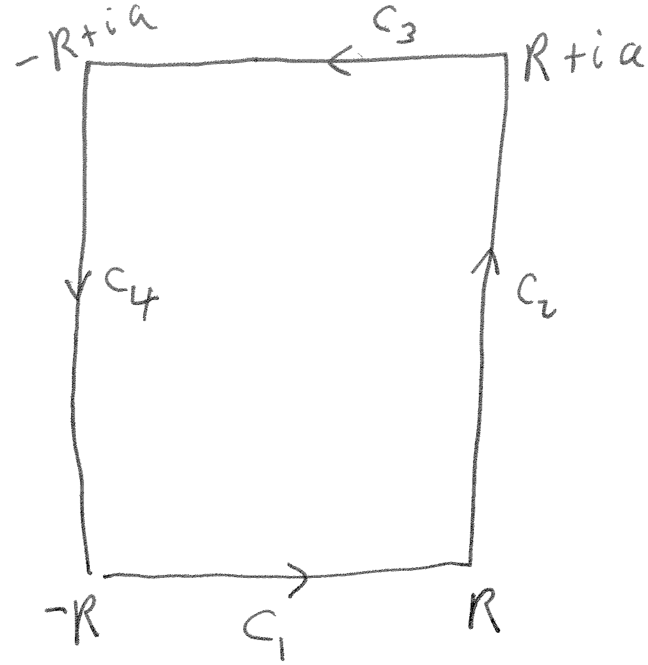
$$\le |e^{ab}|\, 2 \int_0^1 \frac{e^{-AR+\alpha x}}{e^{\pi R} - e^{-\pi R}}\,dx =$$

$$2|e^{ab}| \frac{e^{-AR}}{e^{\pi R}-e^{-\pi R}} \int_0^1 e^{\alpha x}\,dx = \frac{(e^{\alpha}-1)}{\alpha} 2|e^{ab}| \frac{e^{-AR}}{e^{\pi R}-e^{-\pi R}}$$

Since α, a, b are fixed, we need only show $\frac{e^{-AR}}{e^{\pi R}-e^{-R\pi}} \to 0$. Since $|A| < \pi$ this is clear.

The same argument works for $\int$

⑥



C_3 given by $z = x+ia$, x from R to $-R$

$$C_R = C_1 + C_2 + C_3 + C_4$$

$$\int_{C_1} e^{-z^2}dz = \int_{-R}^{R} e^{-x^2}dx$$

$$\int_{C_3} e^{-z^2}dz = \int_{R}^{-R} e^{-(x+ia)^2}dx$$

$$= -\int_{-R}^{R} e^{-(x+ia)^2}dx$$

$$\int_{C_2} = \int_0^a e^{-(R+iy)^2} i\,dy$$

$$C_4 = \int_a^a e^{-(-R+iy)^2} i\,dy = -\int_0^a e^{-(-R+iy)^2} i\,dy$$

So the decomposition is shown.

$$\left|\int_{C_2}\right| \le \int_0^a \left|e^{-(R^2-y^2)-2yRi}\right| dy$$

$$= \int_0^a e^{-R^2} e^{y^2} dy = \underbrace{\left(\int_0^a e^{y^2}dy\right)}_{\text{constant}} \underset{\substack{\downarrow \\ 0 \text{ as } R\to\infty}}{e^{-R^2}}$$

Similarly $\left|\int_{C_4}\right| \to 0$ as $R \to \infty$

Thus $$\int_{-\infty}^{\infty} e^{-(x+ia)^2}\,dx = \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

The latter is easily seen to be $\sqrt{\pi}$. (Why?)

So $$\int_{-\infty}^{\infty} e^{-(x+ia)^2}\,dx = \sqrt{\pi}$$

$$\|$$

$$\int_{-\infty}^{\infty} e^{-x^2} e^{a^2} e^{-2xai}\,dx = \sqrt{\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-x^2}\left(\cos(2ax) - \sin(2xa)i\right)dx = e^{-a^2}\sqrt{\pi}$$

Since sin is odd, we get

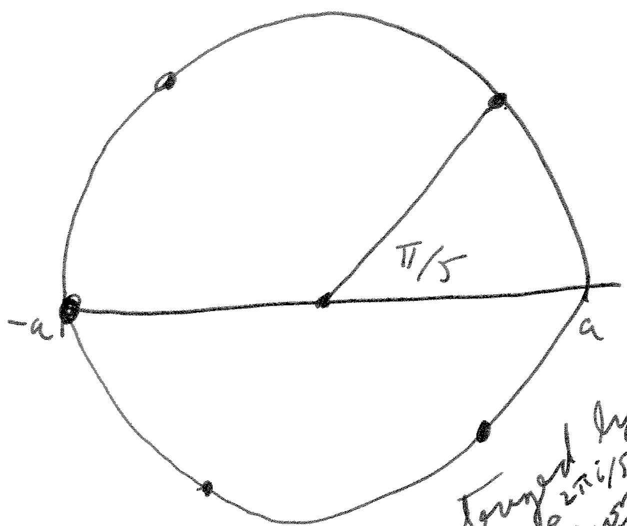
$$\int_{-\infty}^{\infty} e^{-x^2}\cos(2ax)\,dx = e^{-a^2}\sqrt{\pi}$$

7)

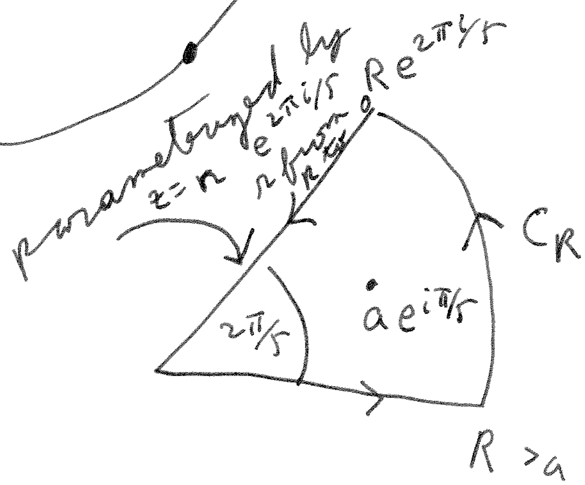
$z^5 + a^5 = 0$, $a > 0$

has solutions $z = a e^{i\pi/5},\ a e^{i3\pi/5},\ a e^{i\pi},\ a e^{i7\pi/5},\ e^{i9\pi/5} a$

$a e^{i\pi} = -a$



So the sector



contains one singularity $ae^{i\pi/5}$ of $\frac{1}{z^5+a^5}$

Thus $\displaystyle\int_{C_R} \frac{dz}{z^5+a^5} = 2\pi i \,\mathrm{Res}_{ae^{i\pi/5}} \frac{1}{z^5+a^5}$

$$= \frac{2\pi i}{5a^4 e^{i4\pi/5}}$$

$\parallel$

$$\underbrace{\int_0^{2\pi/5} \frac{i e^{i\theta} R\, d\theta}{R^5 e^{i5\theta} + a^5}} + \int_0^R \frac{dx}{x^5+a^5} + \int_R^0 \frac{e^{2\pi i/5}\, dr}{r^5+a^5}$$

$$\int_R^0 \frac{e^{2\pi i/5}\, dr}{r^5+a^5} = -e^{2\pi i/5} \int_0^R \frac{dx}{x^5+a^5}$$

$$|\quad| \le \frac{R}{R^5 - a^5}\, 2\pi/5 \to 0$$

So

$$\int_0^\infty \frac{dx}{x^5+a^5} = \frac{2\pi i}{5a^4 e^{i4\pi/5}\left(1 - e^{2\pi i/5}\right)} =$$

$$\frac{2\pi i}{5a^4\left(e^{i4\pi/5} - e^{i6\pi/5}\right)} = \frac{2\pi i}{5a^4\left(-e^{-i\pi/5} + e^{i\pi/5}\right)}$$

since $e^{i\pi} = -1$

$e^{i\pi 4/5} = -e^{-i\pi/5}$

$e^{i6\pi/5} = -e^{i\pi/5}$

$$= \frac{\pi}{5a^4} \cdot \frac{1}{\frac{e^{i\pi/5} - e^{-i\pi/5}}{2i}} = \frac{\pi}{5a^4 \sin(\pi/5)}$$

8a · $\csc(\pi\zeta) = \frac{1}{\sin(\pi\zeta)}$ has singularities at the zeroes of $\sin(\pi z)$; i.e., at $z = -N, \dots, N$

$$\operatorname{Res}_j \frac{\pi}{\sin(\pi\zeta)(z^2 - \zeta^2)} = \frac{\pi}{z^2 - j^2}\,\frac{1}{\pi\cos(\pi j)} = \frac{(-1)^j}{z^2 - j^2}$$

$$\operatorname{Res}_z \frac{\pi}{\sin(\pi\zeta)(z^2 - \zeta^2)} = \frac{\pi}{\sin(\pi z)(-2z)}$$

$$\operatorname{Res}_{-z} \frac{\pi}{\sin(\pi\zeta)(z^2 - \zeta^2)} = \frac{\pi}{\sin(-\pi z)(2z)}$$

So $I(N)$ = sum of residues $= \left(\sum_{j=-n}^{n} \frac{(-1)^j}{z^2 - j^2}\right) - \frac{\pi\csc(\pi z)}{z}$

9 $\quad \cot(\pi z) = \dfrac{\cos(\pi z)}{\sin(\pi z)}$ has singularities at $z =$ any integer

$$\coth(\pi z) = \frac{e^{\pi z}+e^{-\pi z}}{e^{\pi z}-e^{-\pi z}}$$ has singularities

at $2\pi z = 2\pi i \cdot n$ where n is an integer

So $$\frac{1}{2\pi i}\int_{C_N} \frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}\,dz$$

$=$ sum of residues at $-N,\dots,N$
$-iN,\dots,iN$

zero is special

at $j \neq 0$ $\quad \operatorname{Res}_j \dfrac{\pi\cot(\pi z)\coth(\pi z)}{z^3}$

$$= \frac{\pi(-1)^j\coth(\pi j)}{j^3\cdot\pi(-1)^j} = \frac{\coth(\pi j)}{j^3}$$

at ij $\quad \operatorname{Res}_{ij}$ —— $= \dfrac{\pi}{\pi}\dfrac{\cot(ij\pi)}{-j^3 i^3} = -\dfrac{e^{i^2\pi j}+e^{-i^2\pi j}}{e^{i^2\pi j}-e^{-i^2\pi j}}$

$$= \left(\frac{e^{\pi j}+e^{-\pi j}}{e^{\pi j}-e^{-\pi j}}\right)\frac{1}{j^3} = \frac{\coth(\pi j)}{j^3}$$

So assuming the $\int_{C_N} \to 0$ as $N \to \infty$ (it does by the argument in class), we

get
$$2\sum_{j=-\infty}^{\infty}{}' \frac{\coth(\pi j)}{j^3} = -\mathrm{Res}_0\left(\frac{\pi\cot(\pi z)\coth(\pi z)}{z^3}\right)$$

The prime means skip $j=0$

Note $\frac{\coth(\pi(-j))}{(-j)^3} = \frac{\coth(\pi j)}{j^3}$ and we get

$$\sum_{1}^{\infty} \frac{\coth(\pi n)}{n^3} = \frac{1}{4}\mathrm{Res}_0 \frac{-\pi\cot(\pi z)\cdot\coth(\pi z)}{z^3}$$

$$= \frac{7}{180}\pi^3$$ by the hint in the book.

We could hack out the residue, but I am attaching a simple Maple worksheet (only ~~work~~ work if a machine can't do it!)

```
> with(numapprox);
```

$$[chebdeg, chebmult, chebpade, chebsort, chebyshev, confracform, hermite\_pade, hornerform, infnorm, laurent, minimax, pade, remez] \tag{1}$$

```
> laurent(-Pi*cot(Pi*z)*coth(Pi*z)/z^3,z=0);
```

$$-\frac{\frac{1}{\pi}}{z^5} + \frac{\frac{7}{45}\pi^3}{z} + O(z^3) \tag{2}$$

we can go further (the default is to the 1/z term, which is needed for residue calculations.

```
> laurent(-Pi*cot(Pi*z)*coth(Pi*z)/z^3,z=0,4);
```

$$-\frac{\frac{1}{\pi}}{z^5} + \frac{\frac{7}{45}\pi^3}{z} + \frac{19}{14175}\pi^7 z^3 + O(z^7) \tag{3}$$

```
> laurent(-Pi*cot(Pi*z)*coth(Pi*z)/z^3,z=0,15);
```

$$-\frac{\frac{1}{\pi}}{z^5} + \frac{\frac{7}{45}\pi^3}{z} + \frac{19}{14175}\pi^7 z^3 + \frac{2906}{212837625}\pi^{11} z^7 + \frac{13687}{97692469875}\pi^{15} z^{11} + O(z^{15}) \tag{4}$$