Homework 5 (Due Thursday, February 19)

Pg. 216 1, 3, 5.

Pg. 235 1bd, 2abehi.

pg 216

1 a) $2z^3-3z^2-2z = z(2z^2-3z-2) =$

roots 0, $z = \frac{3}{4} \pm \frac{\sqrt{25}}{4} = \frac{4}{2}, -\frac{1}{2}$

so $\int = 2\pi i \left(Res_0 + Res_{-\frac{1}{2}} \right)$

$= 2\pi i \left(\frac{1}{-2} + \frac{\frac{1}{2}}{6\cdot\frac{1}{4}+6\cdot\frac{1}{2}-2} \right) = -\pi i + 2\pi i \frac{\frac{1}{2}}{\frac{5}{2}}$

$= -\frac{3\pi i}{5}$ and $\boxed{\frac{1}{2\pi i}\int = -\frac{3}{10}}$

---

b) 0 only sing.

$\frac{\cosh(\frac{1}{z})}{z} = \frac{e^{1/z}+e^{-1/z}}{2z}$

$= \frac{2}{2z}$ + power of z other than z^{-1}

so $\int = 2\pi i$ and $\boxed{\frac{1}{2\pi i}\int = 1}$

---

c) only singularities at $\pm\frac{\pi}{2} i$ both larger than 1 in abs. value

so $\int = 0$

e)

2

$$\frac{z+\frac{1}{z}}{z\left(2z-\frac{1}{2z}\right)} = \frac{(z^2+1)2z}{z^2(4z^2-1)}$$

$$= 2\,\frac{z^2+1}{z(4z^2-1)}$$

sing. at $0, \pm\frac{1}{2}$

So $\displaystyle\int = 2\pi i\left(\mathrm{Res}_0 + \mathrm{Res}_{\frac{1}{2}} + \mathrm{Res}_{-\frac{1}{2}}\right)$

$$= 2\pi i\left(\frac{2}{-1} + \frac{2\,\frac{5}{4}}{\frac{1}{2}\cdot 4} + \frac{2\cdot\frac{5}{4}}{-\frac{1}{2}\cdot(-4)}\right)$$

$$= 2\pi i\left(-2 + \frac{5}{4} + \frac{5}{4}\right)$$

$$= \pi i \qquad \boxed{\frac{1}{2\pi i}\int = \frac{1}{2}}$$

---

(3) (a) pole of order m

(b) branch point $\frac{1}{w^{\frac{1}{3}}}$

(c) simple pole

$\sqrt{\frac{1}{w^2}+a^2} \qquad a^2>0$

$= \frac{\pm\sqrt{1+aw^2}}{w}$

and so on as in the back of book

5 I did this in class.

pg 235

1 b) $\quad a^2 > 0$

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

$$= \frac{1}{2} \cdot 2\pi i \cdot \mathrm{Res}_{|a|i} \frac{1}{(z^2+a^2)^2}$$

$$= \pi i \left(\mathrm{Res}_{|a|i} \frac{(z+|a|i)^{-2}}{(z-|a|i)^2} \right)$$

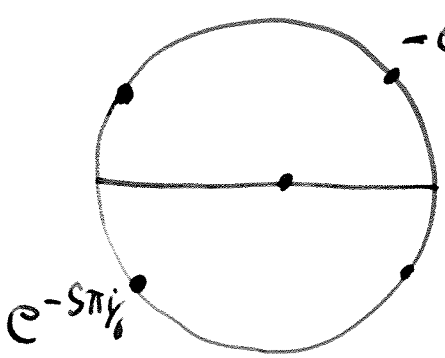
$$= \pi i \frac{-2}{(|a|i+|a|i)^3}$$

$$= \frac{2\pi}{(2|a|)^3} = \frac{\pi}{4|a|^3}$$

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \cancel{2\pi i \left(\mathrm{Res}_{e^{\pi i/6}} + \mathrm{Res}_{e^{2\pi i/3}} \right)}$$

$$\frac{2\pi i}{2}\left(\operatorname{Res}_{e^{\pi i/6}} + \operatorname{Res}_{e^{i\pi/2}} + \operatorname{Res}_{e^{5\pi i/6}}\right)$$

$$= \frac{2\pi i}{2\cdot 6}\left(e^{-5\pi i/6} + e^{-5\pi i/2} + e^{-\pi i/6}\right)$$



$$= \frac{2\pi i}{2\cdot 6}\left(-i + (-e^{i\pi/6} + e^{-\pi i/6})\right)$$

$$= +\frac{2\pi}{2\cdot 6} + \frac{2\pi i}{2\cdot 6}\, 2i \sin(-\pi/6)$$

$$= +\frac{2\pi}{2\cdot 6} + \frac{4\pi}{2\cdot 6}\sin(\pi/6)$$

$$= +\frac{\pi}{6} + \frac{\pi}{3}\sin(\pi/6)$$

$$= \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$$

2a)

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^2+a^2}\,dx$$

Using Jordan =

$$\operatorname{Im}\left(2\pi i \operatorname{Res}_{i|a|} \frac{x e^{ix}}{x^2+a^2}\right)$$

$$= \operatorname{Im}\left(2\pi i \frac{i|a|e^{-|a|}}{2i|a|}\right)$$

$$= \pi e^{-|a|}$$

---

b) $\int_{-\infty}^{\infty} = \operatorname{Re} \int \frac{e^{ikx}}{(x^2+a^2)(x^2+b^2)}\,dx$ (we assume $|a| \neq |b|$, $|b| > |a|$)

$$\underset{\text{by Jordan}}{=} \operatorname{Re}\left(2\pi i \left(\operatorname{Res}_{i|a|} + \operatorname{Res}_{i|b|}\right)\right)$$

$$= \operatorname{Re}\left(2\pi i\left(\frac{e^{-k|a|}}{2i|a|(b^2-a^2)} + \frac{e^{-k|b|}}{2i|b|(a^2-b^2)}\right)\right)$$

$$= \frac{\pi}{\ldots}\left(\frac{e^{-k|a|}}{|a|} - \frac{e^{-k|b|}}{\ldots}\right)$$

e) $$\int_0^\infty \frac{x^3 \sin(k x)}{x^4+a^4}\,dx =$$

$$\frac{1}{2}\,\mathrm{Im}\int_{-\infty}^{\infty}\frac{x^3 e^{ikx}}{x^4+a^4}\,dx$$

$$\underset{\text{Jordan}}{\overset{\uparrow}{=}}\ \frac{1}{2}\,\mathrm{Im}\left(2\pi i\left(\mathrm{Res}_{|a|\cdot e^{i\pi/4}} + \mathrm{Res}_{|a|e^{i3\pi/4}}\right)\right)$$

$$= \cancel{\frac{\pi}{4}\,\mathrm{Im}\left(i\left(e^{ike^{i\pi/4}|a|} + e^{ik|a|3\pi/4}\right.\right.}$$

$$\frac{\pi}{4}\,\mathrm{Im}\left(i\left(e^{ik|a|e^{i\pi/4}} + e^{ik|a|e^{3\pi i/4}}\right)\right)$$

$$\frac{\pi}{4}\,\mathrm{Im}\, i\left(e^{ik|a|\cos(\pi/4)}e^{-k|a|\sin(\pi/4)} + e^{ik|a|\cos(3\pi/4)}\cdot e^{-k|a|\sin(3\pi/4)}\right)$$

$$= \frac{\pi}{4}\cos\left(k|a|\cos\left(\frac{\pi}{4}\right)\right)e^{-k|a|\sin(\pi/4)} + \frac{\pi}{4}\cos\left(k|a|\cos(3\pi/4)\right)e^{-k|a|\sin(3\pi/4)}$$

note $\cos\left(\frac{3\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right)$

so
$$\int_0^\infty = \frac{\pi}{4}\left(\cos\left(k|a|\frac{\sqrt{2}}{2}\right)\cdot\left(2\, e^{-k|a|\frac{\sqrt{2}}{2}}\right)\right)$$

$$= \frac{\pi}{2}\, e^{-k|a|\frac{\sqrt{2}}{2}} \cos\left(k|a|\frac{\sqrt{2}}{2}\right)$$

(h)
$$\int_0^{2\pi} \frac{d\theta}{(5-3\sin\theta)} = \int_0^{2\pi} \frac{dz}{iz}\cdot\frac{1}{\left(5-3\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)\right)}$$

$z=e^{i\theta}$

$$= \int_{\partial\Delta_{1,0}} \frac{dz}{iz}\cdot\frac{1}{5-\frac{3}{2i}\left(z-\frac{1}{z}\right)}$$

$$= \int_{\partial\Delta_1(0)} \frac{1}{i}\,\frac{dz}{\left(\frac{3}{2}iz^2+5z-\frac{3}{2}i\right)}$$

roots of the denominator are ⑧

at $\dfrac{-5 \pm \sqrt{\cancel{16}\,16}}{3i}$

$= -\dfrac{3}{i}, \ -\dfrac{1}{3i}$

only $\frac{i}{3}$ is in disc $\Delta_1(0)$

<u>So</u> $\displaystyle\int = 2\pi i \cdot \mathrm{Res}_{i/3} \frac{1}{i\left(\frac{3}{2}iz^2+5z-\frac{3}{2}i\right)}$

$= \dfrac{2\pi i}{i} \cdot \dfrac{1}{3i\cdot\frac{i}{3}+5}$

$= 2\pi \dfrac{1}{-1+5} = \dfrac{\pi}{2}$

2i

since $e^{i(k+m)x} + e^{i(k-m)x}$

$= 2e^{ikx}\cos(mx)$

$= 2\cos(kx)\cos(mx) + 2i\sin(kx)\cos(mx)$

we have

$$\int_{-\infty}^{\infty} \frac{\cos(kx)\cos(mx)}{(x^2+a^2)}\,dx$$

$$= \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{i(k+m)x}}{x^2+a^2}\,dx$$

$$+ \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{i(k-m)x}}{x^2+a^2}\,dx$$

assume $k > m > 0$ (other cases similar)

$$= \frac{1}{2}\operatorname{Re}\cdot 2\pi i\left(\operatorname{Res}_{i|a|}\frac{e^{i(k+m)z}}{z^2+a^2} + \operatorname{Res}_{i|a|}\frac{e^{i(k-m)z}}{z^2+a^2}\right)$$

$$= \pi \, \mathrm{Re} \left(\frac{e^{-(k+m)|a|}}{2|a|} + \frac{e^{-(k-m)|a|}}{2a} \right)$$

$$= \frac{\pi e^{-k|a|}}{\cancel{2}|a|} \cosh(m|a|)$$