Homework 4 pg 135 Due Tuesday Feb. 10, 2015 1

1 $f(z) = \dfrac{1}{1+z^2}$

a $1 - z^2 + z^4 - z^6 + z^8 \cdots$

b $\dfrac{1}{z^2}\left(\dfrac{1}{1+\frac{1}{z^2}}\right) =$

$$\frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^6} - \frac{1}{z^8} \cdots$$

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2 $f(z) = \dfrac{z}{a^2 - z^2}$ $a > 0$

a $\dfrac{z}{a^2}\left(\dfrac{1}{1-\frac{z^2}{a^2}}\right) = \dfrac{z}{a^2}\displaystyle\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^{2n}$

b $-\dfrac{1}{z}\left(\dfrac{1}{1-\left(\frac{a}{z}\right)^2}\right) = -\dfrac{1}{z} - \dfrac{a^2}{z^3} - \dfrac{a^4}{z^5} \cdots$

3

$$\frac{z}{(z-2)(z+i)} = \frac{a}{z-2} + \frac{b}{z+i}$$

$$az + bz + ai + b(-2)$$

$$= z$$

$$\Rightarrow \quad a + b = 1 \quad \Rightarrow \quad a + \frac{a}{2} i = 1$$

$$ai = 2b$$

$$a = \frac{1}{1 + \frac{i}{2}}$$

$$= \frac{2}{2+i}$$

$$= \frac{4-2i}{5}$$

a

$|z| < 1$

$$\frac{a}{z-2} + \frac{b}{z+i}$$

$$= \frac{-a}{2} \frac{1}{1-\frac{z}{2}} + \frac{b}{i} \frac{1}{1-iz}$$

$$= \sum_{n=0}^{\infty} -\frac{a}{2}\left(\frac{z}{2}\right)^n + \sum_{n=0}^{\infty} \frac{b}{i}(-iz)^n$$

$$= \sum_{n=0}^{\infty} \left(-\frac{a}{2}\frac{1}{2^n} + \frac{b(-i)^n}{i}\right) z^n$$

c

$$\frac{a}{z-2} + \frac{b}{z+i}$$

$$= \frac{a}{z}\frac{1}{1-\frac{2}{z}} + \frac{b}{z}\frac{1}{1+\frac{i}{z}}$$

$$= \frac{a}{z}\sum_{0}^{\infty}\left(\frac{2}{z}\right)^{n} + \frac{b}{z}\sum_{0}^{\infty}\left(\frac{-i}{z}\right)^{n}$$

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4 a $$\int_C \frac{e^z}{z^3}\,dz = \int\frac{dz}{z^3} + \int\frac{dz}{z^2} + \int\frac{dz}{z\cdot 2!}$$

$$+ \int \underbrace{\sum \text{pos powers of } z}_{\text{convergent on whole plane}}\,dz$$

$$= 0 + 0 + \frac{2\pi i}{2} + 0 = \pi i$$

b $$\frac{1}{z^3\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}\ \cdots\right)} = \frac{1}{z^3} + \frac{1}{z^3}\left(\frac{z^2}{3!} - \frac{z^4}{5!}\cdots\right) + \frac{1}{z^3}\left(\frac{z^2}{3}+\cdots\right)^2$$

$$= \cancel{\text{anal}}\ \frac{1}{z^3} + \frac{1}{6z} + \text{analytic near } 0$$

So $\int_C \frac{dz}{z^2 \sin z} = \frac{2\pi i}{6} = \frac{\pi i}{3}$ 4

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4 c) $\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$

$\sinh(z) = \frac{e^z - e^{-z}}{2}$ analytic

$\cosh(z) = \frac{e^z + e^{-z}}{2}$ analytic and

zero iff $e^{2z} = -1$

iff $2z = \pi i + 2n\pi i$

iff $z = \underbrace{\frac{\pi}{2} i + n\pi i}_{\text{none within the unit circle}}$

so $\int_C \tanh(z)\,dz = 0$

4 c) $\int_C \frac{dz}{\cos(2z)}$ $\cos(2z) = 0$ iff

$2z = \frac{\pi}{2} + n\pi$

iff $z = \frac{\pi}{4} + \frac{n}{2}\pi$

so
$$\int_C \frac{dz}{\cos(2z)} = 2\pi i \left(\text{Residue at } \tfrac{\pi}{4} + \text{Res. at } -\tfrac{\pi}{4}\right)$$

$$= 2\pi i \lim_{z \to \frac{\pi}{4}} \frac{z - \frac{\pi}{4}}{\cos 2z} + 2\pi i \lim_{z \to -\pi/4} \frac{z + \pi/4}{\cos 2z}$$

$$\parallel$$

$$2\pi i \frac{1}{-2\sin\left(\frac{\pi}{2}\right)} + 2\pi i \frac{1}{-2\sin\left(-\frac{\pi}{2}\right)}$$

$$= -\pi i + \pi i = 0$$

e) $$\int_C e^{\frac{1}{z}} dz = 2\pi i \text{ res at } 0$$
$$= 2\pi i$$