Homework 2

pg 44

1 b) $u = y^3 - 3x^2y$

$v = x^3 - 3xy^2 + 2$

$$\frac{\partial u}{\partial x} = -6xy = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 = -\frac{\partial v}{\partial x}$$

$$f(z) = iz^3 + 2i$$

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1 c) $u = e^y \cos x$

$v = e^y \sin y$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= -e^y \sin(x) \\ \frac{\partial v}{\partial y} &= e^y \sin(y) + e^y \cos(y) \end{aligned}\right\}$$

Not equal so not analytic

2b

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial x}(2xc - 2xy) = 2c - 2y$$

$$\Rightarrow v = A(x) + 2cy - y^2$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = +2x$$

$$\parallel$$

$$A'(x) \qquad \Rightarrow A(x) = x^2 + \alpha$$

constant of integration

so $u + iv =$

$$2xc - 2xy + i x^2 + i\alpha + 2cyi - y^2 i$$

$$= 2cz + iz^2 + i\alpha$$

2 d)

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x} \frac{\cos(x)e^{y} + \cos(x)e^{-y}}{2}$$

$$= \frac{-\sin(x)e^{y} - \sin(x)e^{-y}}{2}$$

$$\Rightarrow \quad V = A(x) + \frac{-\sin(x)e^{y}}{2} + \frac{\sin(x)e^{-y}}{2}$$

$$\frac{\partial v}{\partial x} = A'(x) - \frac{\cos(x)e^{y}}{2} + \frac{\cos(x)e^{-y}}{2}$$

$$\parallel$$

$$-\frac{\partial u}{\partial y} = -\frac{\cos(x)e^{y}}{2} + \frac{\cos(x)e^{-y}}{2}$$

$$\Rightarrow \quad V = C + \frac{-\sin(x)e^{y} + \sin(x)e^{-y}}{2}$$

$$u + iv = iC + e^{y}\frac{\cos(x) - i\sin(x)}{2} + e^{-y}\frac{\cos(x) + i\sin(x)}{2}$$

$$= ic + \frac{e^{y-ix} + e^{ix-y}}{2} = ic + \frac{e^{-iz}}{2} + \frac{e^{iz}}{2}$$

$$= ic + \cosh(iz)$$

3 a) $\tan(z) = \dfrac{\sin(z)}{\cos(z)}$ analytic

except at $\dfrac{\pi}{2} + \pi n$ n an integer

not entire

(b) $e^{\sin z}$ entire

c) $e^{\frac{1}{z-1}}$ analytic except at $z=1$

d) $e^{\overline{z}}$ not analytic $e^{\overline{z}} = e^{x-iy}$

$= e^{x}\cos y$

$- i e^{x} \sin y$

$\dfrac{\partial}{\partial x} u = e^{x}\cos y$

$\dfrac{\partial}{\partial y} v = -e^{x}\cos y$

e) $\dfrac{z}{z^4+1}$ analytic except at

$z = e^{i\frac{\pi}{4} + j\frac{\pi i}{2}}$

$j = 0, 1, 2, 3$

~~f)~~ $\cancel{\dfrac{\partial}{\partial x} \cos x \cosh y = -\sin x \cosh y}$

$\cancel{\dfrac{\partial}{\partial y} -\sin x \sinh y = -\sin x \cosh y}$

~~and so on or simply not~~

$\cancel{\dfrac{\partial}{\partial x}} \cos(x)\cosh y - i \sin(x)\sinh y = e^{-ix}e^{y}+c$

3 f

$$f(x,y) = \frac{\cos(x)e^{y}}{2} + \frac{\cos(x)e^{-y}}{2}$$

$$- i\,\frac{\sin(x)\,e^{y}}{2} + i\,\frac{\sin x\, e^{-y}}{2}$$

$$= \frac{e^{-ix}e^{y}}{2} + \frac{e^{ix}e^{-y}}{2}$$

$$= \frac{e^{-iz} + e^{iz}}{2} = \cosh(iz)$$

entire

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5 $f(z)$ assume pure imaginary $\Rightarrow$

$f(D) \subseteq$ y axis, a curve $\Rightarrow$ f constant

by results in class

Let $z^* \in D$

$f(z)$ and $f(\bar{z})$ analytic $\Rightarrow$ f constant

Pf $\quad f(z) = u(x,y) + i v(x,y)$

$$f(\bar{z}) = u(x,-y) + i v(x,-y)$$

both analytic $\Rightarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{all } (x,y) \in D$$

$$\left.\frac{\partial u}{\partial x}\right|_{(x,-y)} = -\left.\frac{\partial v}{\partial y}\right|_{(x,-y)} \qquad \text{all } \overset{(x,-y)}{\cancel{(x,y)}} \in D$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x} \qquad \text{and so on}$$

$$\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial y}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

$\Rightarrow$ u, v constant.

pg 59

1 a) $z = 1, \infty$

$z = x \quad x \geq 1$ or $x < 1$

or $z = 1 + re^{i\alpha}$ α fixed, $r \geq 0$

b) $z = -1 + 2i, \infty$

$z = x + 2i \quad x \geq -1$

or

$z = -1 + 2i + re^{i\alpha}$ α fixed, $r \geq 0$

c) $z = 0, \infty$ pos. real axis

neg real axis

pos imag. axis

⋮

d) $z^{\sqrt{2}} = e^{\sqrt{2} \ln z} \Rightarrow z = 0, \infty$ branch point

use the branch cuts from c)

8

(2) (a) $i^{\frac{1}{2}} = \left(e^{\frac{i\pi}{2} + 2\pi N i}\right)^{\frac{1}{2}}$ N integer

$= e^{i\pi/4} e^{\pi N i}$

$= \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$

$i^{\frac{1}{2}} = e^{\frac{1}{2}\ln i}$ ← prin value

$= i\frac{\pi}{2}$

So $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ is principal value

(b) $\frac{1}{(1+i)^{1/2}} = \frac{1}{e^{\frac{1}{2}\ln(1+i)}}$

since $\ln(1+i) = \underbrace{\ln\sqrt{2} + i\frac{\pi}{4}}_{p.v.} + \underset{\substack{\uparrow \\ \text{integer}}}{N} 2\pi i$

$\frac{1}{(1+i)^{\frac{1}{2}}} = \frac{1}{\sqrt[4]{2}\, e^{i\frac{\pi}{8}} \cdot e^{N\pi i}} = \pm \frac{e^{-i\pi/8}}{\sqrt[4]{2}}$

with p.v. $\frac{1}{\sqrt[4]{2}} e^{\frac{-i\pi}{8}}$

c $\ln(1+\sqrt{3}\,i) = \ln\left(2e^{i\frac{\pi}{3}}\right)$

So p.v. $= \ln 2 + i\frac{\pi}{3}$

all poss. values

$$\ln 2 + i\frac{\pi}{3} + 2N\pi i$$

N an integer

d $\ln i^3$

$= \ln -i = \frac{3}{2}i\pi + 2\pi i N$

with p.v.

$$\frac{3}{2}\pi i$$

e) $i^{\sqrt{3}} = e^{\sqrt{3}\ln i} = e^{\sqrt{3}\left(i\frac{\pi}{2} + 2Ni\pi\right)}$

$= e^{i\frac{\pi\sqrt{3}}{2}}$ p.V.

all pos. values $e^{i\pi\left(2N+\frac{1}{2}\right)\sqrt{3}}$

N runs over integers

$= \cos\left(\pi\left(2N+\frac{1}{2}\right)\sqrt{3}\right) + i\sin\left(\pi\sqrt{3}\left(2N+\frac{1}{2}\right)\right)$

(f) $\arcsin \frac{1}{\sqrt{2}}$

$$= -i \ln\left(\frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$= -i\left(\frac{i\pi}{4} + 2N\pi i\right)$$

$$= \frac{\pi}{4} + 2N\pi$$

p.v. when $N = 0$

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(3) (a) $z^5 = 1$

$$z = e^{j\frac{2\pi i}{5}} \qquad j = 0, 1, 2, 3, 4$$

(3b) $\ln e^{z-i} = \ln -1$

$$z - i = i\pi + 2N\pi i$$

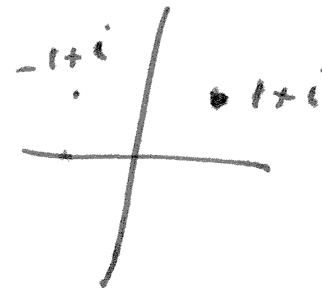
$$z = i\left(1 + (2N+1)\pi i\right)$$

p.v. when $N = 0$

$z = \tan^{-1}(1)$

$$= \frac{1}{2i} \ln\left(\frac{i-1}{i+1}\right)$$

$$= \frac{1}{2i} \ln \frac{e^{\frac{3}{4}\pi i}}{e^{\frac{\pi}{4} i}} =$$



$$\frac{1}{2i} \ln e^{\pi i/2}$$

$$= \frac{1}{2i}\left(i\frac{\pi}{2} + i 2\pi N\right)$$

p.v. $= \frac{\pi}{4}$

N arbitrary integer.

all values $\frac{\pi}{4} + \pi N$

⑥ $$\frac{d}{dz} \sinh^{-1} z = \frac{1 + \frac{z}{\sqrt{1+z^2}}}{z + \sqrt{1+z^2}}$$

$$= \frac{\frac{\sqrt{1+z^2} + z}{\sqrt{1+z^2}}}{z + \sqrt{1+z^2}} = \frac{1}{\sqrt{1+z^2}}$$

⑫

pg 79

① $\int z^2\, dz = 0$

$$\int \bar{z}^2\, dz = \int_0^{2\pi} e^{-2it}\, i e^{it}\, dt$$

$$= \int_0^{2\pi} e^{-it}\, dt$$

$$i e^{-it}\Big|_0^{2\pi} = 0$$

$$\int_C \frac{z+1}{z^2}\, dz = \int \frac{dz}{z} + \int \frac{dz}{z^2}$$

$$= 2\pi i$$

⑧ $$\left|\int \frac{dz}{z^3+1}\right| = \left|\int_{\theta^*}^{\theta^*+\frac{\pi}{3}} \frac{Re^{i\theta} i}{R^3 e^{3i\theta}+1}\right|$$

$$\leq \int_{\theta^*}^{\theta^*+\frac{\pi}{3}} \frac{R}{R^3-1}\, d\theta = \frac{\pi}{3}\frac{R}{R^3-1}$$ clearly $\to 0$ as $R \to \infty$

pg 90

2 a

$$\frac{1}{z(z-2)} = \frac{a}{z} + \frac{b}{z-2}$$

$$a + b = 0$$

$$b - 2a = 1$$

$$b = \frac{1}{3}$$

$$a = -\frac{1}{3}$$

$$\int \frac{dz}{z} = 2\pi i$$

$$\int \frac{dz}{z-2} = 0$$

$\frac{1}{z-2}$ analytic in $\Delta_1(0,$

So $\int \frac{dz}{z(z-2)} = -\frac{2\pi}{3} i$

(b) $$\frac{z}{z^2-\frac{1}{9}} = \frac{a}{z-\frac{1}{3}} + \frac{b}{z+\frac{1}{3}}$$

$$a+b=1 \qquad a=\frac{1}{2}$$
$$a-b=0 \qquad b=\frac{1}{2}$$

So
$$\int_{\partial\Delta_1(0)} \frac{z\,dz}{z^2-\frac{1}{9}} = \frac{1}{2}\int_{\partial\Delta_1(0)} \frac{dz}{z-\frac{1}{3}} = \pi i$$

(c) $$\frac{1}{z(z+\frac{1}{2})(z-2)} = \frac{a}{z} + \frac{b}{z+\frac{1}{2}} + \frac{c}{z-2}$$

$$c = \frac{1}{5} \qquad b = \frac{1}{5/4}$$

$$a = -1$$

So $$\int_C \frac{dz}{z(z+\frac{1}{2})(z-2)} = -2\pi i + \frac{4}{5}2\pi i = -\frac{2\pi i}{5}$$