A Brief Introduction to Euler Characteristic and Genus of Riemann surface

An n-cell is a set homeomorphic to an open ball in $\mathbb{R}^n$, e.g., $\{(x_1, \dots, x_n) \in \mathbb{R}^n / x_1^2 + \dots + x_n^2 < 1\}$

Two open sets $\langle A, B \rangle$ in $\mathbb{R}^N$ are homeomorphic if there is a 1-1, onto continuous map from A to B.

homeomorphic

or by a slight generalization of the definition

upper half sphere

unit disk

unit square

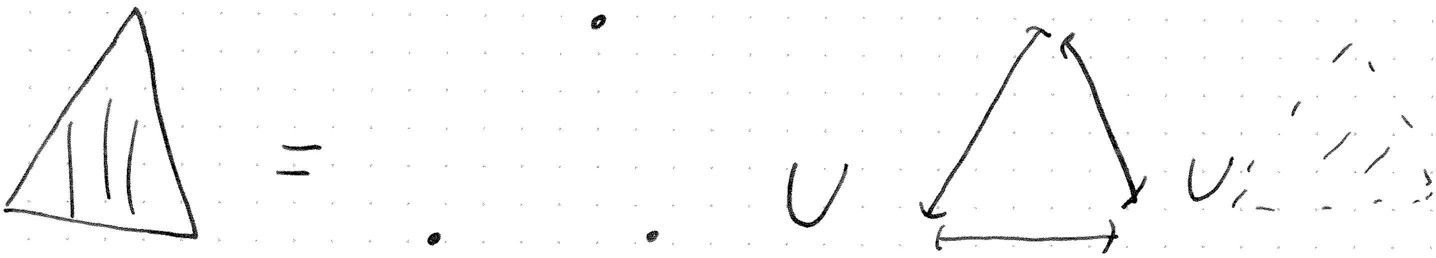
If we can break a space into a finite number of cells, i.e.

$$X = \underbrace{\underset{\text{0-cells}}{\text{vertices}}}_{c_0 \text{ of them}} \cup \underbrace{\text{1-}\cancel{}\text{cells}}_{c_1 \text{ of them}} \cup \underbrace{\text{2-cells}}_{c_2 \text{ of them}} \cup \cdots \cup n\text{-cells}$$

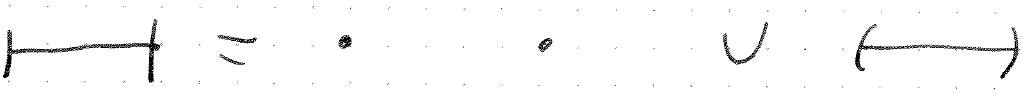
we define the Euler characteristic:

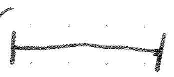
$$e(X) = c_0 - c_1 + c_2 - c_3 \cdots + (-1)^n c_n .$$

The dimension of X is the largest n such that $c_n \neq 0$.



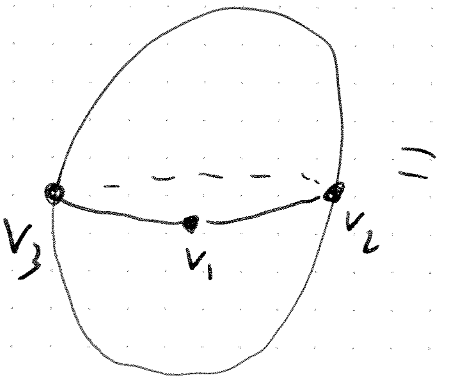
so e() $= 3 - 3 + 1 = 1.$



so e() $= 2 - 1 = 1$

$$e(S^2) = 2$$

the 2 sphere



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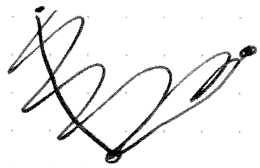


3 vertices ∪ 2 2-cells ∪ 3 1-cells

$$e(\ ) = 3 - 3 + 2$$

Note the n sphere = n-cell ∪ n cell ∪ (n−1) sphere

so $$e(S^n) = e(S^{n-1}) + (-1)^n \cdot 2$$

$n = 0$	$e(S^n) = 2$
1	0
2	2
3	0
4	2
5	0
6	2

also closed n-ball $\overline{B_n}$

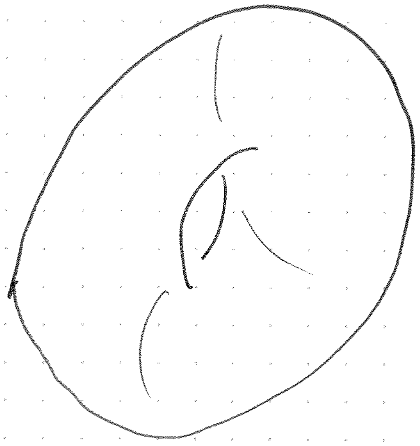
$= \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum x_i^2 \le 1\}$

= $n-1$ sphere ∪ n-cell

so $e(\overline{B_n}) = e(S^{n-1}) + (-1)^n$

$$= \begin{cases} 1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

always 1 (a useful fact to remember.



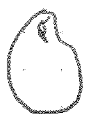
The 2-torus (skin of a donut, hollow inner tube) has euler characteristic zero.

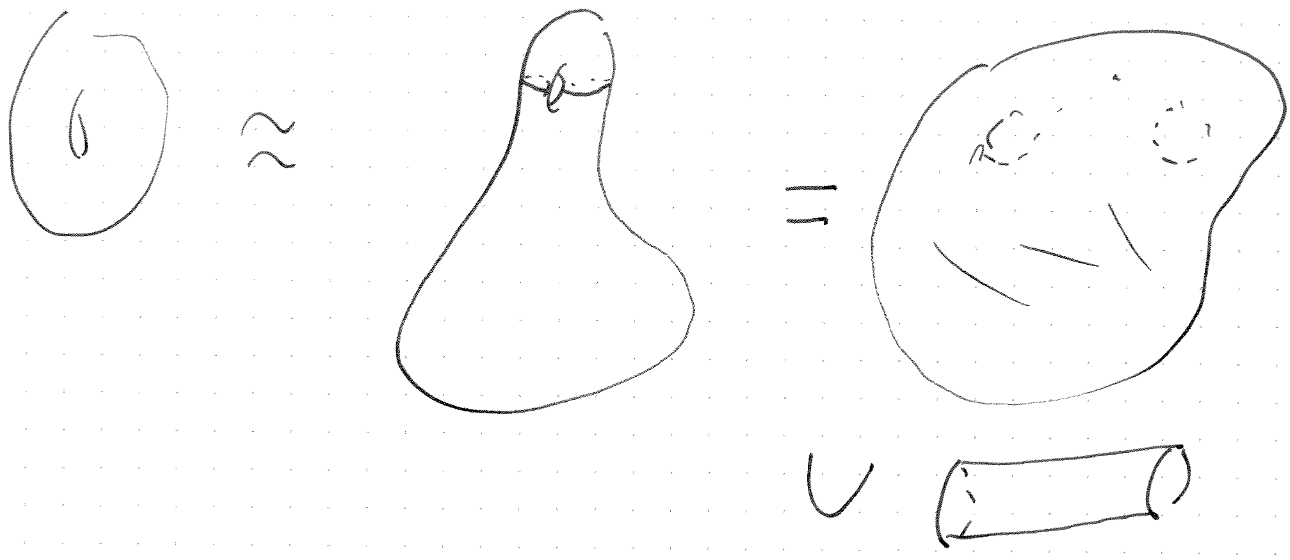
one way of seeing this is to note a torus is homeomorphic to $S^1 \times S^1$ and going back to the definition it can be checked ~~iff~~

$$e(X \times Y) = e(X)\, e(Y)$$

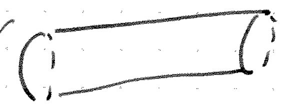
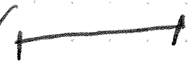
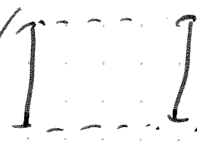
(This suggests we should think of e as nice function on an object consisting of finite sums of spaces!)

A second way is to think of the 2 torus as a sphere with a handle

 . Thus



$$e(S^2 - 2 \text{ closed disks})$$
$$= e(S^2) - 2\,e(\text{closed disk}) = 2-2 = 0$$

$e($  $) = e($  $) + e($  $)$

$$= 1 + (1-2) = = 0$$

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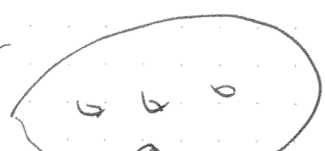
more generally $e($  $) = 2-2g$

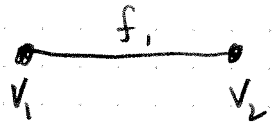
inner tube with g holes

 = Sphere with g handles

$= S^2 - 2g$ disks $\cup$ g cylinders

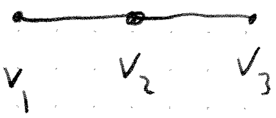
$e(S^2 - g \text{ disks}) = 2-2g$ $e(\text{cylinder}) = 0$

The Euler characteristic is a homeomorphism invariant and well-defined. For 1 an 2 dimensions this is not hard to see.



$= \cdot \quad \cdot \cup \longleftrightarrow$

adding a vertex $\qquad e(\quad) = 2-1=1$



also adds 1 edge so

$e(\quad) = 3-2=1$

we don't change the answer.

The Euler characteristic has the additive property if $X = X_1 \cup X_2$ then $e(X) = e(X_1) + e(X_2)$.

+ multiplicative property

$$e(X \times Y) = e(X) \times e(Y).$$

We will freely use these.

It is in fact an homotopy invariant (homotopy lets us deform more radically without tearing. A closed n Ball is homotopic to a point

$$\begin{array}{c} R \quad \text{genus } g \\ \downarrow \\ S^2 \end{array}$$

k sheeted
with branch points

n-sheeted

$\Delta \xrightarrow{z^k} \Delta$ 0 has ramification $k-1$

ρ sum of ramifications

$2g-2 = -2n + \rho$ Hurwitz formula

$$\begin{array}{ll} R_1 & g_1 \\ f \downarrow & \\ R_2 & g_2 \end{array}$$

$2g_1 - 2 = n(2g_2 - 2) + \rho.$

Pf. Break R_2 up into "triangles" with the points over which there is ramification among the vertices.

v_2, f_2, e_2 $e(R_2) = v_2 - e_2 + f_2$

f is n sheeted $f^{-1}(\text{face}) = n$ faces
$f^{-1}(\text{edge}) = n$ edges
$f^{-1}(\text{vertex}) = n$ vertices unless there is ramification

$2 - 2g_1 = n v_2 - n e_2 + n f_2$
$-\rho$
the overcount
$= n(2-2g_2) - \rho.$

Formula is usually written $2g_1 - 2 = n(2g_2 - 2) + \rho.$