Lecture Two on Topology and Riemann Surfaces

Last time I ended with Hurwitz's

Theorem 1 for compact smooth Riemann surfaces (R.S.): $f: R_1 \to R_2$ analytic, n-sheeted map from a genus g_1 Riemann surface (R.S.) ~~on~~to a genus g_2 R.S. $\Rightarrow 2g_1 - 2 = n(2g_2 - 2) + \rho$ where ρ is the ramification.

I will go over it again, but first we will spend time discussing R.S. and examples.

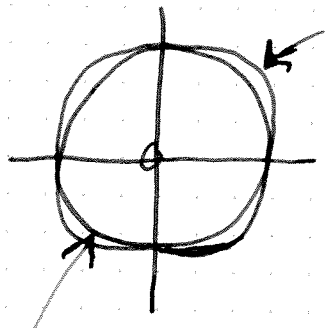
For the moment we have a polynomial $p(z,w)$ on $\mathbb{C}^2$. The set $V(p) = \{(z,w) \in \mathbb{C}^2 \mid p(z,w) = 0\}$ is a complex "algebraic curve". (aka Riemann surface).

Ex. $V(z^n + w^n - 1) = F_n$ is often called the Fermat curve of degree n.

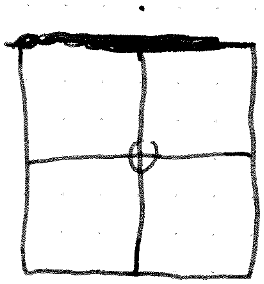
What does it look like?

Over $\mathbb{R}$, i.e., $\{(z,w) \in \mathbb{R}^2 \mid z^n + w^n = 1\}$

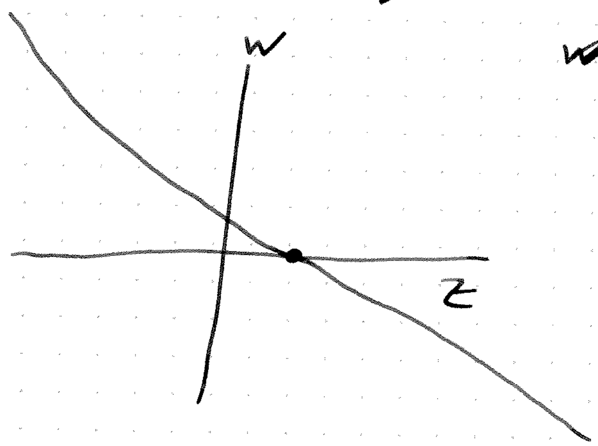
it is pretty dull.



as $\underset{\text{even}}{n} \to \infty$ curve approaches the square



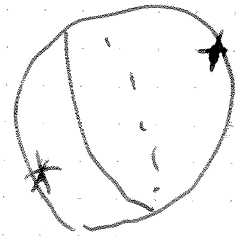
The case of n odd



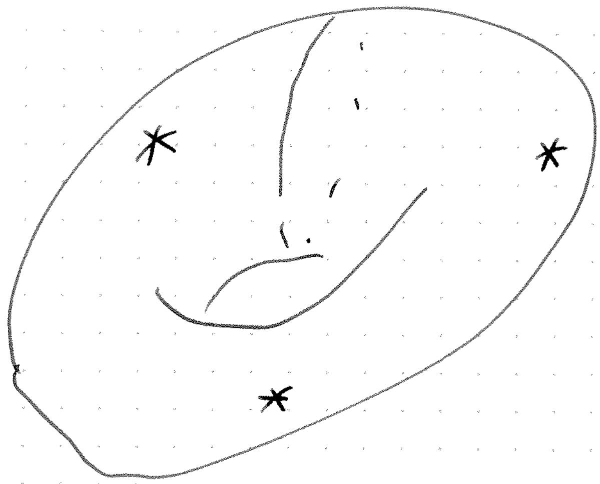
$z < 1 \Rightarrow w > 0$

$z > 1 \Rightarrow w < 0$

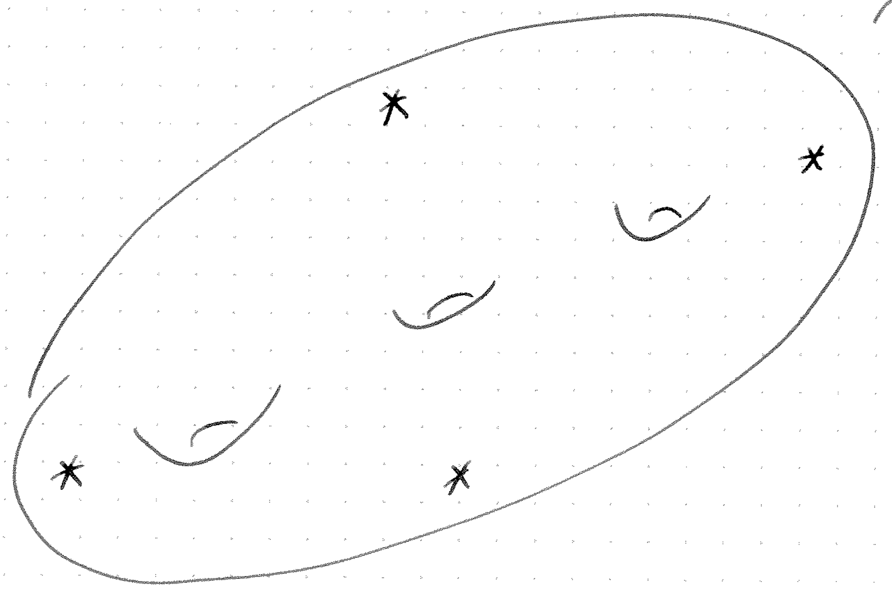
Over $\mathbb{C}$ Hurwitz's formula will let us understand the curve. For now I note if $n = 2$, F_2 looks like the usual sphere minus 2 points; i.e., it is $\mathbb{C}^* = \mathbb{C}$ - origin



F_3 looks like the 2 torus
minus 3 points



F_4 looks like a genus 3 surface
minus 4 points



When we are [redacted] a bit further along
you will be able to do this calculation.

We need some general facts.

Analogue of fundamental theorem of algebra.

THM Let $p(z,w)$ be a polynomial ~~of~~ on $\mathbb{C}^2$. If $p(z,w)=0$ has no solutions then $p(z,w)$ is constant.

Pf. Assume p has no solutions. Let $(z^*,w^*) \in \mathbb{C}^2 - \{(0,0)\}$ be a point. $\{(\lambda z^*, \lambda w^*) \mid \lambda \in \mathbb{C}\}$ is a line through O (the origin $(0,0)$).

The $P_{(z^*,w^*)}(\lambda) = P(\lambda z^*, \lambda w^*)$ is a polynomial in the variable λ with no ~~solutions~~ roots.

Thus $P_{(z^*,w^*)}(\lambda)$ is constant for all $(z^*,w^*) \in \mathbb{C}^2 - O$.

But then $p(z,w)$ is constant on all lines through O. Since all the lines meet in O, the constants are identical and $p(z,w)$ is constant.

The above works in all dimensions.

Note it is ~~as~~ not true for ~~[illegible]~~ real polynomials, e.g. X^2+1 has no solutions over $\mathbb{R}$.

It ~~[illegible]~~ may be shown if $p(z,w)$ has degree d, then for most lines p ~~[illegible]~~ restricted to the ~~the~~ line has d solutions. Ex

$p(z,w) = W - Z^2$ $\quad$ p restricted to $V(Z - z^*)$, e.g., $p(z^*,w) = w - z^{*2}$

has only one solution, but p restricted to $V(W - a^*z - b^*)$ is $a^*z + b^* - z^2$ which has 2 roots counting multiplicity.

This is a topological fact. On $\mathbb{P}^2$, $V(w-z^2)$ meets every line in two points counting multiplicities.

<u>Fact</u>: If $p(z,w)$ is not a constant, then $V(p)$ is the closure of the two dimensional smooth points of $V(p)$.

[A point $(z^*, w^*) \in V(p)$ is a two dimensional smooth point if there is an analytic map $\phi : \Delta_1(0) \to \mathbb{C}^2$
$$\phi = (\phi_1, \phi_2) = (\phi_1(\lambda), \phi_2(\lambda))$$
such ~~that~~ that:

1) $\phi(0) = (z^*, w^*)$ and $p(\phi_1(\lambda), \phi_2(\lambda)) = 0$
2) $d\phi|_0 = \left(\frac{\partial \phi_1}{\partial \lambda}\Big|_0, \frac{\partial \phi_2}{\partial \lambda}\Big|_0\right) \neq (0,0)$]

This is ~~false~~ false over $\mathbb{R}$

E.g., ~~[illegible]~~ $z^2 + w^2 = 0$ has the single root $(0,0)$ on $\mathbb{R}^2$.

(On $\mathbb{C}^2$ $z^2 + w^2 = 0$ vanishes on $V(z+iw) \cup V(z-iw)$)

Fact (consequence of implicit function theorem):

$(z^*, w^*) \in V(p)$ is a 2 dimensional smooth point of $V(p)$ if $\left(\frac{\partial p}{\partial z}\Big|_{(z^*,w^*)}, \frac{\partial p}{\partial w}\Big|_{(z^*,w^*)} \right) \neq (0,0)$.

Ex $V(\underbrace{z^n + w^n - 1}_{p})$ consists entirely of two dimensional smooth points

PF. $$\left.\begin{aligned} \frac{\partial p}{\partial z} &= n z^{n-1} \\ \frac{\partial p}{\partial w} &= n w^{n-1} \end{aligned}\right\}$$ Both vanish on $(0,0)$ alone + this is not a root of $z^n + w^n - 1$.

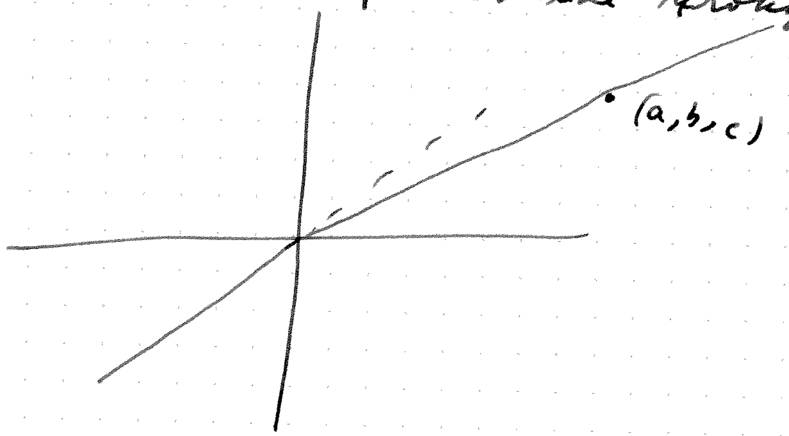
Projective ~~space~~ plane: Lines through origin in $\mathbb{C}^3$ $=$ $\mathbb{C}^2 \cup$ directions

$\mathbb{P}^2$ consists of "points"

$[a, b, c]$ where $(a, b, c) \neq (0, 0, 0)$ and

$[a, b, c] \sim [a\lambda, b\lambda, c\lambda]$ for any $\lambda \neq 0 \in \mathbb{C}$.

points on line through (a, b, c) and 0



We can ~~also~~ map $\mathbb{C}^2 \to \mathbb{P}^2$ in many ways. $(z_1, z_2) \longrightarrow [1, z_1, z_2]$

one-to-one

The only points missed are $[0, z_1, z_2]$ which is the same as a $\mathbb{P}^1$.

There is a correspondence between

$p(z_1, z_2)$ poly. of degree n on $\mathbb{C}^2$ and

$z_0^n \, p\left(\frac{z_1}{z_0}, \frac{z_2}{z_0}\right)$ homogeneous polynomials of degree n on $\mathbb{P}^2$.

$$z_1^n + z_2^n - 1 \longleftrightarrow z_0^N\left(\left(\frac{z_1}{z_0}\right)^n + \left(\frac{z_2}{z_0}\right)^n - 1\right)$$

$$= z_1^n + z_2^n - z_0^n$$

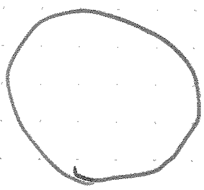
$\xrightarrow{}$ homogenization

$\xleftarrow{}$ dehomogenization

$z_0 = 0$ defines the hyperplane at ∞.

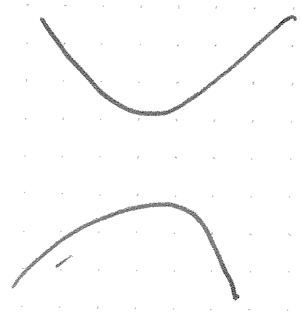
On the projective plane ~~[illegible]~~ many things become easier.

$$x^2+y^2-1 = 0 \longrightarrow x^2+y^2-z^2 = 0$$



homogenizing
$z=0$ ~~hyperpl at~~
hyperplane at ∞

$$\longrightarrow x^2+1-z^2 = 0$$



dehomogenizing
$y=0$ hyperplane at ∞

Fact: If for p homogeneous $\left(\frac{\partial p}{\partial z_0}, \frac{\partial p}{\partial z_1}, \frac{\partial p}{\partial z_2}\right)\Big|_{z^*} \neq (0,0,0)$

then if $p(z^*)=0$, z^* is a two dimensional

smooth point of $V(p(z))$.

So $V(x^n+y^n-z^n)$ consists of

two dimensional smooth points.

The difference between

$$V(x^n+y^n-z^n) \subseteq \mathbb{P}^2$$

and

$$V(x^n+y^n-1) \subseteq \mathbb{C}^2$$ are

the points at ∞, i.e., the

points on $\mathbb{P}^2$ satisfying

$z=0$ and $x^n+y^n-z^n=0$

i.e., $[1,w,0]$ where w is one of the

n solutions of $x^n=-1$.