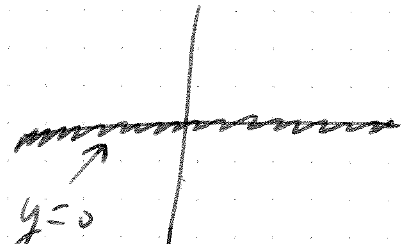
What are the analytic one forms on a smooth curve?

To make everything ~~[illegible]~~ concrete what are the analytic one forms on

x^3+1-y^3 ?

<u>Idea</u> consider the curve $y=0$ and $\frac{dx\,dy}{y}$



We can "take the residue to get rid of the denominator giving dx

More generally $\frac{p(x,y)}{y}dx\,dy$ where $p(x,y)$ is not identically 0 on y giving

$p(x,0)\,dx$.

Is this well defined?

Patches on $\mathbb{P}^2$

$\mathbb{P}^2$ consists of points

$[z_0, z_1, z_2]$ $\quad (z_0, z_1, z_2) \neq (0,0,0)$

and for $\lambda \neq 0$ $[\lambda z_0, \lambda z_1, \lambda z_2] = [z_0, z_1, z_2]$

3 patches covering are

$[1, x, y]$

on overlap

$[1, x, y] = [u, 1, v]$

$= [1, \frac{1}{u}, \frac{v}{u}]$

$[u, 1, v]$

$[u, 1, v] = [a, b, 1] =$

$[\frac{a}{b}, 1, \frac{1}{b}]$

$[a, b, 1]$

$[a, b, 1] = [1, x, y]$

$= [\frac{1}{y}, \frac{x}{y}, 1]$

$$\frac{dx\,dy}{p(1,x,y)} \qquad \frac{dv\,du}{p(u,1,v)}$$

$$\frac{da\,db}{p(a,b,1)}$$

where $p(x,y,z)$ defines C on $\mathbb{P}^2$

$[1,x,y]$ patch	$[u,1,v]$ patch
$p(1,x,y)=0$ defines C on this patch	$p(u,1,v)=0$ defines C on this patch

$x = \frac{1}{u}$

$y = \frac{v}{u}$

Note $p(1,x,y) = p\left(1, \frac{1}{u}, \frac{v}{u}\right)$

$= \dfrac{p(u,1,v)}{u^d}$ where

d = degree of p. E.g.,

$p(x,y,z) = x^3 - y^3 + z^3$

$p(1,x,y) = x^3 - y^3 + 1 = \left(\frac{1}{u}\right)^3 - \left(\frac{v}{u}\right)^3 + 1$

$= \dfrac{1 - v^3 + u^3}{u^3} = \dfrac{p(u,1,v)}{u^3}$

$$\frac{dx\,dy}{p(1,x,y)} = \frac{d\left(\frac{1}{u}\right) d\left(\frac{v}{u}\right) u^d}{p(u,1,v)}$$

$$= \frac{-\frac{du}{u^2}\frac{dv}{u}\,u^d}{p(u,1,v)} = \frac{(dv\,du)\,u^{d-3}}{p(u,1,v)}$$

$$dx\,dy \to \frac{dv\,du}{u^3}$$

So $\dfrac{q(1,x,y)}{p(1,x,y)}\,dx\,dy$ well defined $\deg q = d-3$

$$\frac{dx\,dy}{x^3-y^3+1} = \omega.\,(3x^2dx - 3y^2dy)$$

we want to use x as a "global coordinate"

So

$$\omega = \frac{dx}{(x^3-y^3+1)(-3y^2)}$$ is fine

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Residue on C in $[1, x, y]$ patch is

$$\frac{dx}{\frac{\partial p(1,x,y)}{\partial y}}$$

$$\frac{dx}{-3y^2} = \frac{dx}{-3\sqrt[3]{(x^3+1)^2}}$$

No zeroes

$x = e^{\pi i/3}$

$$\frac{dx}{x^{2/3}}$$

$z^3 = x$

$$\frac{3z^2\,dz}{z^2} = 3\,dz$$

~~More generally
q, p homogeneous
$V(p)$ smooth $\deg p = d$
$\deg q = d-3$~~

$$\frac{q(1,x,y)}{\frac{\partial p}{\partial y}(1,x,y)}\,dx$$ is an analytic one form on C