```
> restart;
```

Problem 9.1

```
> with(LinearAlgebra):
> for j from 1 to 5 do
  Digits:=10*j;
  ConditionNumber(evalf(HilbertMatrix(30)),2);
  od;
```

$$Digits := 10$$

$$9.302314490\ 10^{12}$$

$$Digits := 20$$

$$1.5990645462216154592\ 10^{22}$$

$$Digits := 30$$

$$1.276376098132376839317582685 40\ 10^{32}$$

$$Digits := 40$$

$$1.270218893063568655814686052999263189344\ 10^{41}$$

$$Digits := 50$$

$$4.22780628810262350216614443941258845222746862229 51\ 10^{43} \quad \textbf{(1)}$$

```
> for j from 1 to 5 do
  Digits:=10*j;
  ConditionNumber(evalf(HilbertMatrix(30)),infinity);
  od;
```

$$Digits := 10$$

$$3.633135867\ 10^{13}$$

$$Digits := 20$$

$$5.57659296196397691 67\ 10^{22}$$

$$Digits := 30$$

$$4.339925982622964914217292352 07\ 10^{32}$$

$$Digits := 40$$

$$3.605932883220734882273306249982262478373\ 10^{41}$$

$$Digits := 50$$

$$1.177679544917680362570809423155004867731868410262 4\ 10^{44} \quad \textbf{(2)}$$

Problem 9.2

```
> restart;
> with(LinearAlgebra):
> A:= evalf(HilbertMatrix(8));
```

$$A := [[1., 0.5000000000, 0.3333333333, 0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000], [0.5000000000, 0.3333333333, 0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111], [0.3333333333, 0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000], [0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091], [0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333], [0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333, 0.07692307692], \quad \textbf{(3)}$$

[0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333, 0.07692307692, 0.07142857143],
[0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333, 0.07692307692, 0.07142857143, 0.06666666667]]

```
> T,Z := SchurForm(A,output=['T','Z']);
```

$$T, Z := \left[\left[0.298125211315803, -2.22044604925031\ 10^{-16}, 1.48960542641647\ 10^{-17}, 8.89720325539185\ 10^{-18}, 1.35617255153186\ 10^{-17}, 1.24830005787839\ 10^{-17}, 3.16538303241544\ 10^{-17}, 6.25767918640535\ 10^{-17}\right], \left[-0., 1.69593899694191, -1.71929087882565\ 10^{-17}, 1.17874511383439\ 10^{-16}, -9.57363713143115\ 10^{-17}, 1.80556424052703\ 10^{-17}, -3.03682912507584\ 10^{-17}, -6.00009659738191\ 10^{-17}\right], \left[0., -0., 0.0262128435674622, 3.98986399474666\ 10^{-17}, -8.67903292968283\ 10^{-18}, 1.31418450463798\ 10^{-19}, -3.96822024969536\ 10^{-17}, -6.09407627776179\ 10^{-17}\right], \left[0., 0., -0., 0.00146768811695009, 2.10586138015309\ 10^{-17}, -1.32608009636094\ 10^{-17}, -2.45110582458641\ 10^{-17}, -3.79481104106170\ 10^{-17}\right], \left[0., 0., 0., 0., 0.000543694260751768, 8.42628375928578\ 10^{-18}, 1.23385020605892\ 10^{-17}, 1.59992768951981\ 10^{-17}\right], \left[0., 0., 0., 0., 0., 0.0000129432455494521, -1.54294644852170\ 10^{-18}, -7.07495602645295\ 10^{-18}\right], \left[0., 0., 0., 0., 0., -0., 1.10003559823955\ 10^{-10}, -4.65368503899883\ 10^{-19}\right], \left[0., 0., 0., 0., 0., 0., -0., 1.79972423374871\ 10^{-8}\right]\right], \left[\left[0.629483940288172, 0.720271369430432, -0.277547250856855, -0.0865170304531276, 0.0206555745775761, -0.00381063148535788, -0.0000471572962447299, -0.000523816957421020\right], \left[-0.125670851600514, 0.432526015441124, 0.644928477354285, 0.550138190533486, -0.266125890622646, 0.0857915129531394, 0.00253499253134843, 0.0188072830848564\right], \left[-0.286419020871867, 0.318843646412652, 0.335150531825316, -0.336254015561981, 0.624065792616028, -0.418794332023564, -0.0331777549115658, -0.157995766371225\right], \left[-0.327570106532218, 0.255242887159269, 0.0513071061632137, -0.459247720640691, -0.0319532860419056, 0.579716987098143, 0.179887489485451, 0.494525731698721\right], \left[-0.332094295325504, 0.213861951122070, -0.144024123777325, -0.275649619289560, -0.419384538782755, 0.123202210936319, -0.485066845385979, -0.566479270042503\right], \left[-0.323539966469642, 0.184519788927988, -0.272863053753642, -0.0112717063634575, -0.365529408867358, -0.418513205691169, 0.687274024888002, -0.0747154786648259\right], \left[-0.310265781845791, 0.162514374472369, -0.357044204137071, 0.248964109680522, -0.0180070396012874, -0.352358807353878, -0.489665413819636, 0.568864949704569\right], \left[-0.295616625870828, 0.145343694353780, -0.411580765430296, 0.477278378910096, 0.477966356540585, 0.408259721856491, 0.138295524240993, -0.282911917516730\right]\right] \tag{4}$$

```
> Norm(A - Z .T. HermitianTranspose(Z),infinity);
```

$$3.20576898360513951\ 10^{-15} \tag{5}$$

```
> TT:=Matrix(8):
  i:='i':
  j:='j':
  for i from 1 to 8 do for j from 1 to 8 do
  if (abs(T(i,j))>10^(-10)) then TT(i,j):=T(i,j) fi:
  od:
  od:
  TT;
```

(6)

$$\begin{bmatrix} 0.298125211315803 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.69593899694191 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0262128435674622 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.00146768811695009 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0000543694260751768 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0000129432455494521 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.10003559823955\ 10^{-10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.79972423374871\ 10^{-8} \end{bmatrix} \quad (6)$$

I just want to get the elements in order-if I was going to do this on a larger matrix, I would write a routine.

```
> M:=Matrix(8):
  M(7,8):=1:
  M(8,7):=1:
  M(1,2):=1:
  M(2,1):=1:
  M(3,3):=1:
  M(4,4):=1:
  M(5,5):=1:
  M(6,6):=1:
```

Take U:= Z . M, Sigma = M . TT . M, and V = Z . M

```
> Sigma:= M  .  TT  .  M;
  U:= Z  .  M;
  V:= U:
```

$$\Sigma := \begin{bmatrix} 1.69593899694191 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0.298125211315803 & 0. & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0.0262128435674622 & 0. & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0.00146768811695009 & 0. & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0.0000543694260751768 & 0. & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0.0000129432455494521 & 0. & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1.79972423374871\ 10^{-8} & 0. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1.10003559823955\ 10^{-10} \end{bmatrix}$$

$$U := \begin{bmatrix} 0.720271369430432 & 0.629483940288172 & -0.277547250856855 & -0.0865170304531276 & \\ 0.0206555745775761 & -0.00381063148535788 & -0.000523816957421020 & -0.000471572962447299 \end{bmatrix}, \quad (7)$$

$$\begin{bmatrix} 0.432526015441124 & -0.125670851600514 & 0.644928477354285 & 0.550138190533486 & \\ -0.266125890622646 & 0.0857915129531394 & 0.0188072830848564 & 0.00253499253134843 \end{bmatrix},$$

$$\begin{bmatrix} 0.318843646412652 & -0.286419020871867 & 0.335150531825316 & -0.336254015561981 & \\ 0.624065792616028 & -0.418794332023564 & -0.157995766371225 & -0.0331777549115658 \end{bmatrix},$$

$$\begin{bmatrix} 0.255242887159269 & -0.327570106532218 & 0.0513071061632137 & -0.459247720640691 & \\ -0.0319532860419056 & 0.579716987098143 & 0.494525731698721 & 0.179887489485451 \end{bmatrix},$$

$$\begin{bmatrix} 0.213861951122070 & -0.332094295325504 & -0.144024123777325 & -0.275649619289560 & \\ -0.419384538782755 & 0.123202210936319 & -0.566479270042503 & -0.485066845385979 \end{bmatrix},$$

$$\begin{bmatrix} 0.184519788927988 & -0.323539966469642 & -0.272863053753642 & -0.0112717063634575 & \\ -0.365529408867358 & -0.418513205691169 & -0.0747154786648259 & 0.687274024888002 \end{bmatrix},$$

$$\begin{bmatrix} 0.162514374472369 & -0.310265781845791 & -0.357044204137071 & 0.248964109680522 & \\ -0.0180070396012874 & -0.352358807353878 & 0.568864949704569 & -0.489665413819636 \end{bmatrix},$$

[0.145343694353780, -0.295616625870828, -0.411580765430296, 0.477278378910096,
0.477966356540585, 0.408259721856491, -0.282911917516730, 0.138295524240993]]

```
> Norm(A-U . Sigma . HermitianTranspose(V),infinity);
```

$$2.60902410786911787\ 10^{-15} \quad (8)$$

Problem 9.3

```
> restart;
> with(LinearAlgebra):
> A:= evalf(HilbertMatrix(8));
```

$A :=$ [[1., 0.5000000000, 0.3333333333, 0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000], (9)
[0.5000000000, 0.3333333333, 0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111],
[0.3333333333, 0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000],
[0.2500000000, 0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091],
[0.2000000000, 0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333],
[0.1666666667, 0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333, 0.07692307692],
[0.1428571429, 0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333, 0.07692307692, 0.07142857143],
[0.1250000000, 0.1111111111, 0.1000000000, 0.09090909091, 0.08333333333, 0.07692307692, 0.07142857143, 0.06666666667]]

```
> Q, R :=QRDecomposition(A):
> Norm(A-Q . R,infinity);
```

$$3.74700270810990332\ 10^{-16} \quad (10)$$

Problem 9.4

```
> restart;
> Digits:=15;
```

$$Digits := 15 \quad (11)$$

```
> with(LinearAlgebra):
> A:= evalf(HilbertMatrix(8));
```

$A :=$ [[1., 0.500000000000000, 0.333333333333333, 0.250000000000000, 0.200000000000000, (12)
0.166666666666667, 0.142857142857143, 0.125000000000000],
[0.500000000000000, 0.333333333333333, 0.250000000000000, 0.200000000000000,
0.166666666666667, 0.142857142857143, 0.125000000000000, 0.111111111111111],
[0.333333333333333, 0.250000000000000, 0.200000000000000, 0.166666666666667,
0.142857142857143, 0.125000000000000, 0.111111111111111, 0.100000000000000],
[0.250000000000000, 0.200000000000000, 0.166666666666667, 0.142857142857143,
0.125000000000000, 0.111111111111111, 0.100000000000000, 0.0909090909090909],
[0.200000000000000, 0.166666666666667, 0.142857142857143, 0.125000000000000,
0.111111111111111, 0.100000000000000, 0.0909090909090909, 0.0833333333333333],
[0.166666666666667, 0.142857142857143, 0.125000000000000, 0.111111111111111,
0.100000000000000, 0.0909090909090909, 0.0833333333333333, 0.0769230769230769],
[0.142857142857143, 0.125000000000000, 0.111111111111111, 0.100000000000000,

```
    0.0909090909090909, 0.0833333333333333, 0.0769230769230769, 0.0714285714285714],
    [0.125000000000000, 0.111111111111111, 0.100000000000000, 0.0909090909090909,
    0.0833333333333333, 0.0769230769230769, 0.0714285714285714, 0.0666666666666667]]
> Q, R :=QRDecomposition(A):
> for j from 1 to 10 do
  B:= R . Q:
  Q,R :=QRDecomposition(B):
  od:
> B:=Q . R;
```

$$B := \left[\left[1.69593899692195,\ 3.44209300356216\ 10^{-8},\ 1.25492567655824\ 10^{-16},\ -2.53292041855164\ 10^{-17},\ -1.43844569929438\ 10^{-18},\ 3.13187629408252\ 10^{-17},\ 8.13402030644869\ 10^{-17},\ -1.15931583525015\ 10^{-17}\right],\right.$$
$$\left[3.44209298447741\ 10^{-8},\ 0.298125211316932,\ -1.20995985275652\ 10^{-11},\ 3.04596034665234\ 10^{-17},\ 2.90004964245103\ 10^{-17},\ 9.56780200834514\ 10^{-18},\ 1.02305225332389\ 10^{-17},\ -7.33856346002277\ 10^{-18}\right],$$
$$\left[-1.03102519703884\ 10^{-18},\ -1.20995736923656\ 10^{-11},\ 0.0262128435781190,\ 1.61224501944775\ 10^{-14},\ 2.07152915388106\ 10^{-17},\ 1.27015659950748\ 10^{-17},\ 6.45734267613759\ 10^{-19},\ -2.54863972338176\ 10^{-17}\right],$$
$$\left[-3.75512528666514\ 10^{-31},\ -4.91653062569369\ 10^{-24},\ 1.61199923666269\ 10^{-14},\ 0.00146768811774172,\ -8.17099197019427\ 10^{-19},\ -6.93615561132006\ 10^{-18},\ -3.37450426392623\ 10^{-18},\ -1.05829768184955\ 10^{-18}\right],$$
$$\left[-2.16130392209021\ 10^{-45},\ -2.98204171077192\ 10^{-38},\ 1.16591404328460\ 10^{-28},\ 1.71489398419024\ 10^{-17},\ 0.0000543694336975200,\ -1.37833922218532\ 10^{-18},\ 9.29897919152072\ 10^{-19},\ 1.86361147521150\ 10^{-18}\right],$$
$$\left[-1.36967869025907\ 10^{-61},\ -1.94585797208877\ 10^{-54},\ 8.36160963519558\ 10^{-45},\ 1.53371965516217\ 10^{-33},\ 8.17119659022494\ 10^{-21},\ 0.00000129433209174503,\ -8.55057346117849\ 10^{-19},\ 5.68049050268845\ 10^{-18}\right],$$
$$\left[-3.38537643205394\ 10^{-80},\ -4.89669147970469\ 10^{-73},\ 2.22793198096885\ 10^{-63},\ 4.63533004296222\ 10^{-52},\ 3.17191394462088\ 10^{-39},\ 8.66780809434596\ 10^{-25},\ 1.79887374557506\ 10^{-8},\ 1.20667758895390\ 10^{-18}\right],$$
$$\left[-1.83530052987920\ 10^{-102},\ -2.68619273556100\ 10^{-95},\ 1.26865507278509\ 10^{-85},\ 2.85931409232873\ 10^{-74},\ 2.26959786353736\ 10^{-61},\ 8.13314827189429\ 10^{-47},\ 2.96709542206907\ 10^{-30},\ 1.11153842434037\ 10^{-10}\right]\right] \tag{13}$$

```
> C:=DiagonalMatrix(Eigenvalues(A));
```

$$C := \left[\left[1.69593899692195 + 0.\,I,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right],\right.$$
$$\left[0,\ 0.298125211316931 + 0.\,I,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0\right],$$
$$\left[0,\ 0,\ 0.0262128435781189 + 0.\,I,\ 0,\ 0,\ 0,\ 0,\ 0\right],$$
$$\left[0,\ 0,\ 0,\ 0.00146768811774175 + 0.\,I,\ 0,\ 0,\ 0,\ 0\right],$$
$$\left[0,\ 0,\ 0,\ 0,\ 0.0000543694336975087 + 0.\,I,\ 0,\ 0,\ 0\right],$$
$$\left[0,\ 0,\ 0,\ 0,\ 0,\ 0.0000129433209174197 + 0.\,I,\ 0,\ 0\right],$$
$$\left[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1.79887374623880\ 10^{-8} + 0.\,I,\ 0\right],$$
$$\left[0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1.11153837196241\ 10^{-10} + 0.\,I\right]\right] \tag{14}$$

```
> for j from 1 to 8  do for k from 1 to 8 do
  if (B(j,k)< 10^(-8)) then B(j,k):=0: fi: od: od:
> B;
```

$$\begin{bmatrix} [1.69593899692195,\ 3.44209300356216\ 10^{-8},\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.], \\ [3.44209298447741\ 10^{-8},\ 0.298125211316932,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.], \\ [0.,\ 0.,\ 0.0262128435781190,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.], \\ [0.,\ 0.,\ 0.,\ 0.00146768811774172,\ 0.,\ 0.,\ 0.,\ 0.], \\ [0.,\ 0.,\ 0.,\ 0.,\ 0.0000543694336975200,\ 0.,\ 0.,\ 0.], \\ [0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.0000012943320917450 3,\ 0.,\ 0.], \\ [0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 1.79887374557506\ 10^{-8},\ 0.], \\ [0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.,\ 0.] \end{bmatrix} \tag{15}$$