Problems for ACMS 60690

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10 November 14, 2012: Due November 28, 2012

Problem 10.1. *Consider the 2-point boundary value problem*

$$y'' - 2y = 4x^2 e^{1+x^2} \qquad y(0) = e; \quad y(1) = e^2$$

on $[0, 1]$*. The solution is* $f(x) = e^{1+x^2}$*.*

1. *Using the Galerkin method and the space of piecewise linear functions with nodes at* $x_j = jh$ *for* $j = 0, \ldots, N+1$ *with* $h = \frac{1}{N+1}$ *and* $N = 10$*. Compute the relative error*

$$\frac{\sum_{1=1}^{N} |y_i - f(x_i)|}{\sum_{1=1}^{N} |f(x_i)|}.$$

2. *Repeat with* $N = 20$*.*

Problem 10.2. *Using Newton's method, find an approximate solution of*

$$\begin{bmatrix} \frac{x^2}{9} + \frac{y^2}{16} - 1 \\ \frac{x^2}{16} + \frac{y^2}{9} - 1 \end{bmatrix} = 0.$$

9 October 31, 2012: Due November 7, 2012

Problem 9.1. *The Hilbert matrix*

$$H_n := \begin{bmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-1} \end{bmatrix}$$

is a traditional example of a badly conditioned matrix. In these problem you need to use floating point numbers, e.g., using `with(LinearAlgebra);`*, the Hilbert matrix is built in as* `HilbertMatrix(n)`*, where n is a number of your choice say 30:* `evalf(HilbertMatrix(n))` *will construct the Hilbert matrix with floating point entries.*

1. *Using* $10, 20, 30, 40, 50$ *digits compute the condition number of* H_{30} *in the matrix norm associated to the 2-norm.*

2. *Using* $10, 20, 30, 40, 50$ *digits compute the condition number of* H_{30} *in the matrix norm associated to the* ∞*-norm.*

Any conclusions?
Note trying your calculation using `HilbertMatrix(n)` *and* `\evalf(HilbertMatrix(n))`*, the timings (and answers) will be very different. Why?*

Problem 9.2. *Compute the eigenvalues and eigenvectors of* H_8 *and using this data compute the singular value decomposition* $H_8 = U \cdot \Sigma \cdot V^*$ *of* H_8 *and verify numerically that* $H_8 - U \cdot \Sigma \cdot V^*$ *has small infinity norm, i.e., small norm in the matric norm associated to the infinity norm.*

Problem 9.3. *Compute the* QR *decomposition* $H_8 = Q \cdot R$ *of* H_8 *and verify numerically that* $H_8 - Q \cdot R$ *has small infinity norm.*

Problem 9.4. *Set* $A_0 = H_8$*. Let for j from 1 to 10 let* $A_j = R_{j-1} \cdot Q_{j-1}$ *where the QR-decomposition of* A_{j-1} *is* $A_{j-1} = Q_{j-1} \cdot R_{j-1}$*.*

1. *What is* A_{10}*?*

2. *How does the diagonal of* A_{10} *compare with the eigenvalues of* H_8*?*

8 October 24, 2012: Due October 31, 2012

Problem 8.1. *Let* $\Phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = x_1 + 3x_2 + 4x_3 + 2x_4$ *be a linear function on* $\mathbb{R}^4$*. Assuming we have the norm for*

$$\left\|\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right\| = \max\{|x_1|, |x_2|, |x_3|, |x_4|\}$$

on $\mathbb{R}^4$*, what is* $||\Phi||$ *using the induced norm.*

Problem 8.2. *Let*

$$A := \begin{bmatrix} 3 & 0.1 & 0.2 & 0.2 \\ 0.1 & 3\sqrt{-1} & 0.2 & 0.2 \\ 0 & 0.1 & 0.4 & 0.1 \\ 0.1 & 0.2 & 0.2 & 1 \end{bmatrix}$$

be a matrix.

1. *What is* $||A||$*, where the norm is induced from the infinity norm on* $\mathbb{R}^4$*.*

2. *Graph the Gershgorin disks with respect to this norm for* A*.*

3. *Using 2), what can you say about the invertibility of* A*?*

Problem 8.3. *Let*

$$A := \begin{bmatrix} 3 & 0.3 & 0.3 & 0.3 \\ 1 & 0.1 & 0.1 & 0.1 \end{bmatrix}$$

and let

$$B := \begin{bmatrix} 3 & 1 \\ 0.3 & 0.1 \\ 0.3 & 0.1 \\ 0.3 & 0.1 \end{bmatrix}.$$

1. *What is the column rank of* A*?*

2. *What is the row rank of* A*?*

3. *What is the rank of* A*?*

Answer the same questions for B *using the answers for* A *and the fact that* $B = A^\dagger$*.*

7 October 10, 2012: Due October 24, 2012

Problem 7.1. *Use finite differences with* $h = 0.05$ *to find a numerical solution of*

$$y'' - 4y + 4x = 0; \quad 0 \le x \le 1; \ \ y(0) = 1, \ y(1) = 2.$$

How does the numerical solution $(y_0, \ldots, y_{20})$ *you found compare to the true solution*

$$y = e^2 \left(\frac{e^{2x} - e^{-2x}}{e^4 - 1} \right) + x, \quad \text{i.e.,}$$

1. *what is the relative error* $\dfrac{|y(1) - y_{20}|}{|y(1)|}$; *and*

2. *what is the relative error in the* 1*-norm:*

$$\frac{\sum_{i=0}^{20} \left|y\left(\frac{i}{20}\right) - y_i\right|}{\sum_{i=0}^{20} \left|y\left(\frac{i}{20}\right)\right|}.$$

Problem 7.2. *Use nonlinear shooting with* $h = 0.05$ *to find a numerical solution of*

$$y'' + (y')^2 + y - \ln(x) = 0; \quad 1 \leq x \leq 2; \;\; y(1) = 0, \; y(2) = \ln(2).$$

How does the numerical solution $(y_0, \ldots, y_{20})$ *you found compare to the true solution* $y = \ln(x)$*, i.e.,*

1. *what is the relative error* $\dfrac{|y(2) - y_{20}|}{|y(2)|}$; *and*

2. *what is the relative error in the* 1*-norm:*

$$\frac{\sum_{i=0}^{20} \left|y\left(1 + \frac{i}{20}\right) - y_i\right|}{\sum_{i=0}^{20} \left|y\left(1 + \frac{i}{20}\right)\right|}.$$

6 October 4, 2012: Due October 10, 2012

Problem 6.1. *Find the roots of the polynomial*

$$x^3 - 7x^2 + 3$$

1. *using the solver (with Euler's method and without Newton) and* $N = 10$ *gridpoints;*

2. *using the solver (with the Modified Euler method and without Newton) and* $N = 10$ *gridpoints;*

3. *using the solver (with the fourth-order Runge-Kutta method and without Newton) and* $N = 10$ *gridpoints.*

Compare the results with the solutions found using `fsolve`.

Problem 6.2. *Find the roots of the polynomial*

$$x^3 - 7x^2 + 3$$

1. *using the solver (with Euler's method and with Newton) and $N = 10$ gridpoints;*

2. *using the solver (with the fourth-order Runge-Kutta method and with Newton) and $N = 10$ gridpoints.*

Compare the results with the solutions found using `fsolve`*.*

Problem 6.3. *Find the roots of the polynomial*

$$x^3 - 6x^2 + 12x - 8$$

1. *using the solver (with Euler's method and without Newton) and $N = 10$ gridpoints;*

2. *using the solver (with the fourth-order Runge-Kutta method and with Newton) and $N = 10$ gridpoints.*

Compare the results with the solutions found using `fsolve`*. Why are the results so poor?*

5 September 19, 2012: Due September 26, 2012

Given a sequence knots $[t_{-\ell}, \ldots, t_0, \ldots, t_n]$,

$$B_{i,k}(x) = (-1)^k (t_k - t_0)(x - t)_+^{k-1}[t_i, \ldots, t_k]$$

denotes the order k B-spline with knot sequence $\widehat{t} := [t_i, \ldots, t_{i+k}]$. When we want to emphasize the knot sequence, we use the notation $B_{i,k,\widehat{t}}(x)$ for this spline.

Problem 5.1. *Fix the knot sequence $\widehat{t_1} := [0, 0.5, 1.0]$. Compute $B_{0,2}(x)$ using divided differences. You may use Maple to help in the calculations, but in this problem you may not use Maple's built in B-splines.*

Problem 5.2. *Fix the knot sequences*
$\widehat{t} := [0, 0.5, 0.9, 1.3, 1.9]$*. Verify using Maple that*

$$B'_{0,4}(x) = 3\left(\frac{B_{0,3}(x)}{1.3 - 0} - \frac{B_{1,3}(x)}{1.9 - 0.5}\right).$$

Problem 5.3. *Fix the knot sequences*
$\widehat{t} := [0, 0.5, 0.9, 1.3, 1.9]$*. Express $B_{0,4}(x)$ in terms of $B_{0,3}(x)$ and $B_{1,3}(x)$.*

Problem 5.4. *Fix knot sequences*

$\widehat{k}_1 := [0, 0.5, 0.9, 1.1, 1.3, 1.9, 2.4]$, $\widehat{k}_2 := [0, 0.5, 0.9, 0.9, 1.3, 1.9, 2.4]$,
$\widehat{k}_3 := [0, 0.9, 0.9, 0.9, 0.9, 1.9, 2.4]$, *and* $\widehat{k}_4 := [0, 0.9, 0.9, 0.9, 0.9, 0.9, 2.4]$.

Plot $B_{0,6,\widehat{k}_1}(x)$, $B_{0,6,\widehat{k}_2}(x)$, $B_{0,6,\widehat{k}_3}(x)$, *and* $B_{0,6,\widehat{k}_4}(x)$ *for the range* $-0.5 \leq x \leq 2.9$.

4 September 12, 2012: Due September 19, 2012

Problem 4.1. *On* $[a, b]$, *for* h *be a positive real number such that* $(b-a)/N$ *is a positive integer, let*

$$T(f, h) = \left(f(a) + 2 \sum_{i=1}^{N-1} f(a + ih) + f(b) \right) \frac{h}{2}$$

denote the composite trapezoid rule.

1. *Assume you know* $T(f, b-a)$ *and* $T(f, (b-a)/2)$. *What is the rule you obtain by using Richardson extrapolation.*
2. *Assume you know* $T(f, b-a)$ *and* $T(f, (b-a)/3)$. *What is the rule you obtain by using Richardson extrapolation.*

You may assume that

$$T(f, h) = \int_a^b f(x)dx + a_1 h^2 + O(h^4)$$

where a_1 *is independent of* h.

Problem 4.2. *By hand compute the Bernoulli polynomials* $B_2(x), B_3(x), B_4(x)$ *and the corresponding Bernoulli numbers. You may assume that* $B_0(x) = 1$ *and* $B_1(x) = x - 1/2$.

Problem 4.3. *On the interval* $[-1, 1]$ *find the Gaussian one point, two point and three point rules. Do this by hand. Verify that the three point rule is exact for integration of polynomials of degree at most 5.*

Problem 4.4. *On the interval* $[0, 1]$ *find the Gaussian three point rule with weight function* $w(x) := \sqrt{x}$. *Do this by hand. Verify that the three point rule is exact for integration*

$$\int_0^1 p(x)\sqrt{x}\ \mathrm{d}x$$

where $p(x)$ *is a polynomial of degree at most 5. (You should use Maple for this problem.)*

3 September 5, 2012: Due September 12, 2012

See rungeErrorControl.pdf on the class website.

Problem 3.1. *Fix an interval* $[a,b]$ *and a function* $f \in C^2[a,b]$*. The composite trapezoid rule for the points* $x_i = a + i(b-a)/n$ *with* $i = 0,\ldots,n$ *is*

$$\mathrm{trap}(f,a,b,n) = \left(f(a) + 2\sum_{j=1}^{n-1} f\left(a + \frac{j(b-a)}{n}\right) + f(b) \right)\left(\frac{b-a}{2n}\right)$$

The Runge method of error control asserts that for f *as above with* $f^{(2)}(x)$ *not changing too fast*

$$\frac{|\mathrm{trap(f,a,b,n)} - \mathrm{trap(f,a,b,2n)}|}{3}$$

is close to the error

$$\left| \int_a^b f(x)\mathrm{dx} - \mathrm{trap(f,a,b,2n)} \right|.$$

In particular, this should be true for all sufficiently large n*.*

Compare

$$\frac{|\mathrm{trap(f,a,b,n)} - \mathrm{trap(f,a,b,2n)}|}{3}$$

and

$$\left| \int_a^b f(x)\mathrm{dx} - \mathrm{trap(f,a,b,2n)} \right|$$

for $f(x) = \cos(x)$*,* $a = 0$*,* $b = \pi$*, and* n *from 1 to 10.*

Problem 3.2. *For the composite Simpson rule*

$$\mathrm{Simp}(f,a,b,n) = \left[f(a) + f(b) + 2\sum_{j=1}^{n-1} f(x_j) + 4\sum_{j=1}^{n} f(y_j) \right]\left(\frac{b-a}{6n}\right),$$

with $x_j = a + \frac{j(b-a)}{n}$ *and* $y_j = a + \frac{(2j-1)(b-a)}{2n}$*, make the analogous comparison of*

$$\frac{|\mathrm{Simp(f,a,b,n)} - \mathrm{Simp(f,a,b,2n)}|}{15}$$

and

$$\left| \int_a^b f(x)\mathrm{dx} - \mathrm{Simp(f,a,b,2n)} \right|$$

using $f(x) = \cos(x)$*,* $a = 0$*,* $b = \pi$*, and* n *from 1 to 10.*

Problem 3.3. *Given a function $f(x) \in C^4[a,b]$, write down the integration rule $f(x) \to I(f)$ obtained by using*

$$I(f) := \int_a^b p_3(x)\mathrm{dx}$$

to approximate

$$\int_a^b f(x)\mathrm{dx},$$

where $p_3(x)$ is the polynomial of degree ≤ 3 such that $p(a) = f(a)$, $p'(a) = f'(a)$, $p(b) = f(b)$, $p'(b) = f'(b)$.

Show that

$$\left| \int_a^b f(x)\mathrm{dx} - \mathrm{I(f)} \right| \leq \frac{\max_{[a,b]} |f^{(4)}(x)|}{720}(b-a)^5.$$

2 August 29, 2012: Due September 5, 2012

Problem 2.1. *Let $f(x)$ be function with three continuous derivatives and let $N(h) := \frac{f(x+h)-f(x)}{h}$. From the Taylor expansion, we have*

$$N(h) = f'(x) + f''(x)h/2 + O(h^2).$$

Imitate what we did in class to construct an $O(h^2)$ approximation using Richardson extrapolation with $N(h)$ and $N(h/2)$. Using Maple, compute the absolute error for $f(x) = x^6$ and $x = \sqrt{2}$.

Problem 2.2. *Use Lagrange polynomials to write down a second degree polynomial $p(x)$ that has the values f_0 at 1, f_1 at 1.5, and f_2 at 2. Compute the integral*

$$\int_1^2 p(x)\mathrm{dx}.$$

Problem 2.3. *Write down the Newton form of the second degree polynomial $p(x)$ that has the values f_0 at 1, f_1 at 1.5, and f_2 at 2.*

Problem 2.4. *Write down the Newton form of the third degree polynomial $p(x)$ that has the values $p(0) = f_0$; $p'(0) = f_0'$; $p(1) = f_1$; and $p'(1) = f_1$. Compute the integral*

$$\int_0^1 p(x)\mathrm{dx}.$$

1 August 24, 2012: Due August 29, 2012

Problem 1.1. *Using 3-digit arithmetic compute*

$$(1.72 + 0.005) + 0.002$$
$$1.72 + (0.005 + 0.002)$$
$$(1.73 + 0.005) + 0.002$$
$$1.73 + (0.005 + 0.002)$$

using chopping and using round to even. In each case compute the relative error.

Problem 1.2. *A person wishes to find a zero of $f(x) = x^3 - 5$ for $1 \le x \le 2$. That person decides to use the bisection method to accomplish this.*

1. *What are the first 3 approximations using the bisection method?*

2. *You would like to find an approximation to a zero of $f(x)$ on $[1, 2]$ with an absolute error of no more than 0.001. Using the bisection method and the error estimate for the bisection method, which is the smallest integer n for which you know that on the n-th approximation you will be within 0.001 of the correct answer. Compute this approximation.*

Problem 1.3. *A person wishes to find a zero of $f(x) = x^3 - 5$ for $1 \le x \le 2$. That person decides to use Newton's method for finding a solution of the equation $f(x) = 0$ on the interval, $[1, 2]$.*

1. *Write down the iteration formula that Newton's method gives for solving $f(x) = 0$. Then using 80 digit arithmetic with 2.0 as a starting guess, find the first six approximations to a solution of $f(x) = 0$ given by this formula.*

2. *How do these approximations compare with the tenth approximation of the bisection method computed in the previous problem.*

Problem 1.4. *A person wishes to find a zero of $f(x) = x^3 - 5$ for $1 \le x \le 2$. That person decides to use the secant method for finding a solution of the equation $f(x) = 0$ on the interval, $[1, 2]$.*

1. *Write down the iteration formula that the secant method gives for solving $f(x) = 0$. Then using 32 digit arithmetic with $1.0, 2.0$ as the starting guesses, find the first six approximations to a solution of $f(x) = 0$ given by this formula.*

2. *For i from 1 to 6, compare the i-th approximation with the i-th approximation by Newton's method.*