CFT EXERCISES, 4/26/2019

1. Free fermion

Recall that the space of states of the (chiral) free fermion is $\mathcal{H} = \mathcal{H}_A \oplus \mathcal{H}_P$ where

$$\mathcal{H}_A = \mathrm{Span}\{\cdots (b_{-\frac{3}{2}})^{n_{\frac{3}{2}}} (b_{-\frac{1}{2}})^{n_{\frac{1}{2}}} |\mathrm{vac}_A\rangle\}_{n_{\frac{1}{2}}, n_{\frac{3}{2}}, \cdots \in \{0,1\}}$$

— the Verma module over the Clifford algebra $Cl_A = \mathrm{Span}\{b_n\}_{n \in \mathbb{Z}+\frac{1}{2}}$ with anti-commutation relation $[b_n, b_m]_+ = \delta_{n,-m}\mathbf{1}$ and highest vector $|\mathrm{vac}_A\rangle$ satisfying $b_n|\mathrm{vac}_A\rangle = 0$ for $n = \frac{1}{2}, \frac{3}{2}, \cdots$. Similarly,

$$\mathcal{H}_P = \mathrm{Span}\{\cdots (b_{-2})^{n_2} (b_{-1})^{n_1} (b_0)^{n_0} |\mathrm{vac}_P\rangle\}_{n_0, n_1, n_2, \cdots \in \{0,1\}}$$

the Verma module over the Clifford algebra $Cl_P = \mathrm{Span}\{b_n\}_{n \in \mathbb{Z}}$ with anti-commutation relation $[b_n, b_m]_+ = \delta_{n,-m}\mathbf{1}$ (in particular, $(b_0)^2 = \frac{1}{2}\mathbf{1}$) and highest vector $|\mathrm{vac}_P\rangle$ satisfying $b_n|\mathrm{vac}_P\rangle = 0$ for $n = 1, 2, \cdots$. Further, recall that the fermion field $\hat{\psi}(z)$ acts on $\mathcal{H}_A$ as $\sum_{n \in \mathbb{Z}+\frac{1}{2}} b_n z^{-n-\frac{1}{2}}$ and acts on $\mathcal{H}_P$ as $\sum_{n \in \mathbb{Z}} b_n z^{-n-\frac{1}{2}}$.

(a) Show that

$$\langle \psi(z)\psi(w)\rangle \stackrel{\mathrm{def}}{=} \langle \mathrm{vac}_A|\mathcal{R}\hat{\psi}(z)\hat{\psi}(w)|\mathrm{vac}_A\rangle \quad = \frac{1}{z-w}$$

(b) Compute

$$\langle \mathrm{vac}_P|\mathcal{R}\hat{\psi}(z)\hat{\psi}(w)|\mathrm{vac}_P\rangle \quad = \frac{1}{2}\frac{\left(\frac{z}{w}\right)^{1/2} + \left(\frac{w}{z}\right)^{1/2}}{z-w}$$

(c) Use Wick's lemma[1] to show that

$$(1) \quad \langle \psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle = \\ = \frac{1}{z_1-z_2}\cdot\frac{1}{z_3-z_4} - \frac{1}{z_1-z_3}\cdot\frac{1}{z_2-z_4} + \frac{1}{z_1-z_4}\cdot\frac{1}{z_2-z_3}$$

(assume for simplicity $|z_1| > \cdots > |z_4|$). Show that the r.h.s. coincides with the Pfaffian of the matrix with entries $\frac{1}{z_i - z_j}$.

2. Virasoro generators in free fermion theory

(a) Show that the operators

$$(2) \qquad L_n = \sum_{m \in \mathbb{Z}+\frac{1}{2}} \left(\frac{m}{2} + \frac{1}{4}\right) : b_{n-m} b_m : \quad \in U(Cl_A)$$

[1] Here we need the "odd version" of Wick's lemma – for the universal enveloping of Clifford algebra rather than Heisenberg algebra. It works in the same way as the usual Wick's lemma, with the correction that one needs to take care the sign appearing when one permutes two odd generators (this is the origin of the minus in the second term on the r.h.s. in (1)).

(where U stands for the universal enveloping algebra) and

$$L_n = \sum_{m\in\mathbb{Z}} \left(\frac{m}{2}+\frac{1}{4}\right) : b_{n-m} b_m : -\frac{1}{16}\delta_{n,0}\mathbf{1} \quad \in U(Cl_P) \tag{3}$$

satisfy Virasoro algebra commutation relations with central charge $c=\frac{1}{2}$.

(b) Show that the formulas (2), (3) arise from the stress-energy tensor

$$\hat{T}(z) = \lim_{w\to z}\left(-\frac{1}{2}\mathcal{R}\hat{\psi}(w)\partial\hat{\psi}(z) + \frac{1}{2}\frac{\mathbf{1}}{(w-z)^2}\right) \tag{4}$$

as coefficients in $\hat{T}(z)=\sum_{n\in\mathbb{Z}} L_n z^{-n-2}$. Show that the definition (4) coincides in the A-sector with the normally-ordered expression $: -\frac{1}{2}\hat{\psi}(z)\partial\hat{\psi}(z) :$, while in the P-sector it differs from the normally-ordered expression by $\frac{\mathbf{1}}{16z^2}$.

(c) Show that there is a certain number α such that

$$(L_{-2}+\alpha L_{-1}^2)|\mathrm{vac}_P\rangle = 0 \tag{5}$$

Find the value of α.

(d) Let $\sigma(z)$ be the field corresponding to $|\mathrm{vac}_P\rangle$ by state-field correspondence. Consider the 4-point correlation function[2] $f(z_1,z_2,z_3,z_4)=\langle\sigma(z_1)\sigma(z_2)\sigma(z_3)\sigma(z_4)\rangle$. From global conformal invariance, show that f satisfies the ansatz

$$f(z_1,z_2,z_3,z_4) = \Big(\prod_{1\le i<j\le 4} z_{ij}\Big)^{-\frac{1}{24}} F(\lambda)$$

with $F(\lambda)$ some function of the cross-ratio $\lambda=\frac{z_{12}z_{34}}{z_{13}z_{24}}$ and with $z_{ij}=z_i-z_j$. From (5) and the Ward identity find a second-order ODE in the variable λ that $F(\lambda)$ must satisfy.

[2] It is a multi-valued holomorphic function on the configuration space, or equivalently a function on the universal cover of the configuration space.