Throughput Analysis for Wireless Networks with Full-Duplex Radios

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Abstract—This paper investigates the throughput for wireless network with full-duplex radios using stochastic geometry. Full-duplex (FD) radios can exchange data simultaneously with each other. On the other hand, the downside of FD transmission is that it will inevitably cause extra interference to the network compared to half-duplex (HD) transmission. In this paper, we focus on a wireless network of nodes with both HD and FD capabilities and derive and optimize the throughput in such a network. Our analytical result shows that if the network is adapting an ALOHA protocol, the maximal throughput is always achieved by scheduling all concurrently transmitting nodes to work in FD mode instead of a mixed FD/HD mode or HD mode regardless of the network configurations. Moreover, the throughput gain of using FD transmission over HD transmission is analytically lower and upper bounded.

I. Introduction

Traditionally, radio transceivers are subject to a HD constraint because of the crosstalk between the transmit and receive chains. The self-interference caused by the transmitter at the receiver if using FD transmission overwhelms the desired received signal from the partner node since it is much stronger than the desired received signal. Therefore, current radios all use orthogonal signaling dimensions, i.e., time division duplexing (TDD) or frequency division duplexing (FDD), to achieve bidirectional communication.

FD communication can potentially double the throughput if the self-interference can be well managed. FD radios have been successfully implemented in the industrial, scientific and medical (ISM) radio bands in a laboratory environment in the past few years [1]–[4]. Key to the success are novel analog and digital self-interference cancellation techniques as well as spatially separated transmit and receive antennas. A FD system with only one antenna has also been implemented in [5] by using specially designed circulator. In general, the main idea is to let the receive chain of a node remove the self-interference caused by the known signal from its transmit chain, so that reception can be concurrent with transmission. A novel signaling technique was proposed in [6] to achieve virtual FD with applications in neighbor discovery [7] and mutual broadcasting [8]

From a theoretical perspective, the two-way transmission capacity of wireless ad hoc networks has been studied in [9] for a FDD model. A FD cellular system has been analyzed in [10] where the throughput gain has been illustrated via extensive simulation for a cellular system with FD base station and HD mobile users. The throughput gain of single cell MIMO (Multiple-Input and Multiple-Output) wireless systems with FD radios has been quantified in [11]. A capacity analysis of FD and HD transmissions with bounded radio resources has been presented in [12] with focus on a single-link system. [13], [14] evaluate the capacity of FD ad hoc networks and alleviate the capacity degradation due to the extra interference of FD by using beamforming and ARQ protocol respectively. Both capacity analyses in [13], [14] are based on the approximation that the distances between the desired receiver and the interfering pair are the same.

In this paper, the impacts of FD transmission on the network throughput are explored. On the one hand, FD transmission allows bidirectional communication between two nodes simultaneously and therefore potentially doubles the throughput. On the other hand, the extra interference caused by FD transmissions can degrade the throughput gain over HD, which makes it unclear that FD can actually outperform HD for a given network configuration. This paper utilizes the powerful analytical tools from stochastic geometry to study the throughput performance of a wireless network of nodes with both FD and HD capabilities. Our results analytically show that for an ALOHA MAC protocol, FD always outperforms HD in terms of throughput if perfect self-interference cancellation is assumed. This result holds for arbitrary node densities, path loss exponents, link distances and SINR regimes.

II. Network Model

Consider an independently marked Poisson point process (PPP) [15] $\hat{\Phi} = \{(x_i, m(x_i), s_{x_i})\}$ on $\mathbb{R}^2 \times \mathbb{R}^2 \times \{0, 1, 2\}$ where $\Phi = \{x_i\}$ is a PPP with density λ and $m(x_i)$ and s_{x_i} are the marks of point x_i. The mark $m(x_i)$ defines the node that x_i communicates with. Here, we fix $\|x - m(x)\| = R$, $\forall x \in \Phi$, i.e., R is the distance of all links. Therefore, $m(x_i)$ can also be written as $m(x_i) = x_i + R(\cos\varphi_i, \sin\varphi_i)$, where the φ_is are independent and uniformly distributed on $[0, 2\pi]$. The link distance R can also be random and the main conclusion in this paper is not affected since we can always derive the results by first conditioning on R and then averaging over R. We define $m(\Phi) = \{m(x) : x \in \Phi\}$, which is also a PPP of density λ. The mark s_{x_i} indicates the independently chosen state of the link that consists of x_i and $m(x_i)$: $s_{x_i} = 0$ means the link is silent, $s_{x_i} = 1$ means the link is in HD mode, and $s_{x_i} = 2$ means in FD mode. HD means that in a given time slot the transmission is unidirectional, i.e., only from x_i to $m(x_i)$, while FD means that x_i and $m(x_i)$ are transmitting to each other. Therefore, for any link there are three states: silence, HD and FD. Assume that a link is in the state of silence with probability
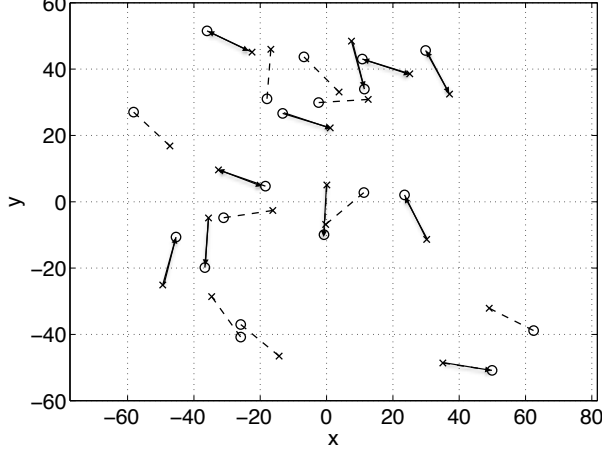
Figure 1: An example of the class of wireless networks considered in this paper where the dashed lines indicate the link is being silent, the arrows mean the link is in HD mode and the double arrows in FD mode. The x's form Φ while the o's form $m(\Phi)$.

p_0, HD with probability p_1 and FD with probability p_2, where $p_0 + p_1 + p_2 = 1$. p_1 and p_2 are the medium access probabilities (MAPs) for HD and FD modes respectively. As a result, $\Phi = \bigcup_{i=0}^{2} \Phi_{[i]}$, where $\Phi_{[i]} = \{x \in \Phi : s_x = i\}$ with density λp_i and $i \in \{0, 1, 2\}$. From the marking theorem [15, Thm. 7.5], these three node sets $\Phi_{[i]}$ are independent from each other.

The marked point process $\hat{\Phi}$ can be used to model a wireless network of nodes with both FD and HD capabilities. The self-interference in the FD links is assumed to be cancelled perfectly. In the following, we will use this model to study the performance of wireless networks with FD radios. An example of such a wireless network is illustrated in Figure 1.

In this network setup, consider the Signal-to-Interference Ratio (SIR) model where a transmission attempt from x to y is considered successful if

$$\text{SIR}_y = \frac{h_{xy}l(x,y)}{\sum_{z \in \tilde{\Phi} \setminus \{x\}} h_{zy}l(z,y)} > \theta,$$

where $\tilde{\Phi}$ is the set of transmitting nodes in a given time slot, θ is the SIR threshold, and h_{xy} and h_{zy} are the fading power coefficients with mean 1 from the desired transmitter x and the interferer z to y respectively. The transmit powers are fixed to 1. We focus on the Rayleigh fading case for both the desired link and interferers. The path loss function $l(x,y)$ between node x and y considered is $l(x,y) = \|x-y\|^{-\alpha}$ where $\alpha > 2$ is the path-loss exponent. If y is at the origin, the index y will be omitted. Also, we define a given set of system parameters $(\lambda, \theta, R, \alpha)$ as one *network configuration*. We will show that some conclusions hold regardless of the network configuration.

III. SUCCESS PROBABILITY

Our first metric of interest is the success probability, defined as

$$p_s = \mathbb{P}\left(\text{SIR}_y > \theta\right), \tag{1}$$

which is also the complementary cumulative distribution function (ccdf) of the SIR. Without changing the distribution

of the point process, assume that the receiver y is at the origin. This implies there is a transmitter at fixed distance R from the origin. The success probability plays an important role in determining the throughput as will be described in the following section.

The following theorem gives the success probability of the FD/HD-mixed wireless network modeled by the marked PPP:

Theorem 1. *In a wireless network described by the marked PPP $\hat{\Phi}$, the success probability defined in (1) is given by*

$$p_s = \exp\left(-\lambda p_1 G(\theta R^\alpha, \alpha)\right) \exp\left(-\lambda p_2 F(\theta R^\alpha, \alpha, R)\right), \tag{2}$$

where $G(s, \alpha) = \frac{\pi^2 \delta s^\delta}{\sin(\pi \delta)}$ with $\delta \triangleq 2/\alpha$ and

$$F(s, \alpha, R) = \int_0^\infty \left(2\pi - \frac{1}{1 + sr^{-\alpha}} K(s, r, R, \alpha)\right) r \, dr$$

with $K(s, r, R, \alpha) = \int_0^{2\pi} \frac{d\varphi}{1 + s(r^2 + R^2 + 2rR\cos\varphi)^{-\alpha/2}}$.

Proof: With Rayleigh fading, the desired signal strength S at the receiver at the origin is exponential, i.e., $S = hR^{-\alpha}$. The interference I consists of two parts: the interference from the HD nodes $\Phi_{[1]}$ and the interference from the FD nodes $\Phi_{[2]}$. It can be expressed in the following form:

$$I = \sum_{x \in \Phi_{[1]}} h_x l(x) + \sum_{x \in \Phi_{[2]}} \left(h_x l(x) + h_{m(x)} l(m(x))\right).$$

The Laplace transform of the interference follows as

$$
\begin{aligned}
L_I(s) &= \mathbb{E}\left(\prod_{x \in \Phi_{[1]}} e^{-sh_x l(x)} \prod_{x \in \Phi_{[2]}} e^{-s\left(h_x l(x) + h_{m(x)} l(m(x))\right)}\right) \\
&\stackrel{(a)}{=} \mathbb{E}\left(\prod_{x \in \Phi_{[1]}} e^{-sh_x l(x)}\right) \cdot \\
&\quad \mathbb{E}\left(\prod_{x \in \Phi_{[2]}} e^{-s\left(h_x l(x) + h_{m(x)} l(m(x))\right)}\right), \tag{3}
\end{aligned}
$$

where (a) comes from the fact that $\Phi_{[1]}$ and $\Phi_{[2]}$ are independent PPPs from the coloring theorem [16, page 53]. The first term in the product of (3) is the Laplace transform of the interference of the PPP $\Phi_{[1]}$, given by [15, page 103]:

$$
\begin{aligned}
L_{I_1}(s) &= \mathbb{E}\left(\prod_{x \in \Phi_{[1]}} e^{-sh_x l(x)}\right) \\
&= \exp\left(-\pi \lambda p_1 \Gamma(1+\delta) \Gamma(1-\delta) s^\delta\right) \\
&= \exp\left(-\lambda p_1 G(s, \alpha)\right),
\end{aligned}
$$

where $\Gamma(\cdot)$ is the gamma function.

The second term in the product of (3) can be written as

follows:

$$L_{I_2}(s) = \mathbb{E}\left(\prod_{x\in\Phi_{[2]}} e^{-s\left(h_x l(x) + h_{m(x)} l(m(x))\right)}\right) \quad (4)$$

$$= \mathbb{E}\left(\prod_{x\in\Phi_{[2]}} \frac{1}{1+sl(x)}\frac{1}{1+sl(m(x))}\right) \quad (5)$$

$$\stackrel{(a)}{=} \exp\left(-\lambda p_2 \int_{\mathbb{R}^2} (1-v(x))\right)$$

$$= \exp\left(-\lambda p_2 \int_0^\infty \left(2\pi - \frac{K(s,r,R,\alpha)}{1+sr^{-\alpha}}\right) r\,dr\right) \quad (6)$$

where (a) follows from the probability generating functional of the PPP with $v(x) = \frac{1}{1+sl(x)}\frac{1}{1+sl(m(x))}$. As a result, the success probability is

$$p_s = L_{I_1}(\theta R^\alpha) L_{I_2}(\theta R^\alpha)$$
$$= \exp\left(-\lambda p_1 G(\theta R^\alpha, \alpha)\right)\exp\left(-\lambda p_2 F(\theta R^\alpha, \alpha, R)\right),$$

which completes the proof. ∎

The fact that the success probability (and the Laplace transform of the interference) are a product of two terms follows from the independence of the point processes $\Phi_{[i]}$. The success probability is not in closed-form due to the integral form of $F(\theta R^\alpha, \alpha, R)$. However, tight bounds can be obtained.

Theorem 2. *The success probability is lower and upper bounded by*

$$\underline{p}_s = \exp\left(-\lambda(p_1+2p_2)G(\theta R^\alpha, \alpha)\right)$$

and

$$\overline{p}_s = \exp\left(-\lambda(p_1+p_2(1+\delta))G(\theta R^\alpha, \alpha)\right).$$

Proof: Bounds only need to be established for the second term of the product in the success probability.

Lower Bound: From (4),

$$L_{I_2}(s) = \mathbb{E}\left(\prod_{x\in\Phi_{[2]}} e^{-s\left(h_x l(x) + h_{m(x)} l(m(x))\right)}\right)$$

$$\stackrel{(a)}{\geq} \mathbb{E}\left(\prod_{x\in\Phi_{[2]}} e^{-sh_x l(x)}\right)\mathbb{E}\left(\prod_{x\in\Phi_{[2]}} e^{-sh_{m(x)} l(m(x))}\right) \quad (7)$$

$$\stackrel{(b)}{=} \exp\left(-\lambda p_2 G(s,\alpha)\right)\exp\left(-\lambda p_2 G(s,\alpha)\right)$$

$$= \exp\left(-2\lambda p_2 G(s,\alpha)\right), \quad (8)$$

where (a) follows from the Fortuin-Kasteleyn-Ginibre (FKG) inequality [15, Thm. 10.13] since both $\prod_{x\in\Phi} e^{-sh_x l(x)}$ and $\prod_{x\in\Phi} e^{-sh_{m(x)} l(m(x))}$ are decreasing random variables. In (7), the first term is similar to the calculation of $L_{I_1}(s)$ with $\Phi_{[1]}$ replaced by $\Phi_{[2]}$ while in the second term, $m(\Phi_{[2]})$ is a PPP with the same density of $\Phi_{[2]}$ due to the displacement theorem [15, page 35]. As a result, the two factors in (7) are equal, and

$$p_s \geq L_{I_1}(\theta R^\alpha)\exp\left(-2\lambda p_2 G(\theta R^\alpha, \alpha)\right) = \underline{p}_s.$$

Upper Bound: The upper bound can be obtained from the Cauchy-Schwarz inequality. From (5),

$$L_{I_2}(s) = \mathbb{E}\left(\prod_{x\in\Phi_{[2]}} \frac{1}{1+sl(x)}\frac{1}{1+sl(m(x))}\right)$$

$$\leq \{K_1(s,\alpha)K_2(s,\alpha)\}^{\frac{1}{2}}$$

which follows from the Cauchy-Schwarz inequality with $K_1(s,\alpha) = \mathbb{E}\left(\prod_{x\in\Phi_{[2]}}\frac{1}{(1+sl(x))^2}\right)$ and $K_2(s,\alpha) = \mathbb{E}\left(\prod_{x\in\Phi_{[2]}}\frac{1}{(1+sl(m(x)))^2}\right)$. We have

$$K_1(s,\alpha) = \exp\left(-2\pi\lambda p_2 \int_0^\infty \left(1-\frac{1}{(1+sr^{-\alpha})^2}\right) r\,dr\right)$$

$$= \exp\left(-\pi\lambda p_2(1+\delta)\Gamma(1+\delta)\Gamma(1-\delta)s^\delta\right)$$

$$= \exp\left(-\lambda p_2(1+\delta)G(s,\alpha)\right).$$

$K_2(s,\alpha) = K_1(s,\alpha)$ because $m(\Phi_{[2]})$ is a PPP with the same density as $\Phi_{[2]}$. As a result,

$$L_{I_2}(s) \leq \{K_1(s,\alpha)K_2(s,\alpha)\}^{\frac{1}{2}} \quad (9)$$

$$= \exp\left(-\lambda p_2(1+\delta)G(s,\alpha)\right). \quad (10)$$

Therefore,

$$p_s \leq e^{-\lambda p_1 G(\theta R^\alpha, \alpha)}e^{-\lambda p_2(1+\delta)G(\theta R^\alpha, \alpha)} = \overline{p}_s.$$

∎

The lower bound can be intuitively understood as lower bounding the interference of the FD nodes (which are formed by two dependent PPPs) by that of two independent PPPs with the same density.

The upper bound turns out to be the same as the result by assuming $l(x) = l(m(x))$ i.e., the distances between the receiver at the origin and the interfering pair from the FD links are the same. Indeed, assuming $l(x) = l(m(x))$, we have

$$\tilde{L}_{I_2}(s) = \mathbb{E}\left(\prod_{x\in\Phi_{[2]}} e^{-s\left(h_x + h_{m(x)}\right)l(x)}\right)$$

$$= \exp\left(-\pi\lambda p_2\mathbb{E}\left[\left(h_x+h_{m(x)}\right)^\delta\right]\Gamma(1-\delta)s^\delta\right)$$

$$= \exp\left(-\pi\lambda p_2\Gamma(2+\delta)\Gamma(1-\delta)s^\delta\right)$$

$$= \exp\left(-\pi\lambda p_2(1+\delta)\Gamma(1+\delta)\Gamma(1-\delta)s^\delta\right)$$

$$= \exp\left(-\lambda p_2(1+\delta)G(s,\alpha)\right)$$

where $\mathbb{E}\left[(h_x+h_y)^\delta\right] = \Gamma(2+\delta)$ comes from the fact that $h_x + h_y$ has an Erlang distribution and $\Gamma(2+\delta) = (1+\delta)\Gamma(1+\delta)$ is due to the property of the gamma function. Hence, the approximated success probability assuming $l(x) = l(m(x))$ is $\tilde{p}_s = L_{I_1}(\theta R^\alpha)\tilde{L}_{I_2}(\theta R^\alpha) = \overline{p}_s$.

This result is not surprising. The equality holds for the Cauchy-Schwarz inequality if $\prod_{x\in\Phi_{[2]}}\frac{1}{(1+sl(x))^2}$ and $\prod_{x\in\Phi_{[2]}}\frac{1}{(1+sl(m(x)))^2}$ are linearly dependent. Obviously, $l(x) = l(m(x))$ satisfies this condition. Therefore, we have $\tilde{p}_s = \overline{p}_s$ as expected. Also, $\overline{p}_s/\underline{p}_s = e^{\lambda p_2(1-\delta)G(\theta R^2,\alpha)} \to 1$ as $\lambda \to 0$ or $R \to 0$. Hence, the bounds are very tight in the asymptotic case.
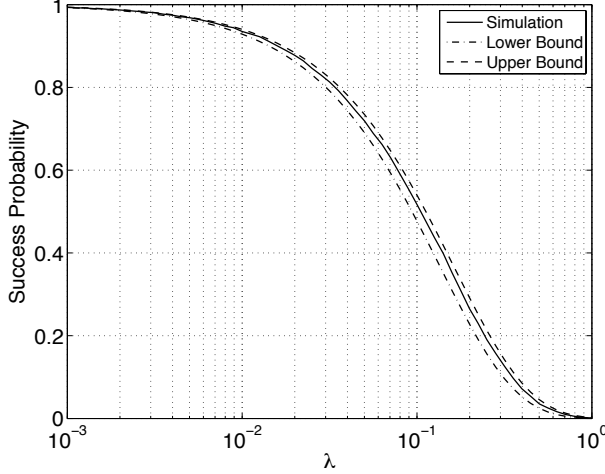
Figure 2: Comparison of success probability between simulation and its bounds as a function of the node density λ: $\alpha = 4$, $\theta = 1$, $R = 1$, $p_0 = 0$, $p_1 = p_2 = 0.5$.

Figure 2 plots the success probability from simulation and its closed-form upper and lower bounds as a function of the node density. As seen, both bounds are tight.

Corollary 3. *The function $F(s, \alpha, R)$ can be bounded as follows:*

$$(1 + \delta) G(s, \alpha) \leq F(s, \alpha, R) \leq 2G(s, \alpha). \quad (11)$$

Proof: From the proof of the upper and lower bounds of the success probability p_s, i.e., (8) and (10), we have

$$\exp\left(-2\lambda p_2 G(s, \alpha)\right) \leq L_{I_2}(s) \leq \exp\left(-\lambda p_2 (1 + \delta) G(s, \alpha)\right), \quad (12)$$

where $L_{I_2}(s) = \exp\left(-\lambda p_2 F(s, \alpha, R)\right)$ from (6). By taking logarithm on both sides of the above, we have

$$-2\lambda p_2 G(s, \alpha) \leq -\lambda p_2 F(s, \alpha, R) \leq -\lambda p_2 (1 + \delta) G(s, \alpha), \quad (13)$$

which leads to (11). ∎

This corollary is useful in calculating the maximal throughput and its bounds in the following section.

IV. THROUGHPUT PERFORMANCE ANALYSIS

A. Problem Statement

The goal of FD transmission in a network is to increase the network throughput. While FD increases the link throughput, it also causes additional interference to the other links. Given a network that consists of nodes of FD and HD capabilities, how should a node choose between FD and HD transmissions as the network configuration varies? Should the node always transmit in a FD mode, or should it always work in HD mode? Or should it sometimes work in FD mode while sometimes in HD mode? It is unlikely to conclude to make the nodes work in HD mode all the time. Otherwise, there is no need for FD. However, it is interesting to figure out which is better between a FD-only network or a FD/HD-mixed network. In our model, the question is equivalent to given a network that consists of nodes with both HD and FD capabilities, how can we choose p_1 and p_2 to optimize the throughput in the network? To see that, we first define the throughput. In a random wireless network described by $\hat{\Phi}$, we can consider the typical link, consisting of a node x_0 and its mark $m(x_0)$. The *typical link* has probability p_1 to be in HD mode and p_2 to be in FD mode. Therefore, its throughput can be defined as follows:

Definition 4. For a wireless network described by $\hat{\Phi}$, the throughput of the typical link is defined as

$$T = (p_1 + 2p_2)\, p_s \quad (14)$$

Inserting p_s from (2) into (14), we have

$$T(p_1, p_2) = (p_1 + 2p_2)\, e^{-(\lambda p_1 G + \lambda p_2 F)}. \quad (15)$$

From now on, we will use G to denote $G(\theta R^\alpha, \alpha)$ and F to denote $F(\theta R^\alpha, \alpha, R)$ for simplicity. Given the definition of throughput, there are two extreme cases that are particularly interesting: one is the case where all concurrently transmitting nodes work in HD mode, i.e., $p_2 = 0$, and the other is where all concurrently transmitting nodes work in FD mode, i.e., $p_1 = 0$. Their throughputs are given as: $T^{\text{HD}}(p_1) = p_1 \exp\left(-\lambda p_1 G\right)$ and $T^{\text{FD}}(p_2) = 2p_2 \exp\left(-\lambda p_2 F\right)$.

B. Throughput Optimization

It is interesting to find the relationship between the maximal values of T^{HD}, T^{FD} and T, denoted as $T^{\text{HD}}_{\max}$, $T^{\text{FD}}_{\max}$ and $T_{\max}$. In other words, we would like to determine how to choose p_1 and p_2 such that the maximal throughput of the network is achieved. First of all, $T^{\text{HD}}_{\max}$ and $T^{\text{FD}}_{\max}$ can be easily obtained by the following lemma.

Lemma 5. *For a wireless network of HD-only network, described by $\hat{\Phi}$ with $p_2 = 0$, $T^{HD}_{\max}$ is given by*

$$T^{\text{HD}}_{\max} = \begin{cases} T^{HD}(\frac{1}{\lambda G}) = \frac{1}{\lambda G} e^{-1} & \text{if } \lambda G \geq 1 \\ T^{HD}(1) = e^{-\lambda G} & \text{if } \lambda G < 1 \end{cases}, \quad (16)$$

with optimal MAP

$$p_1^{opt} = \min\left(\frac{1}{\lambda G}, 1\right). \quad (17)$$

Similarly, for a wireless network of FD-only network, described by $\hat{\Phi}$ with $p_1 = 0$, $T^{FD}_{\max}$ is given by

$$T^{\text{FD}}_{\max} = \begin{cases} T^{FD}(\frac{1}{\lambda F}) = \frac{2}{\lambda F} e^{-1} & \text{if } \lambda F \geq 1 \\ T^{FD}(1) = 2e^{-\lambda F} & \text{if } \lambda F < 1 \end{cases}, \quad (18)$$

with optimal MAP

$$p_2^{opt} = \min\left(\frac{1}{\lambda F}, 1\right). \quad (19)$$

Proof: The proof is straightforward by taking the derivatives of T^{HD} and T^{FD} with respect to p_1 and p_2. ∎

In the following theorem, we show that $T_{\max}$ is achieved by setting all concurrently transmitting nodes to be in FD mode and that $T_{\max} = T^{\text{FD}}_{\max}$.

Theorem 6. *For a wireless network described by $\hat{\Phi}$, the maximal throughput is given by*

$$T_{\max} = T^{FD}_{\max}, \quad (20)$$

with the optimal MAP $(p_1, p_2) = \left(0, p_2^{opt}\right) = \left(0, \min\left(\frac{1}{\lambda F}, 1\right)\right)$. Also, (20) holds regardless of the network configuration $(\lambda, \theta, R, \alpha)$.

Proof: Taking the derivative of T w.r.t. p_1 and p_2 leads to

$$\frac{\partial T}{\partial p_1} = \exp\left(-\lambda\left(p_1 G + p_2 F\right)\right)\left[1 - \lambda G\left(2p_2 + p_1\right)\right], \quad (21)$$

$$\frac{\partial T}{\partial p_2} = \exp\left(-\lambda\left(p_1 G + p_2 F\right)\right)\left[2 - \lambda F\left(2p_2 + p_1\right)\right]. \quad (22)$$

Note that $2p_2 + p_1 \in [0, 2]$ and $\lambda F < 2\lambda G$ from Corollary 3.

1) $\lambda F < 2\lambda G < 1$: $2\lambda G < 1$ leads to $\frac{\partial T}{\partial p_1} > 0$. Therefore, T is an increasing function in p_1. $\lambda F < 1$ implies $\frac{\partial T}{\partial p_2} > 0$. T is also an increasing function in p_2. As a result, $T_{\max} = \max\left\{T^{\text{HD}}(1), T^{\text{FD}}(1)\right\}$. Since

$$\frac{T^{\text{FD}}(1)}{T^{\text{HD}}(1)} = 2e^{\lambda(G-F)} \overset{(a)}{\geq} 2e^{-\lambda F/2} \overset{(b)}{\geq} 2e^{-\frac{1}{2}} > 1,$$

where (a) follows from Corollary 3 and (b) from $\lambda F < 1$, $T_{\max} = T^{\text{FD}}(1) = T^{\text{FD}}_{\max}$ in this case.

2) $\lambda F < 1 < 2\lambda G$: Under this condition, let $\frac{\partial T}{\partial p_1} = 0$ and we have $2p_2 + p_1 = 1/\lambda G$. Also, $\frac{\partial T}{\partial p_2} > 0$ still holds. Therefore, the maximal T is achieved at $(p_1, p_2) = \left(0, \frac{1}{2\lambda G}\right)$ from $2p_2 + p_1 = \frac{1}{\lambda G}$ and $\frac{\partial T}{\partial p_2} > 0$. Note that $T(0, \frac{1}{2\lambda G}) = T^{\text{FD}}(\frac{1}{2\lambda G}) < T^{\text{FD}}(1)$. Hence, $T_{\max} = T^{\text{FD}}(1) = T^{\text{FD}}_{\max}$.

3) $1 < \lambda F < 2\lambda G$: Let $\frac{\partial T}{\partial p_1} = 0$ and $\frac{\partial T}{\partial p_2} = 0$ and we have

$$2p_2 + p_1 = 1/\lambda G \quad (23)$$

and

$$2p_2 + p_1 = 2/\lambda F. \quad (24)$$

From (23), T is maximized at $p_1 = \frac{1}{\lambda G} - 2p_2$, which leads to

$$T = \frac{1}{\lambda G}e^{\lambda p_2(2G-F)-1} \leq T(0, \frac{1}{2\lambda G}) = \frac{1}{\lambda G}e^{-\frac{F}{2G}}$$

From (24), T is maximized at $p_2 = \frac{1}{\lambda F} - \frac{p_1}{2}$, which leads to

$$T = \frac{2}{\lambda F}e^{\lambda p_1\left(\frac{F}{2}-G\right)-1} \leq T(0, \frac{1}{\lambda F}) = \frac{2}{\lambda F}e^{-1}$$

Therefore, $T_{\max} = \max\left\{T(0, \frac{1}{2\lambda G}), T(0, \frac{1}{\lambda F})\right\}$. Since

$$\frac{T(0, \frac{1}{2\lambda G})}{T(0, \frac{1}{\lambda F})} = \frac{F}{2G}e^{1-\frac{F}{G}} \overset{(a)}{\leq} e^{1-\frac{F}{G}} \overset{(b)}{\leq} e^{-\delta} < 1,$$

where (a) follows from $F < 2G$ and (b) from $F > (1+\delta)G$. Hence, $T_{\max} = T^{\text{FD}}(\frac{1}{\lambda F}) = T^{\text{FD}}_{\max}$.

∎

To summarize,

$$T_{\max} = T^{\text{FD}}_{\max}$$

for all the cases, which means $T_{\max}$ is always achieved by setting all transmitting nodes to work in FD mode instead of in HD mode, i.e. $p_1 = 0$, despite the interference caused by the FD nodes. This conclusion is not affected by the network configuration $(\lambda, \theta, R, \alpha)$. The corresponding optimal MAPs is $(p_1, p_2) = \left(0, \min\left(\frac{1}{\lambda F}, 1\right)\right)$, which is equivalent to setting p_2 to be the optimal MAP of FD-only network given in (19) with $p_1 = 0$.

C. Comparison of FD with HD

Since the mixed FD/HD network will achieve maximal throughput at the extreme case of an FD-only network, we can simply focus on the FD-only and HD-only networks and compare their optimal MAPs and maximal throughputs from the results in Lemma 5. Given a fixed set of system parameters (θ, R, α), the optimal MAPs to achieve the maximal throughput for FD and HD networks as a function of node density λ are illustrated in Figure 3. The figure is plotted according to (17) and (19) and it shows that both FD and HD network will make all nodes transmit when the node density is lower than $\frac{1}{F}$ and $\frac{1}{G}$, respectively, and after that the MAPs will be inversely proportional to the node density λ. The throughput gain of FD network with respect to HD
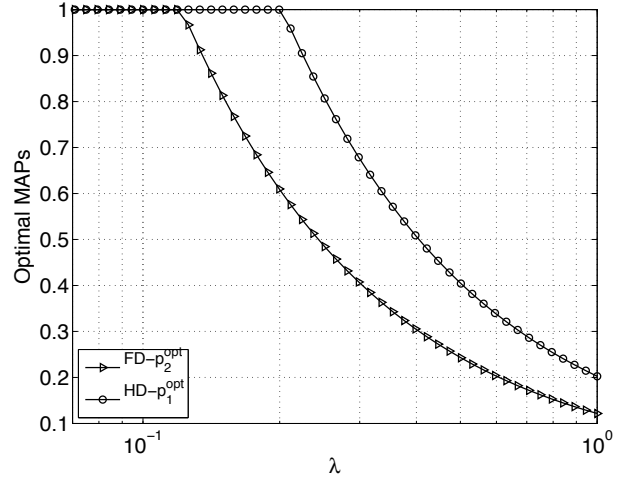

Figure 3: Optimal MAPs that achieve the maximal throughput for FD and HD networks as a function of the node density λ: $\alpha = 4$, $R = 1$, $\theta = 1$.

network is of great interest. In the following, the throughput gain is defined:

Definition 7. The throughput gain (TG) is defined as the ratio between the maximal throughput of FD network and HD network given the same network configuration $(\lambda, \theta, R, \alpha)$:

$$\text{TG} = \frac{T^{\text{FD}}_{\max}}{T^{\text{HD}}_{\max}}.$$

The following corollary gives the theoretical expression of TG in terms of λ, F and G together with its lower and upper bounds in closed-form. Note that F and G are constant given fixed (θ, R, α).

Corollary 8. *The throughput gain is given by*

$$\text{TG} = \begin{cases} 2e^{\lambda(G-F)} & \text{if } \lambda F < 1, \lambda G < 1 \\ \frac{2}{\lambda F}e^{(\lambda G-1)} & \text{if } \lambda F \geq 1, \lambda G < 1 \\ \frac{2G}{F} & \text{if } \lambda F \geq 1, \lambda G \geq 1 \end{cases} \quad (25)$$

and bounded as

$$\begin{cases} 2e^{-\lambda G} < \text{TG} < 2e^{-\delta\lambda G} & \text{if } \lambda F < 1, \lambda G < 1 \\ \frac{e^{(\lambda G-1)}}{\lambda G} < \text{TG} < \frac{2e^{(\lambda G-1)}}{(1+\delta)\lambda G} & \text{if } \lambda F \geq 1, \lambda G < 1 \\ 1 < \text{TG} < \frac{2}{1+\delta} & \text{if } \lambda F \geq 1, \lambda G \geq 1 \end{cases} \quad (26)$$

Proof: From (16) and (18), when $\lambda F < 1$ (it implies $\lambda G < 1$ from Corollary 3), $\text{TG} = T^{\text{FD}}(1)/T^{\text{HD}}(1) =$

$2e^{\lambda(G-F)}$; when $\lambda F > 1$ and $\lambda G < 1$, TG $=$ $T^{\mathrm{FD}}(1/\lambda F)/T^{\mathrm{HD}}(1) = 2e^{(\lambda G-1)}/\lambda F$; and when $\lambda F > 1$ and $\lambda G > 1$, TG $= T^{\mathrm{FD}}(1/\lambda F)/T^{\mathrm{HD}}(1/\lambda G) = 2G/F$. Therefore, we have (25). The upper and lower bounds can be easily proven by using Corollary 3. Note that $G = G(\theta R^\alpha, \alpha)$ and hence both bounds are in closed form. ∎
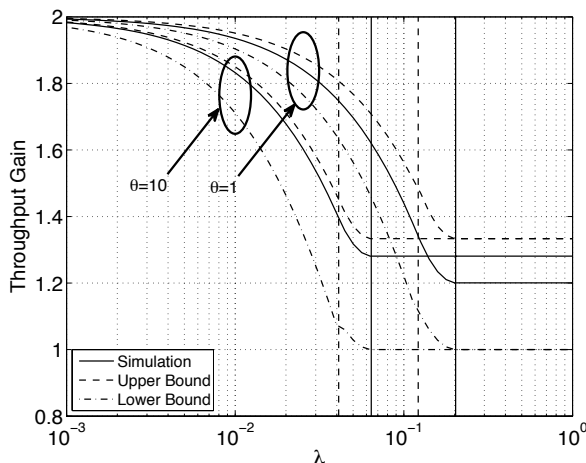

Figure 4: Throughput gain as a function of the node density λ and its bounds: $\alpha = 4$, $R = 1$. The two dashed vertical lines indicate the point where $\lambda = 1/F$ for different θ while the two solid vertical lines indicate $\lambda = 1/G$.

Figure 4 illustrates the throughput gain as a function of the node density together with its upper and lower bounds given in (26). Given θ, R, and α, the estimate of $F(\theta R^\alpha, \alpha, R)$ can be obtained numerically. As seen, the throughput gain is approaching 2 asymptotically as the node density $\lambda \to 0$. As the node density increases, the throughput gain decreases. The throughput gain will become constant after the node density is greater than the threshold $1/G$, which is illustrated by two solid vertical lines for different θ. The reason is that the density of concurrent transmitting nodes will get saturated under ALOHA protocol for both FD and HD networks if λ is great than $1/G$. This constant throughput gain is upper bounded by $\frac{2}{1+\delta}$ and lower bounded by 1. The effect of the SINR threshold θ is that a larger SINR threshold gets the network saturated at lower node density since a larger SINR threshold implies the system can tolerate less interference. As a result, the constant throughput gain will be higher for larger θ. In general, an ALOHA protocol will guarantee the throughput gain of the FD transmission to be greater than 1 for all network configurations, which means that FD transmission always outperforms HD transmission.

V. CONCLUSION

In this paper, we analyzed the throughput of wireless networks with FD radios using mathematical tools from stochastic geometry. Given a wireless network of radios with both FD and HD capabilities, we showed that FD transmission is always preferable compared to HD transmission in terms of throughput. Although the throughput of FD transmission can not be doubled, the gain is considerable in an ALOHA protocol. In general, FD can be a very powerful technique that can be adapted for the next-generation wireless networks if the limitations from the real world can be well mitigated, i.e., the imperfect self-interference cancellation. Moreover, the throughput gain is expected to be larger if more advanced MAC protocols other than ALOHA are used or the interference management can be used for the pairwise interferers in the FD links. Future work will focus on the modeling of imperfect self-interference cancellation in our mathematical framework and the impact that imperfect self-interference cancellation has on the throughput gain of FD over HD.

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