QUASIMONOPHOBIC GRAPHS AND DEGREE SPECTRAL SEQUENCES IN DISCRETE CUBICAL HOMOLOGY

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Abstract. We introduce the degree filtration on the discrete cubical chain complex of a graph, defined in terms of the maximal injective dimension of the facets of singular n-cubes, and study the degree spectral sequence which arises from this filtration. This spectral sequence interpolates between the discrete cubical homology of a graph $H_n(G)$ and the injective homology $H_n^{inj}(G)$, a variant of the discrete cubical homology based on injective singular cubes. Building on the work of Babson et al. we introduce the combinatorial condition of quasimonophobicity on graphs, and show quasimonophobicity implies both the vanishing of the degree spectral sequence in certain bidegrees, and implies $H_n^{inj}(G)$ is isomorphic to the homology of the CW complex obtained by "filling in" subcubes of the graph. These results are applied to compute $H_2(G_n^{sph})$ for the Greene sphere graphs G_n^{sph}.

1. Introduction

In [BCW14], Barcelo, Caparo, and White introduced the discrete cubical homology groups[1]

$$H_n(G)$$

of a graph G. These homology groups capture the higher-dimensional structure beyond that seen in classical homologies (where one considers the homology of the 1-dimensional CW complex associated to a graph, or the homology of a simplicial complex, such as the clique complex, associated to a graph). In contrast to these classical homologies, discrete cubical homology probes a graph using discrete cubes of all dimensions, yielding an interesting theory both from a theoretical perspective and for its applications.

Discrete cubical homology, together with its associated homotopy theory [BKLW01], raises interesting questions in higher category theory. For instance, computational considerations of discrete cubical homology led to the result that the homology of a cubical set is unchanged when a subcomplex consisting of connection cubes is quotiented out from the chain complex [BGJW21a]. Similar studies motivated the comparison between homotopy n-types of graphs and those of cubical and simplicial sets [CK24], [KM24], [CK26]. On the other hand, from an applied viewpoint, discrete cubical homology has also been shown to be more robust to noise than the homology of the flag complex of a graph, making it useful for persistent homology and topological data analysis [KK25].

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[1] These are sometimes denoted $\mathcal{H}_n^{\mathrm{Cube}}(G)$ — see [BGJW19].

Despite its conceptual appeal and broad applications, discrete cubical homology is shockingly difficult to compute in degrees greater than 1, even for some very simple graphs. The primary reason for this is that the hypotheses for excision in discrete cubical homology are much more constrained than in the topological or simplicial contexts [BCW14, Thm. 3.4]. While discrete cubical homology of a finite graph is in principle finitely computable, in practice, even with the implementation of efficient algorithms, actual machine computation of many simple graphs is limited to degrees less than or equal to 3 [KK24]. This difficulty highlights the limited understanding of the nature of the higher-dimensional graph cells contributing to nontrivial homology classes.

At present, it is only known that higher-dimensional cells cannot arise in the absence of triangles and squares in the graph. Indeed, a key result of [BGJW21b] shows that graphs containing no triangles or squares have trivial discrete cubical homology in dimensions 2 and higher. This result is consistent with earlier work in discrete homotopy theory, where triangles and squares play the role of 2-cells: the discrete fundamental group agrees with the fundamental group of the CW complex obtained by attaching 2-cells along all triangles and squares [BKLW01]. Therefore, triangle-free and square-free graphs represent homotopy 1-types. Homotopy n-types for $n \geq 2$ continue to be rather mysterious objects of study [KM24], though Carranza and Kapulkin have recently shown the striking result that every homotopy n-type can be abstractly represented by a graph [CK26].

These results naturally lead to the question of whether there exists an intrinsic notion of dimension for graphs that governs the vanishing of discrete cubical homology in higher degrees. A natural candidate is the *cubical dimension* of a graph, defined as the maximal dimension of an injective singular cube it contains. It is unclear whether this notion alone is sufficient: in principle, non-cubical subgraphs could contribute to homology in degrees exceeding the cubical dimension, though we know of no actual examples of this phenomenon occurring.

In this paper, we will define a *degree filtration* on the discrete cubical singular chains of a graph which arises from consideration of the maximal injective dimension of the facets of a singular cube. Associated to this filtration is a *degree spectral sequence*

$$E^1_{p,q}(G) = H^{[p]}_{p+q}(G) \Rightarrow H_{p+q}(G)$$

where the homology groups $H^{[p]}_*(G)$ are the homology of the pth associated graded of our filtration and satisfy

$$H^{[>n]}_n(G) = 0.$$

If G has cubical dimension d, then

$$H^{[>d]}_n(G) = 0.$$

We will define the *injective homology* $H^{inj}_*(G)$ to be the edge of the E_2-page of this spectral sequence given by the homology of the chain complex

$$H^{[0]}_0(G) \xleftarrow{d_1} H^{[1]}_1(G) \xleftarrow{d_1} \cdots .$$

There are maps

$$H_n(G) \twoheadrightarrow E^\infty_{n,0}(G) \hookrightarrow H^{inj}_n(G). \tag{1.1}$$

The injective homology promises to be much more computable than the discrete cubical homology, because it only involves injective singular cubes. Indeed, we will show that if $CW^{\square}(G)$ is the CW complex whose n-cells correspond to the subgraphs of G isomorphic to n-cubes, then there is a surjection of chain complexes

$$C_n^{cell}(CW^{\square}(G)) \twoheadrightarrow H_n^{[n]}(G) \tag{1.2}$$

resulting in a map

$$H_n(CW^{\square}(G)) \to H_n^{inj}(G). \tag{1.3}$$

We then are motivated to ask:

Question 1.4. *When are the map (1.3) and the maps (1.1) isomorphisms, resulting in an isomorphism*

$$H_n(G) \cong H_n(CW^{\square}(G))?$$

We will see that sometimes affirmative answers to Question 1.4 derive from combinatorial conditions on the graph G. For example, the results of [BGJW21b] imply that

$$H_n(G) \cong H_n(CW^{\square}(G))$$

for triangle-free square-free graphs.[2] Moreover, it follows from [EK25, Thm. 3.4] (see also [BKLW01, Prop. 5.12]) that for triangle-free graphs G we have[3]

$$H_1(G) \cong H_1(CW^{\square}(G)).$$

The groups $H_{\geq 1}^{[0]}(G)$ are all zero for trivial reasons. We will adapt the "covering space" technique used in [BGJW21b] to prove that for G triangle-free we have (Theorem 4.4)

$$H_{\geq 2}^{[1]}(G) = 0.$$

It follows that if G is a triangle-free graph of cubical dimension 2, there are isomorphisms

$$H_n(G) \cong \begin{cases} H_n^{inj}(G), & n \leq 2, \\ H_n^{[2]}, & n > 2. \end{cases}$$

The notion of monophobic cube neighborhoods, introduced in [GJ], provides a different generalization of the triangle-free square-free condition considered by [BGJW21b], where one considers graphs in which every n-cube has a monophobic neighborhood.

Indeed, a triangle-free and square-free graph is precisely one in which every edge (viewed as a 1-cube) has a monophobic neighborhood (such graphs are referred to as 1-monophobic in [GJ]). In this paper, we introduce the class of *n-quasimonophobic* graphs (see Definition 5.2). It will turn out that a graph is 1-quasimonophobic if and only if it is triangle-free.

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[2]For such graphs, $CW^{\square}(G)$ is nothing other than the 1-dimensional CW complex associated to the graph G.

[3]Ender and Kapulkin instead consider the 2-dimensional CW complex X_G obtained by filling in triangles and squares with 2-cells, but for G triangle-free, X_G is just the 2-skeleton of $CW^{\square}(G)$.

We will show (Theorem 5.7) that if G is n-quasimonophobic, then the map (1.2) gives an isomorphism

$$H_n^{[n]}(G) \cong C_n^{cell}(CW^{\square}(G)). \tag{1.5}$$

We will then deduce (Theorem 5.15) that for G n-quasimonophobic, the map (1.2) induces an injection

$$H_n(CW^{\square}(G)) \hookrightarrow H_n^{inj}(G)$$

and this map is an isomorphism if G is $(n-1)$-quasimonophobic.

We then deduce our main result (Theorem 5.17) from the degree spectral sequence. Suppose that G is (≤ 2)-quasimonophobic. Then the map (1.1) and map (1.2) induce an isomorphism

$$H_2(G) \cong H_2(CW^{\square}(G)).$$

This appears to be the first systematic result computing discrete cubical H_2 with the exception of that of [BGJW21b], which establishes that $H_2(G) = 0$ for G triangle-free and square-free.

As an application, we compute the second discrete cubical homology of the Greene spheres G_n^{sph}. The graphs G_n^{sph} $(n \geq 4)$ are a family of graphs of cubical dimension 2 which were constructed by Greene to model the 2-sphere in the setting of discrete cubical homology. Specifically, the graph G_4^{sph} was introduced in [BGJW19], where the authors prove that $H_2(G_4^{sph}) \neq 0$, and machine computation is used in [KK24] to show that[4]

$$H_2(G_4^{sph}) \cong \mathbb{Z}.$$

The Greene spheres turn out to be n-quasimonophobic for every n, and hence are amenable to Theorem 5.17. We will deduce

$$H_2(G_n^{sph}) = \mathbb{Z} \qquad \text{for } n \geq 4. \tag{1.6}$$

Presumably, machine computations could be used to reproduce (1.6) for small n.

We also explain more generally how to "geometrically" produce nontrivial classes in $H_n(G)$ when G is n-quasimonophobic (Theorem 6.6).

Organization of the paper

In Section 2 we set our basic definitions, and recall the definition of discrete cubical homology. In Section 3 we define the degree filtration and the degree spectral sequence. We show in Section 4 that triangle-free graphs G have $H_{\geq 2}^{[1]}(G) = 0$. In Section 5 we introduce the notion of n-quasimonophobicity, and show that for such graphs G we have $H_n^{[n]}(G) \cong C_n^{cell}(CW^{\square}(G))$, and deduce our main results. In Section 6 we apply our results to compute $H_2(G_n^{sph})$. We also give an example to show the necessity of quasimonophobicity for Theorem 5.17 and contemplate possible generalization of Theorem 5.17 for quasimonophobic graphs. We end this section with a geometric construction of classes of $H_n(G)$, and which we prove are non-trivial when G is n-quasimonophobic.

[4] In fact, [KK24] also compute $H_3(G_4^{sph}) = 0$.

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2. Preliminaries

Throughout this document, the term "graph" denotes a simple undirected graph – one without loops or multiple edges. We denote a graph as a pair (V, E) of vertex set V and edge set E, where each edge is a two-point subset of V. We sometimes denote the vertex and edge sets as $V(G)$ and $E(G)$, respectively.

A *graph map*

$$f\colon G \to G'$$

from a graph $G = (V, E)$ to another graph $G' = (V', E')$ is a map $f\colon V \to V'$ between their vertex sets that maps every edge to an edge or a vertex. We shall say f is an isomorphism if it has an inverse. We will let

$$\mathrm{Map}(G, G')$$

denote the set of all graph maps from G to G'.

The *image* $f(G)$ or $\mathrm{Im}(f)$ of a graph map $f\colon G \to G'$ is the subgraph of G' with vertex set $f(V)$ and only those edges of G' that are images of the edges in G.

We shall write

$$G \subseteq G'$$

to indicate that G is a subgraph of G', in the sense that $V(G) \subseteq V(G')$ and $E(G) \subseteq E(G')$. A subgraph $G \subseteq G'$ is *induced* if for each pair of vertices $v, w \in V(G)$, an edge connects them in G if and only if an edge connects them in G'.

The *neighborhood graph* $N(v)$ of a vertex v in a graph G is the induced subgraph of G containing v and the vertices adjacent to v. We say that a graph map $f : G \to G'$ is a local isomorphism for all vertices $v \in V(G)$, f restricts to an isomorphism

$$f\big|_{N(v)} : N(v) \to N(f(v)).$$

For $n \geq 0$, we will let I^n denote the *discrete n-cube graph*, with the vertex set $\{0, 1\}^n$ and two vertices are adjacent if they differ in exactly one coordinate. In particular, I^0 is the graph with a single vertex and no edges.

For each $n \geq 0$, define the *singular cubical n-chains on G* to be the free abelian group

$$Q_n(G) := \mathbb{Z}\,\mathrm{Map}(I^n, G).$$

generated by the *singular n-cubes*

$$\sigma\colon I^n \to G.$$

For $1 \le i \le n$, the *front* and *back i-faces* of such a singular n-cube σ are denoted as $f_i^-\sigma$ and $f_i^+\sigma$, where $f_i^-\sigma$ is obtained by fixing the i-th coordinate equal to 0, while the back i-face $f_i^+\sigma$ is obtained by fixing the i-th coordinate equal to 1. These face operators satisfy the face relation: for σ a singular n-cube and $1 \le i < j \le n$, we have

$$f_i^a f_j^b \sigma = f_{j-1}^b f_i^a \sigma. \tag{2.1}$$

The *boundary* operator $\partial_n\colon Q_n(G) \to Q_{n-1}(G)$ is defined on a singular n-cube σ by

$$\partial_n \sigma = \sum_{i=1}^{n} (-1)^i (f_i^-\sigma - f_i^+\sigma).$$

We have $\partial_n \circ \partial_{n+1} = 0$, so that $(Q_*(G), \partial_*)$ forms a chain complex. However, as in the case with singular cubical chains for topological spaces, the homology of this chain complex is *not* the desired discrete cubical homology of G (e.g. if G has 1 vertex, $H_n(Q_*(G))$ is nonzero in every degree). We instead are required to normalize by modding out by degeneracies.

A singular n-cube σ is said to be *degenerate*, if there exists an index i such that $f_i^-\sigma = f_i^+\sigma$. The boundary of a degenerate cube is a linear combination of degenerate cubes, so the degenerate singular cubes form a subcomplex

$$D_*(G) \subseteq C_*(G).$$

We define the *normalized singular cubical chains* to be the quotient complex

$$C_*(G) := Q_*(G)/D_*(G).$$

The *discrete cubical homology* is defined to be the homology of this quotient complex:

$$H_n(G) := H_n(C_*(G)).$$

It should be noted that even for graphs with few vertices, the rank of these normalized singular cubical chain groups typically grows super-exponentially with respect to dimension, making the computation of discrete cubical homology highly challenging.

3. The degree spectral sequence

We define the *degree* of a singular n-cube

$$\sigma\colon I^n \to G$$

to be the dimension of the maximal injective facet of σ.

Let

$$C_n^k(G) \le C_n(G)$$

denote the subgroup generated by singular n-cubes of degree k or below. This results in a *degree filtration*

$$C_*^0(G) \leq C_*^1(G) \leq \cdots$$

on the chain complex $C_*(G)$. We will denote the associated graded of this filtration by

$$C_n^{[k]}(G) := C_n^k(G)/C_n^{k-1}(G).$$

We let

$$H_*^{[k]}(G) := H_*(C_*^{[k]}(G))$$

denote the homology of this quotient complex.

The *cubical dimension* of a graph G is the dimension of the maximal injective singular cube in G. If G has cubical dimension d, we clearly have

$$C_*^{[k]}(G) = 0 \quad \text{for } k > d$$

and therefore

$$H_*^{[k]}(G) = 0 \quad \text{for } k > d.$$

Associated to the degree filtration is a convergent spectral sequence (see for example, [Wei94])

$$E^1_{p,q} = H^{[p]}_{p+q}(G) \Rightarrow H_{p+q}(G)$$

which we will refer to as the *degree spectral sequence*.

Since for all $k > n$, we have $C_n^k(G) = C_n(G)$, it follows that

$$H_n^{[k]}(G) = 0 \quad \text{for } k > n.$$

For $n > 0$, the degree-0 singular n-cubes are degenerate, hence

$$H_n^{[0]}(G) = 0.$$

For a graph G with cubical dimension d, $H_n^{[p]}(G) = 0$, for $p > d$. The degree spectral sequence is shown in Figure 1, where we write $H^{[p]}_{p+q}$ for $H^{[p]}_{p+q}(G)$.

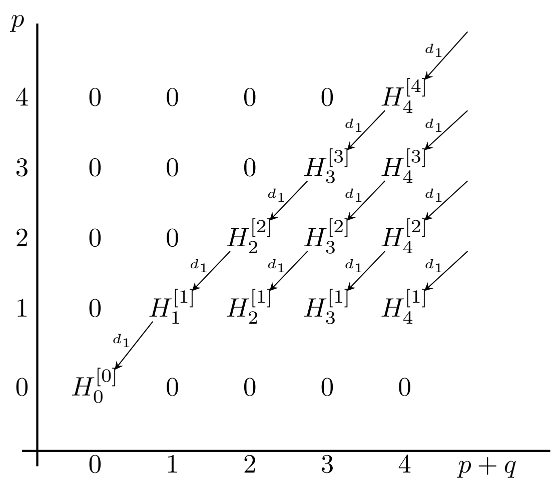

FIGURE 1. E^1-page of the degree spectral sequence

4. Degree-1 homology cycles in a triangle-free graph

In this section, our main goal is to prove that $H_n^{[1]}$ of a triangle-free graph is trivial for $n \geq 2$ (Theorem 4.7). This result extends the main result of [BGJW21b], which asserts that the discrete cubical homology of triangle- and square-free graphs is trivial in dimension 2 and higher.

To provide context and to clarify how the arguments of [BGJW21b] carry over to our setting, we first give a streamlined approach to the main result of [BGJW21b]. The technique these authors used to prove Theorem 4.1 is based on a combinatorial analog of the notion of covering space in the category of graphs.[5]

Given graph G, we will let $G_{\mathbb{R}}$ denote the usual 1-dimensional CW complex (topological graph) whose 0-cells are given by the vertices of G, and 1-cells are given by the edges of G. Let

$$p : \widetilde{G}_{\mathbb{R}} \to G_{\mathbb{R}}$$

denote the universal cover. The CW structure on $G_{\mathbb{R}}$ lifts to a CW structure on $\widetilde{G}_{\mathbb{R}}$, giving $\widetilde{G}_{\mathbb{R}}$ the structure of a topological forest (a disjoint union of topological trees). Let $\widetilde{G}$ be the combinatorial forest associated to this topological forest. Then the local homeomorphism p arises from a local isomorphism of graphs

$$\widetilde{G} \to G.$$

If G is path connected, $\widetilde{G}$ is a tree, and the fundamental group

$$\pi := \pi_1(G_{\mathbb{R}}, v)$$

[5]This combinatorial notion of covering differs from the homotopical covering graph of [KM25], in that $\widetilde{G}$ is always a tree.

acts freely on $\widetilde{G}$ through deck transformations. The following theorem is proven in [BGJW21b].

Theorem 4.1 ([BGJW21b])**.** *If G is triangle-free and square-free, there is an isomorphism*

$$H_*(G) \cong H_*(G_{\mathbb{R}})$$

between the discrete cubical homology of G and the singular homology of $G_{\mathbb{R}}$. In particular,

$$H_{\geq 2}(G) = 0.$$

Proof. We present a different proof which shares the philosophy of [BGJW21b] in its use of the covering graph $\widetilde{G}$. Without loss of generality, assume G is path connected. Let

$$\sigma : I^n \to G$$

denote a singular n-cube. By [BGJW21b, Lem. 4.3(3-4)], since G is triangle-free and square-free, a lift

$$\widetilde{\sigma} : I^n \to \widetilde{G}$$

is uniquely determined by a lift of its base vertex. Thus the group π acts freely and transitively on the collection of lifts, and there is an isomorphism

$$C_*(G) \cong C_*(\widetilde{G}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}. \tag{4.2}$$

By our remarks above, $C_*(\widetilde{G})$ is free as a $\mathbb{Z}[\pi]$-module. Since $\widetilde{G}$ is a tree, it is acyclic, and we therefore have

$$H_*(C_*(\widetilde{G})) = H_*(\widetilde{G}) \cong \begin{cases} \mathbb{Z}, & * = 0, \\ 0, & * > 0. \end{cases}$$

This is because $\widetilde{G}$ is a union of finite trees $\widetilde{G}_i$, each of the $\widetilde{G}_i$ are discretely contractible, and

$$H_*(\widetilde{G}) \cong \varinjlim_i H_*(\widetilde{G}_i).$$

It follows that $C_*(\widetilde{G})$ is a free $\mathbb{Z}[\pi]$-resolution of $\mathbb{Z}$, and hence by (4.2)

$$H_*(G) \cong H_*(\pi; \mathbb{Z}) \cong H_*(G_{\mathbb{R}}).$$

□

In Theorem 4.1, the graph G is required to be triangle- and square-free because this is the condition required to guarantee the existence of lifts of all singular n-cubes. Topologically, the obstruction to lifting a singular n-cube

$$\sigma : I^n \to G$$

is the non-triviality of the image of the induced map

$$\sigma_* : \pi_1(I^n_{\mathbb{R}}) \to \pi_1(G_{\mathbb{R}}).$$

Lemma 4.3. *The fundamental group* $\pi_1(I^n_{\mathbb{R}})$ *is generated by the images*

$$\pi_1(I^2_{\mathbb{R}}) \to \pi_1(I^n_{\mathbb{R}})$$

arising from the 2*-dimensional faces*

$$f : I^2 \hookrightarrow I^n$$

of I^n*.*

Proof. To alleviate any ambiguities arising from basepoints, choose a maximal tree $T \subseteq I^n$, and let $I^n_{\mathbb{R}}/T_{\mathbb{R}}$ be the quotient with basepoint $*$. Since $I^n_{\mathbb{R}} \to I^n_{\mathbb{R}}/T_{\mathbb{R}}$ is a homotopy equivalence, it suffices to show that $\pi_1(I^n_{\mathbb{R}}/T_{\mathbb{R}})$ is generated by the images of the composites

$$\pi_1(I^2_{\mathbb{R}}, 0) \xrightarrow{f_*} \pi_1(I^n_{\mathbb{R}}, f(0)) \to \pi_1(I^n_{\mathbb{R}}/T_{\mathbb{R}}, *).$$

Let $\square^n$ be the solid topological n-cube (contractible). The Van Kampen theorem implies that since $I^n_{\mathbb{R}}$ is the 1-skeleton of $\square^n$, and the 2-cells of $\square^n$ are its 2-dimensional faces, we have

$$0 = \pi_1(\square^n/T_{\mathbb{R}}, *) = \pi_1(I^n_{\mathbb{R}}/T_{\mathbb{R}}, *)/\langle \mathrm{Im}\, f_* \ : \ f \text{ a 2-dimensional face}\rangle.$$

The result follows. □

We deduce the following (compare with [BGJW21b, Lem. 4.3(3-4)]).

Theorem 4.4. *For a graph* G*, let*

$$\sigma \colon I^n \to G$$

be a singular n*-cube. Suppose that for every* 2*-dimensional face* τ *of* σ*,* $\mathrm{Im}(\tau)$ *is not a* 3*- or* 4*-cycle. Then the lifting problem shown below as the solid diagram has a unique solution:*

$$\begin{array}{ccc} I^0 & \xrightarrow{\widetilde{q}} & \widetilde{G} \\ {\scriptstyle 0}\downarrow & \overset{\exists!\ \widetilde{\sigma}}{\nearrow} & \downarrow \\ I^n & \xrightarrow{\sigma} & G. \end{array}$$

Proof. The hypothesis implies that for any 2-dimensional face

$$I^2 \hookrightarrow I^n$$

the induced composite

$$\pi_1(I^2_{\mathbb{R}}) \to \pi_1(I^n_{\mathbb{R}}) \to \pi_1(G_{\mathbb{R}})$$

is trivial. By Lemma 4.3, it follows that

$$\sigma_* : \pi_1(I^n_{\mathbb{R}}) \to \pi_1(G_{\mathbb{R}})$$

is trivial. Thus covering space theory implies

$$\sigma_{\mathbb{R}} : I^n_{\mathbb{R}} \to G_{\mathbb{R}}$$

lifts uniquely to a map

$$\widetilde{\sigma}_{\mathbb{R}} : I^n_{\mathbb{R}} \to \widetilde{G}_{\mathbb{R}}$$

with $\widetilde{\sigma}_{\mathbb{R}}(0) = \widetilde{q}$. Because $\widetilde{\sigma}_{\mathbb{R}}$ is a lift of the topological graph map $\sigma_{\mathbb{R}}$, and because the topological graph structure on $\widetilde{G}_{\mathbb{R}}$ is lifted from $G_{\mathbb{R}}$, we deduce that there exists a unique graph map

$$\widetilde{\sigma} : I^n \to \widetilde{G}$$

which underlies $\widetilde{\sigma}_{\mathbb{R}}$. □

Corollary 4.5. *Suppose that G is triangle-free. Then lifts of a degree-1 singular n-cube*

$$\sigma : I^n \to G$$

to $\widetilde{G}$ are determined by lifts of $\sigma(0)$.

We deduce the following generalization of Theorem 4.1[6]

Proposition 4.6. *If G is triangle-free, then there is an isomorphism*

$$H_*(C^1_*(G)) \cong H_*(G_{\mathbb{R}}).$$

Proof. Since

$$C^1_*\left(\coprod_i G_i\right) \cong \bigoplus_i C^1_*(G_i)$$

we may assume without loss of generality that G is path connected. Let

$$\pi := \pi_1(G_{\mathbb{R}}).$$

By Corollary 4.5, $C^1_*(\widetilde{G})$ is a chain complex of free $\mathbb{Z}[\pi]$-modules, and we have

$$C^1_*(G) \cong C^1_*(\widetilde{G}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}.$$

Furthermore, since $\widetilde{G}$ is a tree, we have

$$C^1_*(\widetilde{G}) = C_*(\widetilde{G})$$

and, like in the proof of Theorem 4.1, we deduce

$$H_*(C^1_*(\widetilde{G})) = H_*(\widetilde{G}) = \begin{cases} \mathbb{Z}, & * = 0, \\ 0, & * > 0. \end{cases}$$

It follows that $C^1_*(\widetilde{G})$ is a free $\mathbb{Z}[\pi]$-resolution of $\mathbb{Z}$, and we have

$$H_*(C^1_*(G)) \cong H_*(\pi; \mathbb{Z}) \cong H_*(G_{\mathbb{R}}).$$

□

The main result of this section is the following:

Theorem 4.7. *Let G be a triangle-free graph. Then we have*

$$H^{[1]}_{\geq 2}(G) = 0.$$

[6]Note that if G is square-free, then $C^1_*(G) = C_*(G)$.

Proof. Consider the long exact sequence

$$\cdots H_n^{[0]}(G) \to H_n(G_{\mathbb{R}}) \to H_n^{[1]}(G) \to H_{n-1}^{[0]}(G) \cdots$$

associated to the short exact sequence

$$0 \to C_*^0(G) \to C_*^1(G) \to C_*^{[1]}(G) \to 0$$

and Proposition 4.6. The result follows from the fact that $H_{\geq 2}(G_{\mathbb{R}}) = 0$ and $H_{\geq 1}^{[0]}(G) = 0$. □

The degree spectral sequence immediately gives the following corollary.

Corollary 4.8. *For a triangle-free graph G, we have*

$$H_2(G) \cong \frac{\ker(H_2^{[2]}(G) \to H_1^{[1]}(G))}{\operatorname{Im}(H_3^{[3]}(G) \to H_2^{[2]}(G))}.$$

Furthermore, for a triangle-free graph G of cubical dimension 2, $H_{\geq 3}(G) \cong H_{\geq 3}^{[2]}(G)$.

5. $H_n^{[n]}$ of n-quasimonophobic graphs

In this section, we show that for a certain class of graphs, which we call *n-quasimonophobic graphs*, the homology group $H_n^{[n]}(G)$ admits a particularly simple description. We begin by introducing the notion of n-quasimonophobic graphs.

The concept of *monophobic cube neighborhoods* was introduced by Babson, Greene, and Jamil in [GJ], a sufficient condition for nontriviality of certain elements in the discrete homotopy groups and in the discrete cubical homology groups of a graph G is established. In the present section, we give the definition of n-quasimonophobic graphs and develop several results building on those in [GJ], to facilitate the computation of $H_n^{[n]}$ for n-quasimonophobic graphs.

A *rigid n-cube* in a graph is an induced subgraph isomorphic to I^n, the standard discrete n-cube. A singular n-cube σ is said to be *supported by* a rigid n-cube if the image of σ is equal to that cube.

A rigid n-cube Q is *monophobic* in a graph G if, whenever a singular $(n+1)$-cube contains a face supported by Q, it also contains at least one additional face supported by Q. This relates to the framework of [GJ] in the sense that a rigid n-cube $Q \subseteq G$ is monophobic if and only if it posesses a monophobic neighborhood.

We shall say that a rigid n-cube Q is *quasimonophobic* in a graph G if, whenever a noninjective singular $(n+1)$-cube contains a face supported by Q, it also contains at least one additional face supported by Q.

Definition 5.1 (n-monophobic graph)**.** *Define a graph G to be* n-monophobic *if every n-cube Q in G is rigid and monophobic.*

Definition 5.2 (n-quasimonophobic graph)**.** *Define a graph G to be* n-quasimonophobic *if every n-cube Q in G is rigid and quasimonophobic. We shall say it is* quasimonophobic *if it is n-quasimonophobic for every n.*

Note the following:

- If a graph is n-monophobic, then it is n-quasimonophobic.
- Every graph is 0-quasimonophobic, but the only graphs which are 0-monophobic are those without edges.
- A graph is 1-monophobic if and only if it is triangle-free and square-free, and it is 1-quasimonophobic if and only if it is triangle-free.
- If G has cubical dimension d, it is tautologically $(> d)$-monophobic (and therefore $(> d)$-quasimonophobic).

In this section, we focus on the degree-n quotient complex $C_*^{[n]}(G)$ associated with an n-quasimonophobic graph G and are primarily concerned with its homology in dimension n:

$$H_n^{[n]}(G) = H_n(C_*^{[n]}(G)).$$

All boundary computations pertaining to the boundary operator

$$\partial_{n+1}^{[n]} : C_{n+1}^{[n]}(G) \to C_n^{[n]}(G)$$

are understood to be taken modulo singular cubes of degree less than n, and only injective singular n-cubes can contribute nontrivially to $H_n^{[n]}(G)$.

Now, we proceed to show that $H_n^{[n]}$ of n-quasimonophobic graphs is given by $C_n^{[n]}(G)$ modulo the equivalence relation defined in the following theorem.

Theorem 5.3. *Let Q be an n-cube in any graph G, and let σ and γ be singular n-cubes in G with $\mathrm{Im}(\sigma) = \mathrm{Im}(\gamma) = Q$. Then σ is homologous to γ up to a sign in $C_n^{[n]}$.*

We will prove Theorem 5.3 through a sequence of lemmas. These lemmas were originally proven in [Jam23] in the context of graphs which are digital images; we include the proofs here for completeness.

Note that σ and γ are related by a finite sequence of compositions of cube automorphisms $T_i,\ T_{i,j} \colon I^n \to I^n,\quad i, j \in \{1, \dots, n\},\ i < j$, where T_i reflects the i-th coordinate via $t_i \mapsto 1 - t_i$, and $T_{i,j}$ permutes the i-th and j-th coordinates. To prove the desired result, we show that σ, $\sigma \circ T_i$, and $\sigma \circ T_{i,j}$ are homologous to each other, up to a sign, where the homology consists solely of singular $(n+1)$-cubes of degree n.

Lemma 5.4 ([Jam23])**.** *For a singular n-cube σ in $C_n^{[n]}(G)$, the chain $\sigma + \sigma \circ T_i$ lies in $\partial_{n+1}\big(C_{n+1}^{[n]}(G)\big)$.*

Proof. We will show that $\sigma + \sigma \circ T_i = (-1)^i \partial\rho$ in $C_n^{[n]}(G)$, where $\rho \in C_{n+1}^{[n]}(G)$ is the singular $(n+1)$-cube given by:

$$\rho(t_1, \dots, t_{n+1}) = \begin{cases} \sigma(t_1, \dots, t_{i-1}, t_{i+1}, t_{i+2}, \dots, t_{n+1}) & \text{if } t_i = 0 \\ \sigma(t_1, \dots, t_{i-1}, 0, t_{i+2}, \dots, t_{n+1}) & \text{if } t_i = 1. \end{cases}$$

We must compute the various faces of ρ.

First we compute $f_i^- \rho$ and $f_i^+ \rho$:

$$f_i^- \rho(t_1, \ldots, t_n) = \rho(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) = \sigma(t_1, \ldots, t_{i-1}, t_i, \ldots, t_n)$$
$$f_i^+ \rho(t_1, \ldots, t_n) = \rho(t_1, \ldots, t_{i-1}, 1, t_i, \ldots, t_n) = \sigma(t_1, \ldots, t_{i-1}, 0, \ldots, t_n)$$

Here we see that $f_i^- \rho = \sigma$, and $f_i^+ \rho$ is degenerate since it does not depend on coordinate t_i.

Next we compute $f_{i+1}^- \rho$ and $f_{i+1}^+ \rho$:

$$\begin{aligned} f_{i+1}^- \rho(t_1, \ldots, t_n) &= \rho(t_1, \ldots, t_i, 0, t_{i+1}, \ldots, t_n) = \sigma(t_1, \ldots, t_{i-1}, 0, t_{i+2}, \ldots, t_{n+1}) \\ f_{i+1}^+ \rho(t_1, \ldots, t_n) &= \rho(t_1, \ldots, t_i, 1, t_{i+1}, \ldots, t_n) \\ &= \begin{cases} \sigma(t_1, \ldots, t_{i-1}, 1, t_{i+1}, t_{i+2}, \ldots, t_n) & \text{if } t_i = 0 \\ \sigma(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) & \text{if } t_i = 1 \end{cases} \\ &= \sigma \circ T_i(t_1, \ldots, t_n) \end{aligned}$$

Thus $f_{i+1}^- \rho$ is degenerate, and $f_{i+1}^+ \rho = \sigma \circ T_i$.

Now we compute $f_j^- \rho$ and $f_j^+ \rho$ for $j \notin \{i, i+1\}$. For simplicity in the notation below, we will assume that $j < i$:

$$\begin{aligned} f_j^- \rho(t_1, \ldots, t_n) &= \rho(t_1, \ldots, 0, t_j, \ldots, t_n) \\ &= \begin{cases} \sigma(t_1, \ldots, 0, t_j, \ldots, t_{i-1}, t_{i+1}, t_{i+2}, \ldots, t_n) & \text{if } t_i = 0 \\ \sigma(t_1, \ldots, 0, t_j, \ldots, t_{i-1}, 0, t_{i+2}, \ldots, t_n) & \text{if } t_i = 1 \end{cases} \end{aligned}$$

From the two cases above, we see that $f_j^- \rho$ can take on at most $2^{n-1} + 2^{n-2} < 2^n$ values, so $f_j^- \rho$ is not injective, and thus $f_j^- \rho = 0 \in C_n^{[n]}(G)$. A similar computation shows that $f_j^+ \rho = 0 \in C_n^{[n]}(G)$.

Summing the faces of ρ in $C_n^{[n]}(G)$ gives only the two nonzero terms $f_i^- \rho = \sigma$ and $f_{i+1}^+ \rho = \sigma \circ T_i$, and we have

$$\partial \rho = \sum_j (-1)^j (f_j^- \rho - f_j^+ \rho) = (-1)^i \sigma - (-1)^{i+1} \sigma \circ T_i = (-1)^i (\sigma + \sigma \circ T_i)$$

as desired. □

The following two lemmas show that the chain $\sigma + \sigma \circ T_{i,j}$ is a boundary in $C_n^{[n]}(G)$. To this end, we first define the cube automorphism $R_{i,j} = T_{i,j} \circ T_j$, which acts on the cube I^n as a quarter-turn rotation in the anticlockwise direction. Then Lemma 5.5 shows that $\sigma - \sigma \circ R_{i,j}$ is a boundary in $C_n^{[n]}(G)$; subsequently, Lemma 5.6 establishes that $\sigma + \sigma \circ T_{i,j}$ is likewise a boundary in $C_n^{[n]}(G)$.

Lemma 5.5 ([Jam23])**.** *For a singular n-cube σ in $C_n^{[n]}(G)$, the chain $\sigma - \sigma \circ R_{i,j}$ lies in $\partial_{n+1}\big(C_{n+1}^{[n]}(G)\big)$.*

Proof. We show that $\sigma - \sigma \circ R_{i,j} = (-1)^{n+1} \partial \rho$ in $C_n^{[n]}(G)$, where $\rho \in C_{n+1}^{[n]}(G)$ is the singular $(n+1)$-cube whose front and back $(n+1)$-faces are σ and $\sigma \circ R_{i,j}$, respectively. That is,

$$\rho(t_1,\dots,t_{n+1}) = \begin{cases} \sigma(t_1,\dots,t_n), & \text{if } t_{n+1}=0, \\ \sigma\circ R_{i,j}(t_1,\dots,t_n), & \text{if } t_{n+1}=1. \end{cases}$$

Thus, it suffices to show that all remaining faces of ρ are trivial in $C_n^{[n]}(G)$. Observe that for $1 \le k \le n$, $f_k^- \rho$ has $f_k^- \sigma$ and $f_k^-(\sigma \circ R_{i,j})$ as a pair of opposite n-faces. The images of these two singular cubes have a nonempty intersection, because, $f_k^- \sigma(I^{n-1})$ is a face of the elementary cube $\sigma(I^n)$, while $f_k^-(\sigma\circ R_{i,j})(I^{n-1})$ is the same face of a quarter-turn rotation of that cube. Consequently, $f_k^- \rho$ is not injective and is therefore trivial in $C_n^{[n]}(G)$. The argument for $f_k^+ \rho$ is analogous. □

Lemma 5.6 ([Jam23])**.** *For a singular n-cube σ in $C_n^{[n]}(G)$, the chain $\sigma + \sigma\circ T_{i,j}$ lies in $\partial_{n+1}\big(C_{n+1}^{[n]}(G)\big)$.*

Proof. Since $T_{i,j} = R_{i,j}\circ T_j$, we have:

$$\sigma + \sigma\circ T_{i,j} = (\sigma - \sigma\circ R_{i,j}) + (\sigma\circ R_{i,j} + (\sigma\circ R_{i,j})\circ T_j),$$

and the above is a chain in $\partial_{n+1}\big(C_{n+1}^{[n]}(G)\big)$ by Lemmas 5.4 and 5.5. □

These lemmas conclude the proof of Theorem 5.3.

We are now ready to prove one of the main result of this section, which provides a convenient description of $H_n^{[n]}(G)$ for n-quasimonophobic graphs.

Theorem 5.7. *For an n-quasimonophobic graph G, we have*

$$H_n^{[n]}(G) = C_n^{[n]}(G)/\sim \;\cong C_n^{\text{cell}}(CW^{\square}(G)),$$

where for injective singular n-cubes σ and γ with $\operatorname{Im}(\sigma)=\operatorname{Im}(\gamma)$, we declare

$$\sigma \sim (-1)^{[\sigma,\gamma]}\,\gamma.$$

Here $[\sigma,\gamma]$ denotes the minimal number of factors of the form T_i and $T_{i,j}$ appearing in any composition that expresses σ in terms of γ.

The proof of Theorem 5.7 is somewhat involved. We pause to explain the strategy of the argument. For any graph G, $C_{n-1}^{[n]}(G)$ is trivial, therefore, the the n-th homology group $H_n^{[n]}(G)$ of the quotient complex $C^{[n]}(G)$, is given by

$$H_n^{[n]}(G) \;=\; \frac{C_n^{[n]}(G)}{\partial_{n+1}^{[n]}\Big(C_{n+1}^{[n]}(G)\Big)} \;=\; \frac{C_n^{[n]}(G)/\sim}{\partial_{n+1}^{[n]}\Big(C_{n+1}^{[n]}(G)\Big)/\sim}.$$

To obtain the desired result, we will use monophobicity arguments, following [GJ], to show that for an n-quasimonophobic graph G, the group

$$\partial_{n+1}^{[n]}\Big(C_{n+1}^{[n]}(G)\Big)/\sim$$

is trivial.

Let σ be a singular $(n+1)$-cube in G with degree equal to n. To obtain the desired result, we analyze the boundary $\partial_{n+1}^{[n]}(\sigma)$ in the quotient complex $C_n^{[n]}(G)$. First note that σ is a noninjective singular cube with at least one face injective, that is, σ contains at least one face supported by a rigid n-cube in G. Since G is n-quasimonophobic, for every face of σ supported by an n-cube, there is at least one more face of σ, supported by the same n-cube. We show that, except possibly for four faces of σ, all the remaining faces are noninjective and therefore vanish in $C_n^{[n]}(G)$ (Lemma 5.12). We also prove that the injective faces of σ occur in pairs such that each face in a pair is related to the other by a sequence of compositions of operators of the form T_i or $T_{i,j}$ (Lemma 5.11). Finally, we show that the relations among the injective faces of σ imply that the boundary $\partial_{n+1}^{[n]}(\sigma)$ is trivial modulo the equivalence relation $\sim$. This completes the proof.

Before proving Lemmas 5.11 and 5.12, we establish two preliminary results. The following result appears in [JSA22, Lemma 5.2], and a similar result is also proved in [GJ].

Lemma 5.8. *Let σ and γ be injective singular n-cubes with images equal to the same rigid n-cube. Suppose that for all $t \in I^n$, $\sigma(t)$ is either equal or adjacent to $\gamma(t)$. Then σ and γ are either equal to each other, or related by a flip or a rotation operator.*

Lemma 5.9. *Let σ be a noninjective nondegenerate singular $(n+1)$-cube with a face supported by an n-cube Q in an n-quasimonophobic graph G. Then exactly one of the following holds for σ:*

- [Case-1] *A pair of adjacent faces $f_i^a\sigma$ and $f_j^b\sigma$, with $i < j$ and $a, b \in \{-, +\}$, are supported by Q such that $f_i^{-a} f_j^b \sigma = f_i^a f_j^{-b} \sigma$.*
- [Case-2] *A pair of opposite faces of σ are supported by Q, with one face being a rotation $T_i T_{i,j}$ of the other.*

Proof. Since G is n-quasimonophobic and σ is noninjective with a face supported by Q, there exists at least one additional face of σ supported by Q. Any two such faces are either adjacent or opposite to each other. We first show that if a pair of adjacent faces are supported by Q, then Case-1 holds. We then consider the case in which a pair of opposite faces is supported by Q and show that this yields either Case-1 or Case-2.

Adjacent case. Suppose that a pair of adjacent faces $f_i^a\sigma$ and $f_j^b\sigma$, with $i < j$, are supported by Q. Since both $f_i^a\sigma$ and $f_j^b\sigma$ are supported by the rigid cube Q, the image of their common face

$$f_i^a f_j^b \sigma = f_{j-1}^b f_i^a \sigma$$

is a face of Q.

Because Q is rigid, an $(n-1)$-dimensional face of Q determines its opposite $(n-1)$-dimensional face uniquely. Therefore, the faces of $f_i^a\sigma$ and $f_j^b\sigma$ opposite to their common $(n-1)$-dimensional face must coincide, *i.e.*, using the face relation (2.1),

$$f_i^{-a} f_j^b \sigma = f_{j-1}^{-b} f_i^a \sigma = f_i^a f_j^{-b} \sigma.$$

Opposite case. Now suppose that a pair of opposite faces $f_i^a\sigma$ and $f_i^{-a}\sigma$ are supported by Q. These faces are injective singular n-cubes satisfying the hypothesis of Lemma 5.8. Since σ is nondegenerate, $f_i^a\sigma$ and $f_i^{-a}\sigma$ are either related by a rotation or by a flip.

If they are related by a rotation, then Case-2 holds.

Suppose, the faces $f_i^a\sigma$ and $f_i^{-a}\sigma$ are related by a flip T_k where $k \in \{1, \ldots, n\}$ is an index i.e.

$$f_i^{-a}\sigma = f_i^a\sigma \circ T_k.$$

A direct computation shows that

$$\begin{array}{rcll} f_i^{-a} f_{k+1}^b\sigma & = & f_i^a f_{k+1}^{-b}\sigma & \text{if } i \leq k, \\ f_k^b f_i^{-a}\sigma & = & f_k^{-b} f_i^a\sigma & \text{if } i > k. \end{array} \tag{5.10}$$

For example, assuming $i \leq k$: for any $t = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k, t_{k+2} \ldots, t_{n+1}) \in I^{n-1}$, setting $\delta_a = 0$ if $a = -$ and $\delta_a = 1$ if $a = +$, we get

$$\begin{aligned} & f_i^{-a} f_{k+1}^b\sigma(t) \\ & = \sigma(t_1, \ldots, t_{i-1}, 1-\delta_a, t_{i+1}, \ldots, t_k, \delta_b, t_{k+2}, \ldots, t_{n+1}) \\ & = f_i^{-a}\sigma(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k, \delta_b, t_{k+2}, \ldots, t_{n+1}) \\ & = f_i^a\sigma \circ T_k(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k, \delta_b, t_{k+2}, \ldots, t_{n+1}) \\ & = f_i^a\sigma(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_k, 1-\delta_b, t_{k+2}, \ldots, t_{n+1}) \\ & = \sigma(t_1, \ldots, t_{i-1}, \delta_a, t_{i+1}, \ldots, t_k, 1-\delta_b, t_{k+2}, \ldots, t_{n+1}) \\ & = f_i^a f_{k+1}^{-b}\sigma(t). \end{aligned}$$

When $i \leq k$, $f_i^a\sigma$ and $f_{k+1}^b\sigma$ have a common face and (5.10) implies that

$$f_i^{-a} f_{k+1}^b\sigma = f_i^a f_{k+1}^{-b}\sigma,$$

showing that the faces of $f_i^a\sigma$ and $f_{k+1}^b\sigma$ opposite to their common face are equal. This means that $f_i^a\sigma$ and $f_{k+1}^b\sigma$ are both supported by the same cube Q. Thus, Case-1 holds in this situation, with $j = k + 1$.

Similarly, if $i > k$, $f_i^a\sigma$ and $f_k^b\sigma$ are both supported by the same cube Q. Thus, Case-1 holds in this situation, with the role of (i, j) and (a, b) played by (k, i) and (b, a), respectively. □

The preceding lemma shows that in case-2, faces supported by the same quasimonophobic n-cube are related by a composition of the form $T_iT_{i,j}$. A different relation holds in case-1, as described in the next lemma; in that case, the faces are related by a composition of $j - i - 1$ factors.

Lemma 5.11. *If $f_i^a\sigma$ and $f_j^b\sigma$ are a pair of adjacent faces of σ with $i < j$, both supported by the same rigid n-cube. Then the following hold:*

- [Case-1(a)] *If* $a = b$*, then*

$$f_i^a\sigma = f_j^a\sigma \circ T_{i,i+1} \circ T_{i+1,i+2} \circ \cdots \circ T_{j-2,j-1}.$$

- [Case-1(b)] *If* $a = -b$*, then*

$$f_i^a\sigma = f_j^{-a}\sigma \circ T_i \circ T_{i,i+1} \circ T_{i+1,i+2} \circ \cdots \circ T_{j-2,j-1}.$$

Proof. When $a = b$, we first claim that $\sigma = \sigma \circ T_{i,j}$. Indeed, the maps σ and $\sigma \circ T_{i,j}$ agree on any point $s = (s_1, \dots, s_n) \in I^n$ with $s_i = s_j$. They also agree when $s_i = 1 - s_j$, since $f_i^{-a} f_j^a \sigma = f_i^a f_j^{-a}\sigma$ by Lemma 5.9.

Similarly, for the case $a = -b$, we can show that $\sigma \circ T_i = \sigma \circ T_i \circ T_{i,j}$, using the relation $f_i^a f_j^a \sigma = f_i^{-a} f_j^{-a}\sigma$ of Lemma 5.9.

The proofs of the relations between faces are analogous in the cases $a = b$ and $a = -b$. We therefore prove only the relation between $f_i^a\sigma$ and $f_j^{-a}\sigma$ in the case $a = -b$, omitting the proof for the other case.

Let $t = (t_1, \dots, t_{i-1}, t_{i+1} \dots, t_{n+1}) \in I^n$ and set $\delta_a = 0$ if $a = -$ and $\delta_a = 1$ if $a = +$.

$$\begin{aligned}
f_i^a\sigma(t) &= \sigma(t_1, \dots, t_{i-1}, \delta_a, t_{i+1}, \dots, t_{n+1}) \\
&= \sigma \circ T_i(t_1, \dots, t_{i-1}, 1 - \delta_a, t_{i+1}, \dots, t_{n+1}) \\
&= \sigma \circ T_i \circ T_{i,j}(t_1, \dots, t_{i-1}, 1 - \delta_a, t_{i+1}, \dots, t_{n+1}) \\
&= \sigma(t_1, \dots, t_{i-1}, 1 - t_j, t_{i+1}, \dots, t_{j-1}, 1 - \delta_a, t_{j+1}, \dots, t_{n+1}) \\
&= f_j^{-a}\sigma \circ T_i(t_1, \dots, t_{i-1}, t_j, t_{i+1}, \dots, t_{j-1}, t_{j+1}, \dots, t_{n+1}) \\
&= f_j^{-a}\sigma \circ T_i \circ T_{i,i+1} \circ T_{i+1,i+2} \circ \cdots \circ T_{j-2,j-1}(t).
\end{aligned}$$

□

The following lemma shows that all the other faces are noninjective.

Lemma 5.12. *A nondegenerate degree-*n *singular* $(n{+}1)$*-cube in an* n*-quasimonophobic graph has exactly* 2 *or* 4 *injective faces.*

Proof. Let σ be a nondegenerate degree-n singular $(n{+}1)$-cube in an n-quasimonophobic graph. Note that σ is noninjective, and has a face supported by an n-cube Q. Then by Lemma 5.9, exactly one of the two cases holds.

Case-1 of Lemma 5.9: We claim that the faces $f_k^c\sigma$, for all indices $k \notin \{i, j\}$ and $c \in \{-, +\}$ are noninjective singular $(n-1)$-cubes. Note that since we have $f_i^a f_j^{-b}\sigma = f_i^{-a} f_j^b\sigma$, two codimension-2 faces of $f_k^c\sigma$ are equal, i.e.

$$\begin{aligned}
f_i^a f_j^{-b}(f_k^c\sigma) &= f_i^{-a} f_j^b(f_k^c\sigma) && \text{if } i < j < k, \\
f_i^a f_{j-1}^{-b}(f_k^c\sigma) &= f_i^{-a} f_{j-1}^b(f_k^c\sigma) && \text{if } i < k < j, \\
f_{i-1}^a f_{j-1}^{-b}(f_k^c\sigma) &= f_{i-1}^{-a} f_{j-1}^b(f_k^c\sigma) && \text{if } k < i < j.
\end{aligned}$$

Note that the shift in indices in the second and third equations above comes from the face relations. For example, for $i < k < j$, using face relations, we get

$$\begin{aligned} f_i^a f_{j-1}^{-b}(f_k^c\sigma) &= f_i^a f_k^c f_j^{-b}\sigma \\ &= f_{k-1}^c(f_i^a f_j^{-b}\sigma) \\ &= f_{k-1}^c(f_i^{-a} f_j^b\sigma) \\ &= f_i^{-a} f_{j-1}^b(f_k^c\sigma). \end{aligned}$$

Thus if $i < j < k$, the image of $f_k^c\sigma$ agrees on points $s, t \in I^{n-1}$ such that $s_\ell = t_\ell$ for all $\ell \notin \{i, j\}$, and $(s_i, s_j) = (\delta_a, 1-\delta_b)$ while $(t_i, t_j) = (1-\delta_a, \delta_b)$, and similarly, for other cases on the ordering of i, j, k.

The four other faces $f_k^a\sigma$, where $k \in \{i, j\}$ and $a \in \{+, -\}$, are either all injective, or else any noninjective face forces the face related to it by a bijection (by Lemma 5.11) to be noninjective as well.

Case-2 of Lemma 5.9: We will establish that all the faces are noninjective except the two which are supported by the cube. For simplicity, we write the proof only for the case when we have indices $j, m \in \{1, \ldots, n+1\}$, with the ordering $i < j < m$ and the $f_i^a\sigma$ and $f_i^{-a}\sigma$ satisfy:

$$f_i^-\sigma = f_i^+\sigma \circ T_j T_{j,m}. \tag{5.13}$$

The proof for other cases when $f_i^-\sigma$ is related to $f_i^+\sigma$ by $T_{j,m}T_j$ or when we have a different ordering for i, j and m, can be done similarly.

For example, When $f_i^-\sigma$ is related to $f_i^+\sigma$ by $T_{j,m}T_j$, then instead of (5.13) we have:

$$f_i^- f_j^a f_m^b\sigma = f_i^+ f_j^b f_m^{-a}\sigma,$$

When the ordering is different, we just change the ordering in the face maps in codim 3 faces of eq (5.14), correspondingly. For example, if $i < m < j$, and $f_i^-\sigma$ is related to $f_i^+\sigma$ by $T_j T_{j,m}$, we get the following in place of (5.14)

$$f_i^- f_m^b f_j^a\sigma = f_i^+ f_m^a f_j^{-b}\sigma,$$

First note that using (5.13), we get

$$f_i^- f_j^a f_m^b\sigma = f_i^+ f_j^{-b} f_m^a\sigma, \tag{5.14}$$

as shown below: Let $\delta_a = 0$ if $a = -$ and $\delta_a = 1$ if $a = +$. Then

$$\begin{aligned} & f_i^- f_j^a f_m^b\sigma(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{m-1}, t_{m+1}, \ldots, t_{n+1}) \\ = \;& f_i^-\sigma(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j-1}, \delta_a, t_{j+1}, \ldots, t_{m-1}, \delta_b, t_{m+1}, \ldots, t_{n+1}) \\ \overset{(5.13)}{=} \;& f_i^+\sigma \circ T_j T_{j,m}(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j-1}, \delta_a, t_{j+1}, \ldots, t_{m-1}, \delta_b, t_{m+1}, \ldots, t_{n+1}) \\ = \;& f_i^+\sigma(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j-1}, 1-\delta_b, t_{j+1}, \ldots, t_{m-1}, \delta_a, t_{m+1}, \ldots, t_{n+1}) \\ = \;& f_i^+ f_j^{-b} f_m^a\sigma(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{m-1}, t_{m+1}, \ldots, t_{n+1}) \end{aligned}$$

Next, consider the faces $f_j^-\sigma$, $f_j^+\sigma$, $f_m^-\sigma$, $f_m^+\sigma$. We claim that these faces of σ are noninjective singular $(n-1)$-cubes. Equation (5.14) gives the noninjectivity of each of these faces. For $a = b$ in (5.14), we get that two of the (i, j)-faces of $f_m^a \sigma$ are equal:

$$f_i^- f_j^a f_m^a \sigma = f_i^+ f_j^{-a} f_m^a \sigma.$$

Similarly, putting $a = -b$ in (5.14), we get the noninjectivity of $f_j^a \sigma$:

$$f_i^- f_{m-1}^{-a}(f_j^a \sigma) = f_i^- f_j^a f_m^{-a} \sigma = f_i^+ f_j^a f_m^a \sigma = f_i^+ f_{m-1}^a (f_j^a \sigma).$$

Now it remains to show the noninjectivity of $f_k^c \sigma$ for indices $k \notin \{i, j, m\}$ and $c \in \{-, +\}$. Such a face is a noninjective, because from (5.14) four of its codimension-3 faces coincide with four others. For example, if $k > m$, a direct computation using (5.13) gives:

$$f_i^- f_j^a f_m^b f_k^c \sigma = f_i^+ f_j^{-b} f_m^a f_k^c \sigma,$$

and similarly for other cases where $m > k \notin \{i, j\}$. □

Completion of the proof of Theorem 5.7. Now we show that the boundary $\partial_{n+1}^{[n]}(\sigma)$ is trivial modulo the equivalence relation $\sim$ in all the three cases $1(a)$, $1(b)$, and 2 of Lemmas 5.9 and 5.11.

Case $1(a)$. In this case, at least one or both of the two pairs of faces $(f_i^- \sigma, f_j^- \sigma)$ and $(f_i^+ \sigma, f_j^+ \sigma)$ consists of injective faces related to each other by the relation stated in Lemma 5.11. In the boundary computation of $\partial_{n+1}^{[n]}(\sigma)$, all the terms corresponding to injective faces $(f_i^a \sigma, f_j^a \sigma)$, for $a \in \{-, +\}$, cancel each other modulo the equivalence relation $\sim$: Set $\delta_a = 0$ if $a = -$ and $\delta_a = 1$ if $a = +$. Then the term

$$\begin{aligned}(-1)^{i+\delta_a} f_i^a \sigma &= (-1)^{i+\delta_a} f_j^a \sigma \circ T_{i,i+1} \circ T_{i+1,i+2} \circ \cdots \circ T_{j-2,j-1} \\ &\sim (-1)^{i+\delta_a} (-1)^{j-i-1} f_j^a \sigma \\ &= (-1)^{j+\delta_a-1} f_j^a \sigma,\end{aligned}$$

cancels with the term $(-1)^{j+\delta_a} f_j^a \sigma$.

The argument for Case $1(b)$ is entirely similar.

Case 2. In this case, the boundary $\partial_{n+1}^{[n]}(\sigma)$ consists of the injective faces are $f_i^a \sigma$ and $f_i^{-a} \sigma$, and since each is a rotation of the other, we have $f_i^a \sigma \sim f_i^{-a} \sigma$. Consequently, the corresponding terms cancel modulo $\sim$, and the boundary is trivial. □

This completes the proof of Theorem 5.7. We deduce the following.

Theorem 5.15. *Suppose that G is n-quasimonophobic. Then the map*

$$H_n(CW^{\square}(G)) \to H_n^{inj}(G)$$

is an injection. If in addition G is $(n-1)$-quasimonophobic, then this map is an isomorphism.

Proof. For any graph G, the cellular chains of $CW^{\square}(G)$ in dimension n is naturally identified with $C_n^{[n]}(G)/\sim$:

$$C_n^{cell}\big(CW^{\square}(G)\big) = \frac{C_n^{[n]}(G)}{\sim}.$$

Moreover, it follows from Theorem 5.3, Lemma 5.4, and Lemma 5.6 that there is a surjection

$$C_i^{[i]}(G)/\sim \twoheadrightarrow H_i^{[i]}(G).$$

Defining K_i to be the kernel of this map, there is a short exact sequence of chain complexes given by

$$0 \to K_i \to C_i^{\text{cell}}(CW^{\square}(G)) \to H_i^{[i]}(G) \to 0. \tag{5.16}$$

If G is i-quasimonophonic, Theorem 5.7 implies that

$$K_i = 0.$$

The result follows from the long exact sequence of homology groups arising from the short exact sequence (5.16). □

Theorem 5.17. *Let G be a graph that is (≤ 2)-quasimonophobic. Then*

$$H_2(G) = H_2\big(CW^{\square}(G)\big).$$

Proof. Since G is 1- and 2-quasimonophobic Theorem 5.15 implies that

$$H_2^{inj}(G) \cong H_2(CW^{\square}(G)).$$

Moreover, by Theorem 4.7, we have $H_2^{[1]}(G) = 0$ because G is 1-quasimonophobic (hence triangle-free).

Therefore, for $H_2(G)$, the only nontrivial contribution in the degree spectral sequence comes from $H_2^{inj}(G)$, and this agrees with the cellular homology group. Hence

$$H_2(G) = H_2\big(CW^{\square}(G)\big).$$

□

6. Applications and questions

Definition 6.1. *Define the nth Greene sphere G_n^{sph} as the graph obtained from a $2n$-cycle (with vertices $0, \ldots, 2n-1$) by adjoining two vertices u and v, where u is adjacent to all odd vertices and v to all even vertices.*

Example 6.2. *Suppose that $n \geq 4$. Then $CW^{\square}(G_n^{\text{sph}})$ is homoemorphic to S^2 (the graph $G_3^{\text{sph}} = I^3$, and hence is contractible). Moreover, the graphs G_n^{sph} are triangle-free (1-quasimonophobic), have cubical dimension 2, and are 2-monophobic* [GJ]. *It follows that these graphs are quasimonophobic.*

By Theorem 5.17,

$$H_2(G_n^{\text{sph}}) \cong \mathbb{Z}.$$

Moreover, since G_n^{sph} contains no 3-cubes, $H_3^{[3]}(G_n^{\text{sph}})$ and $H_4^{[3]}(G_n^{\text{sph}})$ are both trivial, and by Corollary 4.8,

$$H_3(G_n^{\text{sph}}) = H_3^{[2]}(G_n^{\text{sph}}).$$

Note that the computer computations of [KK24] actually establish that

$$H_3(G_4^{\mathrm{sph}}) \cong H_3(CW^{\square}(G_4^{\mathrm{sph}})) = 0.$$

We also saw that 1-quasimonophobicity eliminated the contributions of $H^{[1]}_{\geq 2}$ from the degree spectral sequence. Motivated by these data points, it is natural to ask the following:

Question 6.3. *Suppose G be a graph that is $(\leq n)$-quasimonophobic. Does this imply that $H_n(G)$ is isomorphic to the homology group $H_n(CW^{\square}(G))$?*

The following example shows that the assumption of 2-quasimonophobicity is necessary in Theorem 5.17.

Example 6.4. *Consider the complete bipartite graph $K_{2,3}$ with bipartition $\{v_1, v_2\}$ and $\{u_1, u_2, u_3\}$. In the sense of discrete homotopy theory, $K_{2,3}$ is contractible, since it deformation retracts onto one of its edges (see* [BKLW01]*). Consequently,*

$$H_2(K_{2,3}) = 0.$$

The cubical CW–complex $CW^{\square}(K_{2,3})$ obtained by filling in the cubes of $K_{2,3}$ is homeomorphic to S^2.

The graph $K_{2,3}$ is not 2-quasimonophobic. Indeed, the non-injective singular 3-cube

$$\sigma\colon I^3 \to K_{2,3}$$

given by

$$\begin{aligned} &\sigma(0,0,0) = v_1, \quad &&\sigma(0,1,0) = u_1, \quad &&\sigma(1,0,0) = u_2, \quad &&\sigma(1,1,0) = v_2, \\ &\sigma(0,0,1) = u_3, \quad &&\sigma(1,1,1) = u_3, \quad &&\sigma(0,1,1) = v_1, \quad &&\sigma(1,0,1) = v_1 \end{aligned}$$

witnesses the nonmonophobicity of the 2-cubes in $K_{2,3}$. Its boundary kills the injective 2-cycle in $C_2^{[2]}(K_{2,3})/\sim$ corresponding to the generator of

$$H_2(CW^{\square}(K_{2,3})) = \mathbb{Z},$$

showing that the map

$$C_2^{[2]}(K_{2,3})/\sim \to H_2^{[2]}(K_{2,3})$$

is not injective.

Thus, Theorem 5.17 fails without the assumption of 2-quasimonophobicity.

Finally we explain how quasimonophobicity allows one to geometrically produce non-trivial classes in discrete cubical homology.

Suppose that

$$X_\bullet : \square_{inj}^{op} \to \mathrm{Sets}$$

is a *regular* semi-cubical set in the sense that its geometric realization $|X|$ is a regular CW complex: each characteristic map

$$\square^n \to |X_\bullet|$$

is injective. Let

$$G(X_\bullet)$$

denote the graph given by its 1-skeleton. Suppose that

$$x = \sum_i a_i \sigma_i \in C_n(X_\bullet) = \mathbb{Z}X_n$$

is an n-cycle in the cubical n-chains of $X_\bullet$.

The class x can be regarded as giving a cycle in the cellular chain complex

$$x \in C_n^{\mathrm{cell}}(|X_\bullet|),$$

giving a class

$$[x] \in H_n(|X_\bullet|).$$

By regularity, there is an inclusion of CW-complexes

$$|X_\bullet| \hookrightarrow CW^{\square}(G(X_\bullet)) \tag{6.5}$$

and we let

$$[x]_{CW} \in H_n(CW^{\square}(G(X_\bullet)))$$

denote the image of $[x]$ under this map. Alternatively, we may regard each $\sigma_i \in X_n$ as a map

$$\sigma_i : \square_\bullet^n \to X_\bullet$$

giving a map of graphs

$$\sigma_i : I^n = G(\square_\bullet^n) \to G(X_\bullet).$$

The resulting class

$$x = \sum_i a_i \sigma_i \in C_n(G(X_\bullet))$$

is a cycle in the discrete cubical chains in $G(X_\bullet)$, yielding a cycle

$$[x]_{\mathrm{graph}} \in H_n(G(X_\bullet)).$$

Under the maps (1.1) and (1.2)

$$H_n(G(X_\bullet)) \to H_n^{inj}(G(X_\bullet)) \leftarrow H_n(CW^{\square}(G(X_\bullet)))$$

the classes $[x]_{\mathrm{graph}}$ and $[x]_{CW}$ map to the same element.

Given a graph G and a graph map

$$\alpha : G(X_\bullet) \to G$$

the image of $[x]_{\mathrm{graph}}$ under the map $\alpha_* : H_n(G(X_\bullet)) \to H_n(G)$ gives "geometric" discrete cubical homology class

$$\alpha_*[x]_{\mathrm{graph}} \in H_n(G).$$

If the map α is an injection, it induces an inclusion of CW complexes

$$CW^{\square}(G(X_\bullet)) \hookrightarrow CW^{\square}(G),$$

and the image of $[x]_{CW}$ under this map gives a homology class

$$\alpha_*[x]_{CW} \in H_n(CW^{\square}(G)).$$

Theorem 6.6. *Suppose that G is n-quasimonophobic, $X_\bullet$, x are as above,*

$$\alpha : G(X_\bullet) \hookrightarrow G$$

is an inclusion of graphs, and homology class

$$\alpha_*[x]_{CW} \in H_n(CW^\square(G))$$

is non-trivial. Then the homology class

$$\alpha_*[x]_{\mathrm{graph}} \in H_n(G)$$

is non-trivial.

Proof. Consider the diagram

$$\begin{array}{ccccc}
H_n(G(X_\bullet)) & \longrightarrow & H_n^{inj}(G(X_\bullet)) & \longleftarrow & H_n(CW^\square(G(X_\bullet))) \\
\downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_*} \\
H_n(G) & \longrightarrow & H_n^{inj}(G) & \underset{(*)}{\hookleftarrow} & H_n(CW^\square(G))
\end{array}$$

induced by (1.1) and (1.2). Note that n-quasimonophobicity implies the map $(*)$ in the above diagram is an injection by Theorem 5.15. The result follows. $\square$

For example, for $n \geq 4$ there are clearly regular semi-cubical sets $S^2(n)_\bullet$ whose realization is S^2, we have

$$G_n^{\mathrm{sph}} = G(S^2(n)_\bullet),$$

and for $n \geq 4$ the cannonical map gives a homeomorphism

$$S^2 = |S^2(n)_\bullet| \xrightarrow{\approx} CW^\square(G_n^{\mathrm{sph}}).$$

The graph G_n^{sph} is 2-quasimonophonic, the fundamental class

$$[S^2] \in H_2(S^2(n)_\bullet)$$

has non-trivial image

$$0 \neq [S^2]_{CW} \in H_2(CW^\square(G_2^{\mathrm{sph}})) \cong \mathbb{Z}$$

and hence gives the (nontrivial) generator of

$$[S^2]_{\mathrm{graph}} \in H_2(G_n^{sph}) \cong \mathbb{Z}.$$

However, in the case of $n = 3$, while $G_3^{\mathrm{sph}} = I^3$ is still 2-quasimonophobic, we have

$$CW^\square(G_3^{\mathrm{sph}}) = \square^3$$

and hence the image of $[S^2]$,

$$[S^2]_{CW} \in H_2(CW^\square(G_3^{\mathrm{sph}})) = 0,$$

is trivial. The element

$$[S^2]_{\mathrm{graph}} \in H_2(G_3^{\mathrm{sph}}) = 0$$

is also trivial, since $G_3^{\mathrm{sph}} = I^3$ is contractible.

Finally, we saw that the graph $K_{2,3}$ is not 2-quasimonophobic. There is a semi-cosimplicial set $\widetilde{S}^2_\bullet$, obtained by gluing three $\square^2_\bullet$'s together, with the property that:

- $|\widetilde{S}^2_\bullet| \approx S^2$,
- $G(\widetilde{S}^2_\bullet) = K_{2,3}$.

Since $CW^\square(K_{2,3}) \approx S^2$, the fundamental class

$$[S^2] \in H_2(\widetilde{S}^2_\bullet)$$

gives a non-trivial class

$$0 \neq [S^2]_{CW} \in H_2(CW^\square(K_{2,3})) = \mathbb{Z}.$$

However, since $K_{2,3}$ is contractible, the class

$$[S^2]_{\text{graph}} \in H_2(K_{2,3}) = 0$$

is trivial. This demonstrates the necessity of the quasimonophobicity condition in Theorem 6.6.

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