THE ADAMS SPECTRAL SEQUENCE FOR tmf

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The Adams-Novikov spectral sequence is the easiest way to compute tmf_*, and that is how most people do the computation. However, sometimes you want the classical Adams spectral sequence. That is what is contained in these notes.

1. $p = 3$

The homology of tmf as a comodule over the Steenrod algebra is given by

$$H_*(tmf) = \mathbb{F}_3[b_4, \xi_1^3 - b_4\xi_1, \xi_2, \xi_3, \ldots] \otimes E[\tau_3, \tau_4, \ldots]$$

where $|b_4| = 8$. Define a Hopf algebroid (B, Γ) by

$$B = E[b_4]$$
$$\Gamma = B[\xi_1] \otimes E[\tau_0, \tau_1, \tau_2]/(\xi_1^3 - b_4\xi_1)$$

This is actually a Hopf algebra, because $\eta_L = \eta_R$. The coproduct formulas are just as in the Steenrod algebra. A change of rings theorem implies

$$\mathrm{Ext}_A(\mathbb{F}_3, H_*(tmf)) = \mathrm{Ext}_\Gamma(B, B).$$

We will compute the polynomial part of the latter with the May spectral sequence. Then we will proceed with a sequence of Bockstein spectral sequences. Let $\Gamma(2)$ denote the subalgebra

$$B[\xi_1]/(\xi_1^3 - b_1\xi_1)$$

The E_1 term of the May spectral sequence for $H^*(\Gamma(2))$ is given by

$$B[b_0] \otimes E[h_0].$$

Here the bifiltrations (s, t) are given by

$$|h_0| = (1, 4)$$
$$|b_0| = (2, 12)$$

There are no differentials, and we have $H^*(\Gamma(2))$. Since we had the relation $\xi^3 = b_4\xi_1$ in Γ, we shall define

$$h_1 := h_0 b_4$$

in $H^*(\Gamma(2))$. Observe that it is the image of the element $h_1 = [\xi_1^3]$ in $H^*(A)$.

Letting $\Gamma(1) = \Gamma(2) \otimes E[\tau_2]$, since τ_2 is primitive modulo (τ_0, τ_1), we have

$$H^*(\Gamma(1)) = H^*(\Gamma(2)) \otimes P[v_2]$$

with $|v_2| = (1, 17)$. For $\Gamma(0)\Gamma(1) \otimes E[\tau_1]$, we have a v_1-BSS

$$H^*(\Gamma(1)) \otimes P[v_1] \Rightarrow H^*(\Gamma(0)).$$

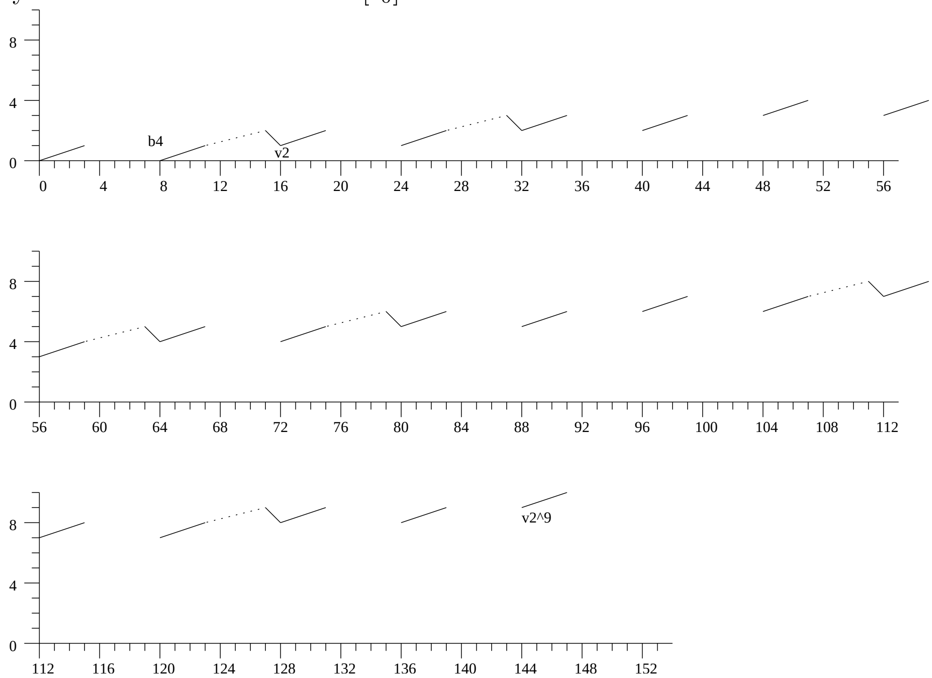
The main differential is

$$d_1(v_2) = v_1 h_1$$

which follows immediately from the coproduct formula for τ_2. It follows that

$$d_1(v_2^i) = \pm h_1 v_1 v_2^{i-1} = \pm h_0 b_4 v_1 v_2^{i-1} \quad \text{for } i \not\equiv 0 \pmod 3.$$

There is no room for any more differentials. The v_1 BSS with differentials is displayed below as a module over $P[b_0]$.



Next we compute the v_0-BSS. The v_0 BSS breaks up into patterns of two types. The first pattern (pattern 1) arises from the elements in $\mathrm{Ext}(tmf \wedge V(0))$ in

$$P[v_0, b_0] \otimes E[h_0]\{v_2^k\}$$

or those in

$$P[v_0, b_0] \otimes E[h_0]\{b_4 v_2^{k-1}\}$$

for a fixed $k \equiv 0 \pmod 3$ The second pattern (pattern 2) arises from the elements of $\mathrm{Ext}(tmf \wedge V(0))$ in

$$P[b_0, v_0] \otimes E[h_0]/(v_0 h_0)\{b_4 v_2^{k-1}\} \oplus P[b_0, v_0]\{v_2^k h_0\}$$

for $k \equiv 1, 2 \pmod 3$.

The basic differential is given by

$$d_1(v_1) = v_0 h_0. \tag{1.1}$$

The fundamental observation is that in the v_0-BSS, v_1^3 is a permanent cycle. This is because after the differential $d_1(b_4 v_1) = v_0 b_4 h_0$, there are no remaining targets for a Bockstein differential supported by v_1^3. Thus, on v_1 multiples, any pattern of differentials is periodic on v_1^3. We shall henceforth refer to v_1^3 as c_6, since that is what it detects in homotopy.

Let us first consider pattern 1, based at $v_2^k b_4^e$ for $e = 0, 1$ and $k \cong -e \pmod p$. There is no target for a Bockstein differential supported by $v_2^k b_4$, so it must be a permanent cycle in the BSS. There are differentials

$$d_1(v_1^i \cdot v_2^k b_4^e) = \pm v_1^{i-1} v_0 h_0 v_2^k b_4^e$$

for $i = 1, 2$ which follow from 1.1 and these are then propagated by multiplication by b_0, and c_4.

Turning our attention to those elements of the form $v_1^i h_0 v_2^k b_4^e$, the only such elements which can support Bockstein differentials are those for which $i \equiv 2 \pmod 3$. This is because for $i \not\equiv 2 \pmod 3$ these elements were targets of Bockstein differentials. There is no room for a d_1. In the E_2 term we can use the Massey product to obtain

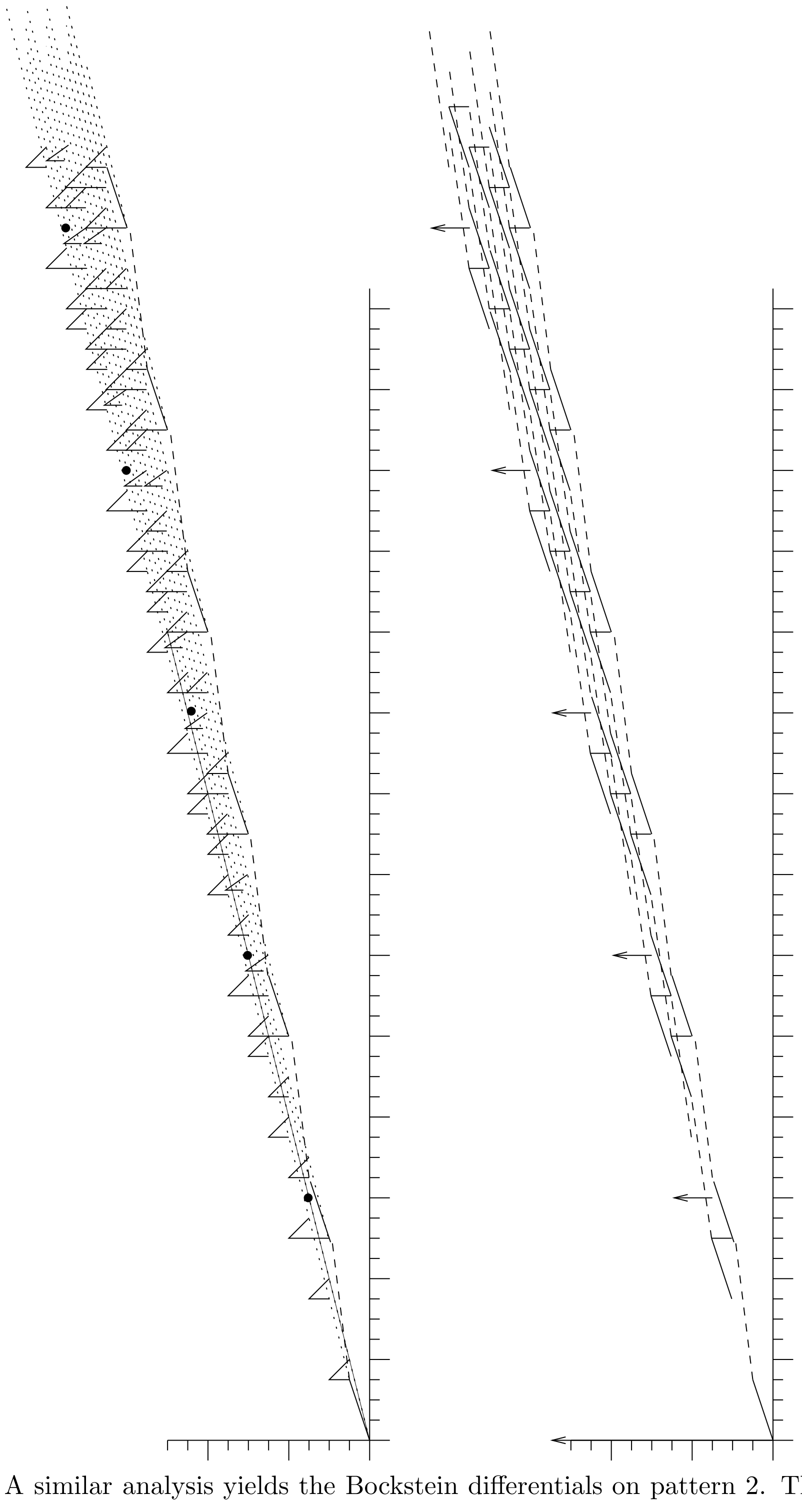
$$\begin{aligned} d_2 v_1^2 h_0 &= \pm d\langle v_1, v_0 h_0, h_0\rangle \\ &= \pm\langle v_0 h_0, v_0 h_0, h_0\rangle \\ &= \pm v_0^2\langle h_0, h_0, h_0\rangle \\ &= \pm v_0^2 b_0. \end{aligned}$$

Therefore

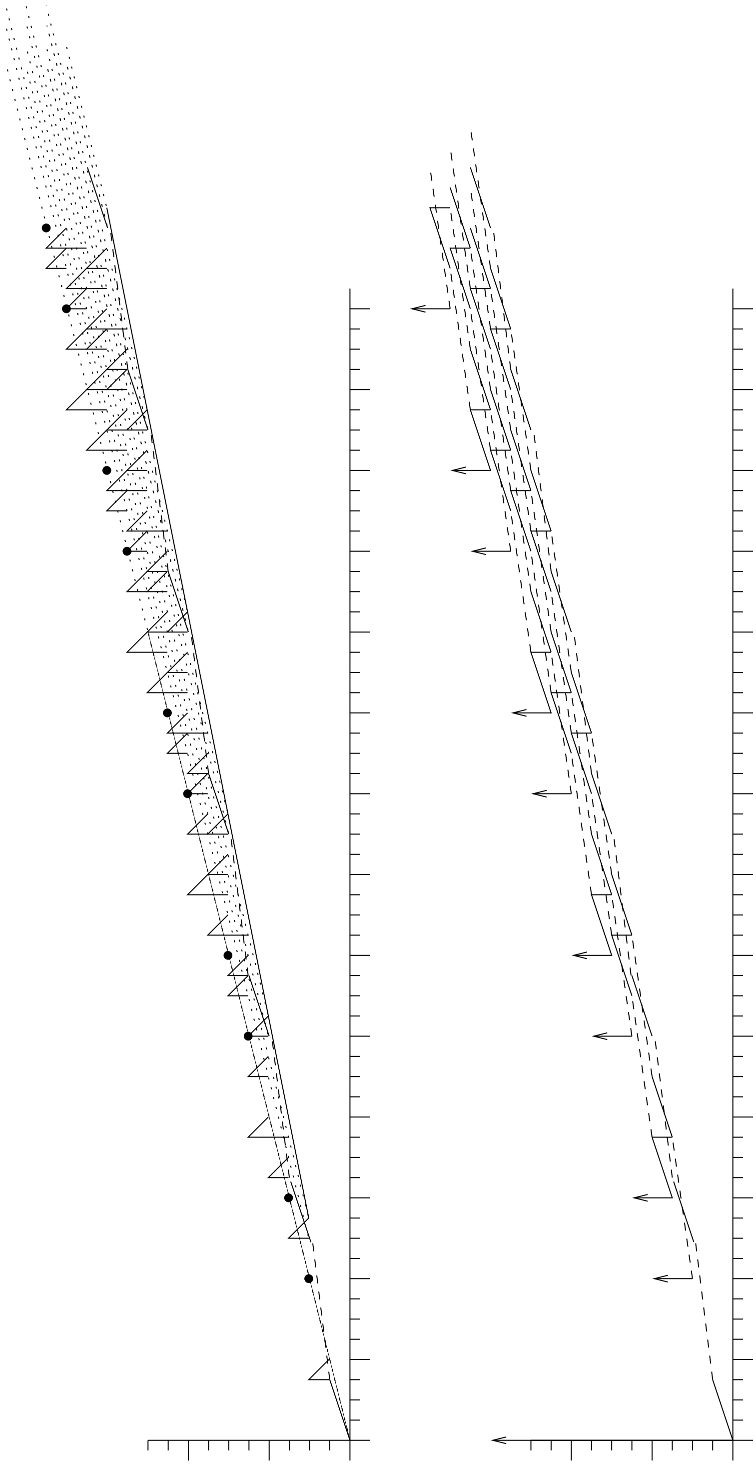
$$d_2 v_1^2 h_0 \cdot v_2^k b_4^e = \pm v_0^2 b_0 \cdot v_2^k b_4^e.$$

This differential is then propagated by c_6 and b_0 multiplication.

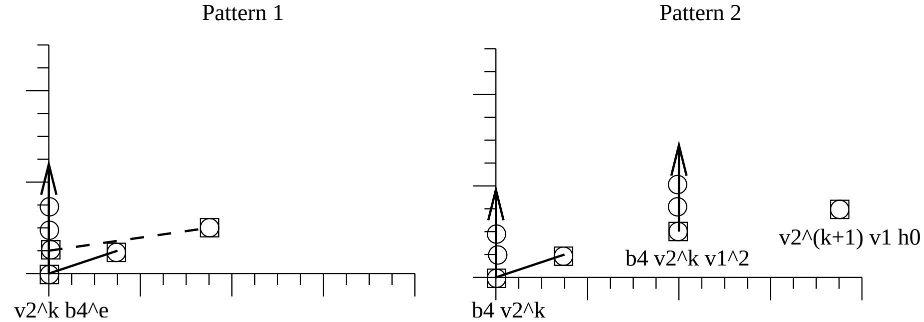
The result of these differentials on pattern 1 is displayed below. The top chart displays the Bockstein differentials, and the bottom chart displays the result after the differentials are taken. In the top chart, multiplication by h_0 is given by solid lines, as are multiplications by v_0 and Bockstein differentials. The solid circles indicate that the element supports an infinite v_0 tower, i.e. it is not the target of a Bockstein differential. The Toda bracket $\langle h_0, h_0, -\rangle$ is indicated with a dashed line. Multiplication by b_0 may be implicitly read off from the composition of h_0 multiplication and $\langle h_0, h_0, -\rangle$. In the first chart, the v_1-towers are indicated with dotted lines.



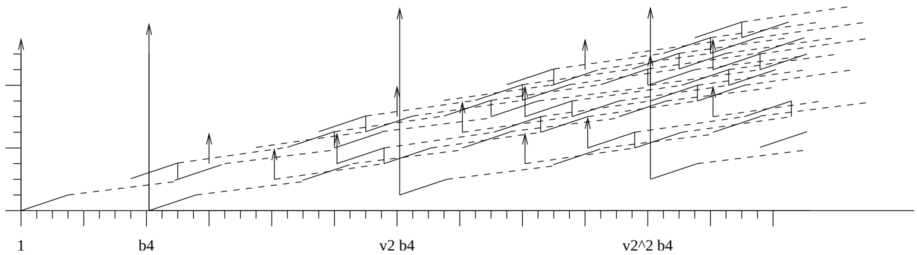
A similar analysis yields the Bockstein differentials on pattern 2. The result is displayed below (here b_0 multiplication is indicated with solid lines).



The results of our analysis can be summarized in the following two charts. This is effectively a summary of the ASS E_2-term. In these charts, circles represent a polynomial algebra $P[c_6]$, squares represent a polynomial algebra $P[b_0]$ and circles and squares represent a $P[c_6, b_0]$.



For instance, the E_2-term begins in the following manner.



We now determine Adams differentials. We will first determine Adams d_2's. This will leave a pattern that is essentially the Adams-Novikov E_2 term. The remaining differentials are then well known.

The primary differential that we need to consider is

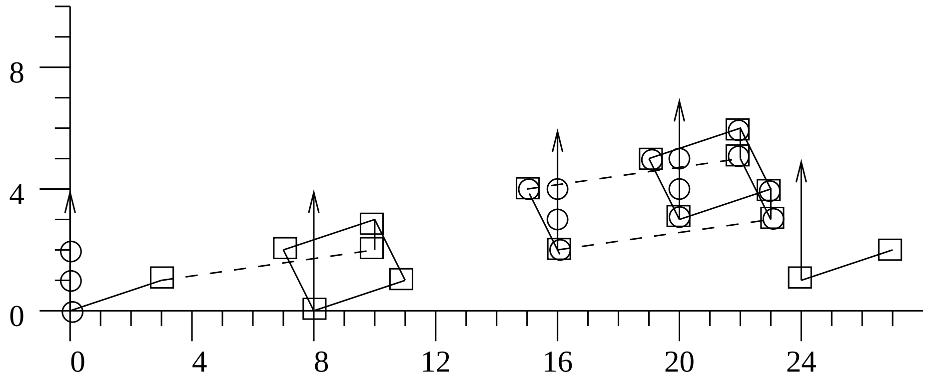
$$d_2 b_4 = v_1 h_0. \tag{1.2}$$

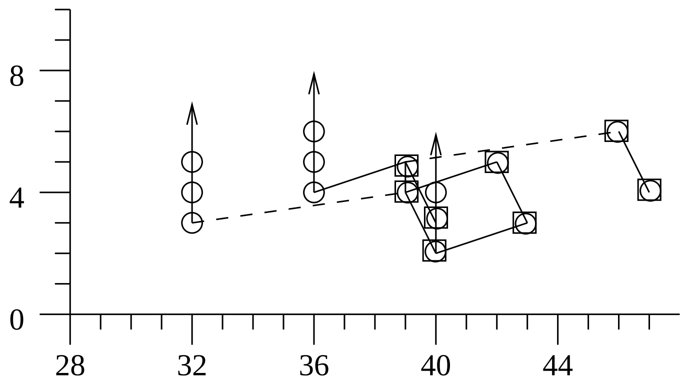
This differential can be deduced from the fact that in the ASS for the sphere

$$d_2 h_1 = v_0 b_0.$$

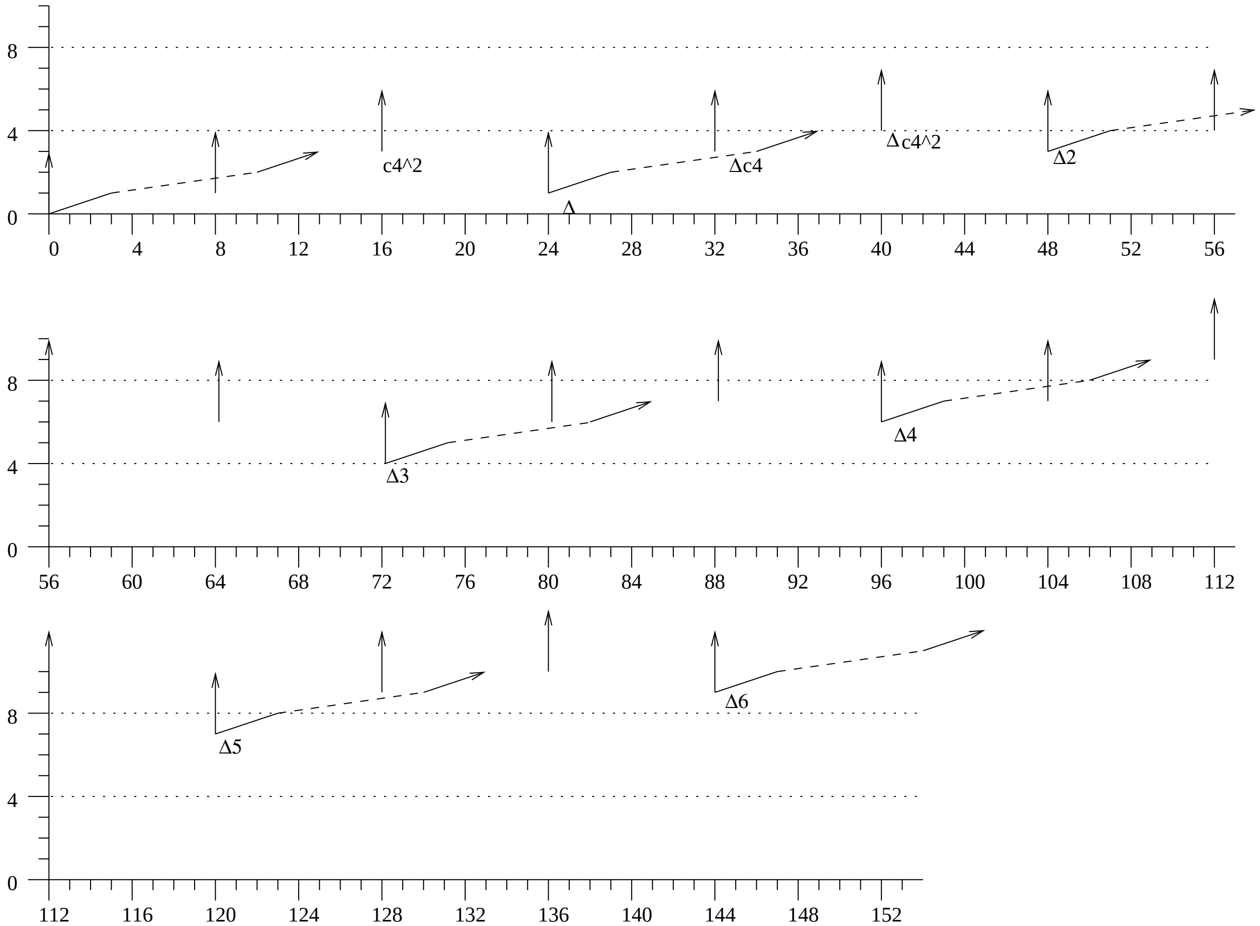
The Hurewicz image of h_1 is $b_4 h_0$, so the differential 1.2 must occur. Alternatively, one can use the formula for $\eta_R(b_4)$ in the elliptic curve Hopf algebroid.

The differential 1.2 propagates to give all of the rest of the d_2's in the ASS. This pattern below is periodic on v_2^3. The notation is the same as before.





What is left is the E_3-term, which is isomorphic to the ANSS. It is displayed below. All of the v_0 towers are periodic on c_6. The labels c_4^i and Δ^j are ANSS names. Most of the multiplicative extensions are not seen in the ASS.


c4^2
Δ
Δc4
Δc4^2
Δ2
Δ3
Δ4
Δ5
Δ6

All of the differentials, modulo the different filtrations, are exactly as those in the ANSS. Namely, they are those differentials generated by

$$d_4(\Delta) = h_0 b_0^2$$
$$d_3(\Delta^2) = h_0 b_0^2 \Delta$$
$$d_6(\Delta^2 h_0) = b_0^5$$
$$d_3(\Delta^4) = \Delta^3 h_0 b_0^2$$
$$d_4(\Delta^5) = \Delta^4 h_0 b_0^2$$
$$d_6(\Delta^5 h_0) = \Delta^3 b_0^5$$

These differentials are propagated by b_0 multiplication. The whole spectral sequence is periodic on the permanent cycle $\Delta^3 = v_2^9$.