Nadler - loop spaces and representations

Note Title 10/17/2009

(jt w/ David Ben-Zvi)

Outline

1.) Motivation from Rep thy

2.) Background on D-modules, DAG

3.) Connections via loop spaces
(D-modules)

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1. Motivation

two cats from Geom rep thy:

G = cx reductive gp

$\cup$

B Borel

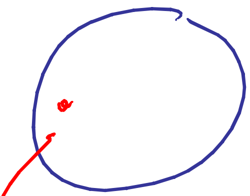
$\cup$

U unipotent radical

G/B flag variety

K/T for a topologist

$U \backslash G/B$ = Schubert stratification e.g. SL_2, $\mathbb{P}^1$, stratification:



point, complement

Want to understand

A) $$D\text{-mod}(U\backslash G/B) = D\text{-mod}^U(G/B)$$

"sheaves which are locally constant on U-orbits"

Beilinson-Bernstein:

$$D\text{-mod}(U\backslash G/B) \simeq \text{"highest weight modules for } \mathfrak{g}\text{"}$$

Only cares about topology of the stratified space

[topology]

B) Steinberg Variety

$$St = \left\{ u \in G \ \text{unipotent}, \ B_1, B_2 \ \text{Borels} \atop u \in B_1 \cap B_2 \right\}$$

[algebra] $QCoh^G(St)$

equivariant quasi-coherent sheaves

note $\mathbb{G}_m$ acts on $\{u\}$

$$QCoh^{G \times \mathbb{G}_m}(St)$$

Kazhdan-Lusztig, Bezrukavnikov

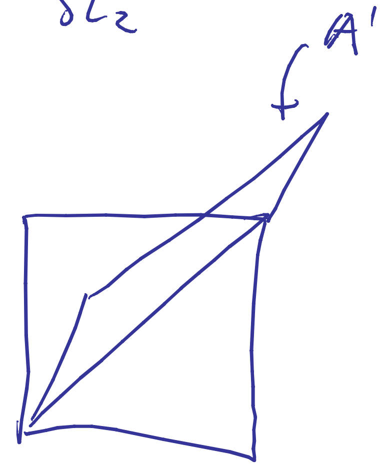
"Affine Hecke algebra"

<u>Point</u>: these structures appear in topology.

explain how related

picture of St for Sl_2

$\mathbb{P}^1 \times \mathbb{P}^1$



flags not equal, no unipotents

G moves Borels, conjugates n

2.) D-modules:

X = sm sch/$\mathbb{C}$, smooth stack/$\mathbb{C}$

e.g. G/B

D_X = ring of diff'l operators

In local coordinates $x_1, \dots, x_n$

$$D_X = \mathbb{C}\langle x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n} \rangle$$
$$[\partial_{x_1}, x_1] = 1$$

Examples: of D-modules

(1.) D_X

(2.) $\mathcal{O}_X$ (left)

Ω_X (right) are D-modules

(3.) Delta functions along subvarieties

Where did D come from?

Ways to think about D-modules:

Def: A flat connection on an $\mathcal{O}$-module is a formalization of M along infinitesimal paths

de Rham stack = De Rham space

$$X_{dR} \longleftarrow X \leftleftarrows \text{formal nbhd of } \Delta \subset X \times X$$

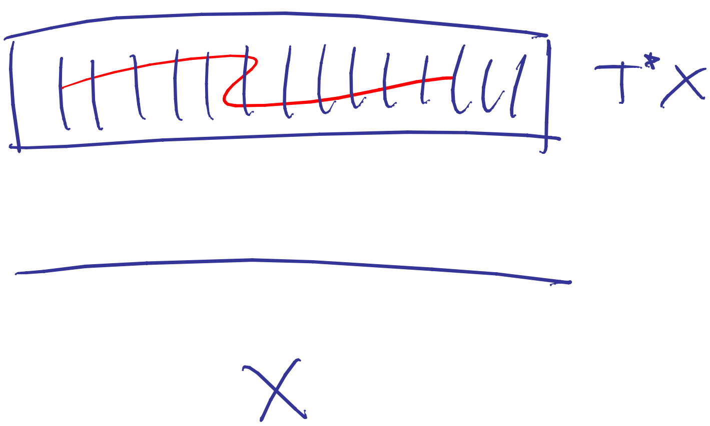
D_X is the algebra governing descent from X to X_{dR}

(dual to a coalgebra which governs descent)

2. Quantization:

D_X is filtered by degree of differential operator

$$\mathrm{gr}\, D_X = \mathcal{O}_{T^*X}$$



"classical support"

"D-modules are non-commutative modules on T^*X"

Must have "coisotropic support"

3. Topological picture (Riemann-Hilbert)

M is a D-module

$\rightsquigarrow$ $\mathcal{F}$ = flat sections of M

D-modules $\longrightarrow$ Constructible sheaves

holonomic D-modules (Lagrangian support) (regular) $\xrightarrow[\sim]{\text{Derived equivalence}}$ Constructible sheaves

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Derived alg geometry

Stacks $\xrightarrow[\text{linearization}]{}$ qcoh sheaves

Usual alg geom: discrete rings + gluing

DAG: generalize to commutative, connective ring spectra

Functors of points

$$\begin{tikzcd}
\text{Rings} \arrow[r, "\text{schemes}"] \arrow[dr, "\text{stacks}"] \arrow[ddr, "\text{higher stacks}"] \arrow[dd, hook] & \text{Sets} \arrow[d, hook] \\
& \text{Groupoids} \arrow[d, hook] \\
E_\infty\text{-rings} \arrow[r, "\text{derived stack}"'] & \text{Spaces}
\end{tikzcd}$$

Examples: 1.) Affine derived schemes

representable

2.) Top'l spaces Σ: "higher stacks"

constant functor

3.) Mapping stack

$$\mathrm{Map}(\Sigma, X)$$

Q coh sheaves:

$$X = \operatorname{Spec} R \longmapsto \mathrm{Mod}_R$$

$\mathrm{QCoh}(X)$ unique extension of this assignment

gives a functor: derived stacks $\xrightarrow[\mathrm{QCoh}(-)]{}$ $(\infty,1)$-cats

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Goal: understand:

(1) $D\text{-mod}^{u}(G/B)$

(2) $\mathrm{QCoh}^{G \times \mathbb{G}_m}(St)$

Key idea: Affinization:

$\Gamma\mathcal{O}$: Schemes $\rightleftarrows$ Rings^{op} : Spec

Def: the affinization of a scheme

$$\mathrm{Aff}(X) := \mathrm{Spec}\ \mathcal{O}_X(X)$$

$$\begin{tikzcd} & \mathrm{Aff}(X) \arrow[dr, dashed] & \\ X \arrow[ur] \arrow[rr] & & \mathrm{Spec}(R) \end{tikzcd}$$

For derived stacks:

$\Gamma\mathcal{O}$: Derived Stacks $\rightleftarrows$ Ring Spectra^{op} : "Spec"

Right adjoint

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & \mathrm{Aff}(X) \arrow[d, dashed] \\ & \mathrm{Spec}(R) \end{tikzcd}$$

e.g. X = top'l space S^1 regarded as derived stack

$$\mathcal{O}(S^1) = C^*(S^1) \simeq H^*(S^1)$$

$$= k[\eta]/(\eta^2)$$

$$|\eta| = 1$$

Char $\neq 2$

$$\mathrm{Aff}(S^1) = B\mathbb{G}_a = K(\mathbb{G}_a, 1)$$

Def: X = derived stack

loop space $LX = \mathrm{Map}(S^1, X)$

unipotent loop space

$$L^u X = \mathrm{Map}(B\mathbb{G}_a, X)$$

If $X = \operatorname{Spec} R$, then $L^u X \xrightarrow{\simeq} LX$

<u>Prop</u>: HKR equivalence

$X = \operatorname{Spec} R$

(1) $$\underset{\substack{\shortparallel \\ THH(R)}}{\int_{S^1} R} = R \otimes S^1 = R \underset{R \otimes R}{\otimes} R \simeq \operatorname{Sym}(\Omega^1_R[1])$$

(2) S^1 action on LHS

$\Updownarrow$

de Rham diff'l

"$B\mathbb{G}_a$-action" (Formal)

Thm: X = artin stack w/ affine diagonal

$$QCoh^{BG_a \times G_m}_{\text{finite}}(L^u X)_{\substack{\text{periodic} \\ \text{localization}}} \simeq D_X\text{-mod}_{\text{fin}}$$

"cyclic homology"

by inverting c_1 under $H^*(\mathbb{CP}^\infty) \simeq k[c_1]$

Categorification of a cyclic picture.

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example:

$$X = {}_U\backslash G/B$$

$$L^u X = LX = St/G$$

$$U\backslash G/B \leftarrow G/B \leftleftarrows U \times G/B \leftleftarrows \cdots$$

Cor: $QCoh^{\mathbb{G}_m \times B\mathbb{G}_a}(St/G)_{per} \simeq D\text{-modules}(G\backslash G/B)$

affine Hecke alg.

Borel-Weil-Bott

Special case

Line bundle $\longleftrightarrow$

$\Updownarrow$

twisted D-module