Madsen - homology operations + moduli spaces

Note Title 10/17/2009

$$\Gamma_g = \pi_0 \operatorname{Diff}(F_g) \simeq \operatorname{Diff}(F_g) \xrightarrow[\rho]{} Sp_{2g}(\mathbb{Z}) \to Sp_{2g}(\mathbb{R}) \simeq U_g$$

orientation preserving

oriented surface genus g

get $\quad B\Gamma_g \longrightarrow BU_g$

s_n = primitive generator of $H^{2n}(BU)$

Riem-Roch $\Rightarrow$ $\rho^*(s_{2n-1}) = (-1)^{n-1} \dfrac{B_n}{2n} \kappa_{2n-1}$

$$\kappa_{2n-1} \in H^*(B\Gamma_g ; \mathbb{Q})$$

Q! is this true in integral coh when we clear denom's?

Conj! (Akita)

$$\operatorname{Denom}\left(\frac{B_n}{2n}\right) \rho^*(s_{2n-1}) = (-1)^{n-1} \operatorname{Num}\left(\frac{B_n}{2n}\right) \kappa_{2n-1}$$

in $H^*(B\Gamma_g)$

$$B\Gamma_g \simeq B\mathrm{Diff}(F_g)$$

$B \to B\Gamma_g$

$$\longrightarrow \quad \begin{tikzcd} & N^\pi E \arrow[dl, squiggly] & \\ E \arrow[d, "\pi"', "F_g"] \arrow[rr, hook] & & B \times \mathbb{C}^{N+1} \arrow[dll, "pr_1"] \\ B & & \end{tikzcd}$$

$$T^\pi E \oplus N^\pi E = \varepsilon_{\mathbb{C}}^{N+1}$$ of cx bundles

P-T collapse map

$$B_+ \wedge S^{2N+2} \longrightarrow Th(N^\pi E)$$

get $\quad \pi_! : K(E) \longrightarrow K(B)$

$$\pi_* : H^*(E) \longrightarrow H^{*-2}(B)$$

$$B \to B\Gamma_g \xrightarrow[\rho]{} BU_g$$

get: $\quad s_n(E) \in H^{2n}(B)$

$$\mathcal{K}_n(E) = \pi_*\left(c_1(T^\pi E)^{n+1}\right) \in H^{2n}(B)$$

Another interpretation coming from fiberwise index theory:

$E_b = \pi^{-1}(b)$ Riemann surface

$$\bar\partial_b : C^\infty(E_b) \longrightarrow C^\infty(E_b, T^{0,1}E_b)$$

$$\phi \longmapsto \frac{\partial \phi}{\partial \bar z}\, d\bar z$$

$\ker(\bar\partial_b) = \mathbb{C}$

Hodge $*$ operator

$\operatorname{Coker} \partial_b = (H^1(E_b;\mathbb{R}), *)$ $\quad *^2 = -1$

$\Rightarrow$ $*$ gives this a $\mathbb{C}$-structure

$\bar\partial = \{\bar\partial_b\}$

Hodge bundle

$$\operatorname{Ind}(\bar\partial) = 1 - H(E)$$

$H(E)_b = (H^1(E_b;\mathbb{R}), *)$

Index thm $\Longrightarrow$ $\quad \mathrm{Ind}(\bar\partial) = \pi_!(1)$
(families)

L_N = canonical line bundle

$$\begin{tikzcd} N^\pi E \arrow[r, "\hat t"] \arrow[d] & L_N^\perp \arrow[d] \\ E \arrow[r, "t"'] & \mathbb{C}P^N \end{tikzcd}$$

$$L_N^\perp \oplus L_N = \varepsilon_{\mathbb{C}}^{N+1}$$

$$t(z) = (T^\pi E)_{\pi(z)}$$

$$B_+ \wedge S^{2N+2} \longrightarrow Th(N^\pi E) \longrightarrow Th(L_N^\perp)$$

$$B \longrightarrow \Omega^{2N+2} Th(L_N^\perp) \xrightarrow[N\to\infty]{} \Omega^\infty MT(2)$$

Madsen-Weiss:

$$\mathbb{Z} \times B\Gamma_\infty^+ \xrightarrow[\simeq]{} \Omega^\infty MT(2)$$

Calculation in $K_{(p)}(Th(L_N^\perp))$ $\qquad \langle k \rangle = \mathbb{Z}_p^\times$

$$k^{-N}(\psi^k - Id)(\underbrace{\lambda_{L_N^\perp}}_{\text{K-thy Thom class}}) = \omega^*\left(r^*(\bar L_N) \cdot \lambda_{L_N^\perp \oplus L_N}\right)$$

Then:

$$\omega : Th(L_N^\perp) \longrightarrow Th(L_N \oplus L_N^\perp) = S^{2N+2} \wedge \mathbb{C}P^N_+$$

$$\rho^k : \mathbb{Z} \times BU \longrightarrow BU^\otimes$$

∞-loop map

$$r^k = \Omega^2(\rho^k - 1)$$

Now just need to check this

(know everything about $K(\mathbb{C}P^\infty)$)

$$\begin{tikzcd}
Th(L_N^\perp) \arrow[r, "\omega"'] \arrow[d, "\lambda^\perp_{L_N}"] & S^{2N+2} \wedge \mathbb{C}P^N_+ \arrow[r, "\bar{L}_N"'] & S^{2N+2} \wedge \mathbb{Z} \times BU \arrow[d, "r^k"] \\
\mathbb{Z} \times BU \arrow[r, "k^{-N}(\psi^k - 1)"'] & \mathbb{Z}_{(p)} \times BU_{(p)} & S^{2N+2} \wedge \mathbb{Z}_{(p)} \times BU_{(p)} \arrow[l, "B\#"']
\end{tikzcd}$$

apply Ω^{2N+2}, take $N \to \infty$

get:

$$\begin{tikzcd}
 & \Omega^\infty MT(2) \arrow[r, "\omega"] \arrow[dd, "\lambda_{-L}"] \arrow[rrd, bend left, "\delta"] & Q(\mathbb{C}P^\infty_+) \arrow[rr, "Q(L)"] & & Q(\mathbb{Z}\times BU) \arrow[dl, "B"'] \arrow[dd, "Q(r^k)"] \\
B \arrow[ur, "\alpha"] \arrow[dr, "\pi_!(1)"'] & & & \mathbb{Z}\times BU \arrow[dl, dashed, "r^k"] \arrow[r, phantom, "\text{Ⓐ}"] & \\
 & \mathbb{Z}\times BU \arrow[r, "k\psi^k-1"'] & \mathbb{Z}_{(p)}\times BU_{(p)} & & Q(\mathbb{Z}_{(p)}\times BU_{(p)}) \arrow[ll, "B"]
\end{tikzcd}$$

$$\hat{s}_n = s_n(\pi_!(1))$$

$$\hat{\kappa}_n = \omega^*\sigma^*(x^n)$$

$$\bar{\kappa}_n = \delta^*(s_n)$$

where

$$\sigma^* : H^*(\mathbb{C}P^\infty) \longrightarrow H^*(Q(\mathbb{C}P^\infty_+))$$

$\hat{s}_n$ pulls back to s_n

$\tilde{\kappa}_n$ pulls back to κ_n

$\bar{\kappa}_n = (-1)^n \hat{\kappa}_n$ in $H^*(-;\mathbb{Q})$

$\bar{\kappa}_n$ pulls back to some other class

In other words: just need to study the problem in terms of $H^*(MT(2))$

Evaluate the diagram in cohomology.

apply $H^*(- ; \mathbb{Z}_{(p)})$

Starting w/

$s_{2n-1} \in H^*(\mathbb{Z}_{(p)} \times BU_{(p)})$

pretend r^k is an ∞-loop map;

(it ain't)

Facts

- $\textcircled{A}$ is not commutative, but $\exists$!

is commutative on $\mathrm{Prim}\, H^*(\mathbb{Z} \times BU)$.

Comes from: r^k is a double loop map.

- $(k\psi^k - 1)(s_{2n-1}) = (k^{2n} - 1)\, s_{2n-1}$

- $(r^k)^*(s_{2n-1}) = (k^{2n}-1)\, \mathrm{Num}\left(B_n / 2n\right)$ Adams J(X) II

"primitive generators of $H^*(BU \times \mathbb{Z})$ evaluate non-trivially on $\pi_* KU$ elts"

- $(k^{2n}-1) = 2\, \mathrm{Denom}\left(\frac{B_n}{2n}\right) \cdot (\text{unit in } \mathbb{Z}_{(p)})$

Thm:

(1) $2\,\mathrm{Denom}(B_n/2n)\,\hat{S}_{2n-1} = 2\,\mathrm{Num}\,(B_n/2n)\,\bar{K}_{2n-1}$

(2) $K_n \neq \hat{K}_n$ in $H^*(-;\mathbb{F}_p)$

$\Rightarrow$ Akita's conj is ~~false~~

(3) But: it's almost true, because

$$p\bar{K}_n = p\,K_n \quad \text{in} \quad H^*(-;\mathbb{Z}_{(p)})$$

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all of this uses H_* operators:

proof of (2)

$\langle \hat{K}_n , Q^i(a) \rangle = 0$ since its in the image of suspension.

$\langle \bar{K}_{2p-1} ; Q^1(a_1) \rangle \neq 0$

Thm:

$$a_n \in H_{2n}(\mathbb{C}P^\infty; \mathbb{F}_p) \to H_{2n}(Q(\mathbb{C}P^\infty_+); \mathbb{F}_p)$$

$$\downarrow \bar{\iota}$$

$$H_{2n}(\mathbb{Z} \times BU, \mathbb{F}_p)$$

$$\Omega^\infty MT(2) \xrightarrow[\omega]{} Q\mathbb{C}P^\infty_+ \longrightarrow \Omega QS^0$$

$$a_1 \longmapsto a_1$$

$$Q^1(a_1) = a_{2p-1} \qquad \text{(Kochman)}$$

$$\langle s_{2p-1}, a_{2p-1} \rangle \neq 0$$

Thm $\quad X = Q(Y)$

$$\gamma \in \mathrm{Prim}\, H^*(X; \mathbb{Z}_{(p)})$$

s.t.

- $\gamma = 0 \quad$ in $\quad H^*(X; \mathbb{F}_p)$
- $\iota^*(\gamma) = 0 \quad$ in $\quad H^*(Y; \mathbb{Z}_{(p)})$

$$\iota : Y \longrightarrow Q(Y)$$

- $\langle \gamma, \beta^{(r)}(a) \rangle = 0$ in $\mathbb{Z}/p^{r+1}$ $\quad \forall\, r \quad \forall\, a \in H_*(Y; \mathbb{F}_p)$

<u>why:</u>

$$\wp : H_*(Q(Y); \mathbb{Z}/p^r) \to H_{*p}(Q(Y); \mathbb{Z}/p^{r+1})$$

Know $H_*(QY; \mathbb{F}_p)$ and complete structure of BSS

interaction of $\wp$ w/ Bockstein diff's

$$\Longrightarrow \quad \gamma = 0$$

(A) uses logarithm (tom Dieck)

$$L_{(p)} : BU^{\otimes} \longrightarrow BU \times \mathbb{Z}$$

∞-loop map

$$L_{(p)}(1-x) = \left(\frac{\psi^p}{p} - 1\right) \log(1-x)$$

<u>and</u>: $\Omega^2(L_{(p)}) = 1 + \psi^p$