Lurie - Non-abelian Poincare duality

Note Title 10/17/2009

(I got here late!)

$\mathcal{U}(M)$ = cat of open sets $U \subseteq M^n$

$$\mathcal{F}_X : \mathcal{U}(M) \longrightarrow \text{Spaces}$$

$$\mathcal{F}_X(U) = \mathrm{Map}_c(U, X)$$

$\mathcal{F}$ is a htpy co-sheaf

Note! U, V disjoint

$\Rightarrow$ there is a canonical map

$$\mathcal{F}_X(U) \times \mathcal{F}_X(V) \xrightarrow[\sim]{} \mathcal{F}_X(U \cup V)$$

"addition" new structure

Throw in "disjoint unions of disks"

$\mathcal{U}_0(M)$ = open sets $U \subseteq M$

homeo to finite unions of disks

Thm 1 $X = (n-1)$-connected, $M = n$-mfld

$$\mathrm{Map}_c(M,X) \xleftarrow{\simeq} \operatorname*{hocolim}_{U \in \mathcal{U}_0(M)} \mathcal{F}_X(U)$$

"cohomology"

"homology"

loses information of $\pi_i X$ for $i \leq n-1$

Plausibility Argument

Surjectivity on π_0:

Any cptly supported map

$$f : M \to X$$

can be modified so

$\mathrm{supp}(f)$ is in a finite corr of disjoint disks

pf Triangulate M

$f|_{sk^{n-1}M}$ null homotopic ✓

e.g. $M = S^1$ $\quad X$ = pointed connected space

$$LX \xleftarrow{\simeq} \text{cyclic Bar construction on } \Omega X$$

$$\begin{gathered} LX \\ \| \\ \mathrm{Map}(S^1, X) \end{gathered}$$

Why is RHS "homology"?

M connected

$$\mathrm{Ran}(M) = \left\{ \begin{gathered} \text{non-empty finite subsets} \\ S \subseteq M \end{gathered} \right\}$$

Topology: Basis of opens:

$U_1, \dots, U_n$ disjoint connected opens in M

$$\mathcal{B}(U_1, \dots, U_n) = \left\{ S \in \mathrm{Ran}(M) \,\middle|\, \begin{gathered} S \subseteq \bigcup U_i \\ S \cap U_i \neq \emptyset \end{gathered} \right\}$$

$$= \mathrm{Ran}(U_1) \times \cdots \times \mathrm{Ran}(U_n)$$

Define

$\mathcal{F}_{Ran}$: basis of opens in $Ran(M)$ $\longrightarrow$ Spaces

$$\mathcal{F}_{Ran}\big(B(U_1,\dots,U_n)\big) = \mathcal{F}_X(U_1)\times\cdots\times\mathcal{F}_X(U_n)$$

$$\cong$$

$$\mathcal{F}_X(U_1\cup\cdots\cup U_n)$$

"cosheaf"

Thm 2

$\mathcal{F}_{Ran}$ is a htpy cosheaf on $Ran(M)$

Can always recover global section of cosheaf by applying to basis of opens

Thm 2

$$\mathcal{F}_{Ran}(Ran(M)) \xleftarrow{\quad} \underset{\substack{B = B(U_1,\dots,U_n) \\ U_i \text{ is a disk}}}{\operatorname{hocolim}} \mathcal{F}_{Ran}(B)$$

$$\cong$$

$$\mathcal{F}_X(M)$$

slight difference: $m \geq 1$ above,

before m allowed to be 0

(this is why we assumed M is connected)

<u>Thm</u> 1 $\Longrightarrow$ poincare <u>duality</u>

$X = K(A, m)$ $\quad m \geq n$

"add" $\quad X = \Omega^\infty E$

$U \subseteq M$

$$\mathcal{F}_E(U) = \mathrm{Map}_c(U, E)$$

$$\mathcal{F}_E(M) \xleftarrow{\simeq} \operatorname*{hocolim}_{U \subseteq \mathcal{U}_0(M)} \mathcal{F}_E(U)$$

"ordinary poincare duality"

Apply Ω^∞

$$\mathcal{F}_X(M) \xleftarrow{\simeq} \Omega^\infty \underrightarrow{\operatorname{hocolim}} \mathcal{F}_E(U)$$

Want to commute Ω^∞ and $\underrightarrow{\operatorname{hocolim}}$

$\Omega : \mathrm{Spaces}_* \longrightarrow \mathrm{Spaces}_*$

does not commute w/ $\underrightarrow{\operatorname{hocolim}}$

but sometimes you get lucky....

$Y_\bullet$, $Z_\bullet$ simplicial spaces

$$\begin{array}{ccc} |Y_\bullet \times Z_\bullet| & \xrightarrow{\simeq} & |Y_\bullet| \times |Z_\bullet| \\ \wr\wr & & \wr\wr \\ \operatorname{hocolim}_{[n]} Y_n \times Z_n & \longrightarrow & \operatorname{hocolim}_{[m],[n]} Y_m \times Z_n \end{array}$$

What's going on?

Something special w/ cat Δ

Def $f : \mathcal{C} \to \mathcal{D}$

f is <u>cofinal</u> if $\forall\ g : \mathcal{D} \to$ Spaces

$$\operatorname{hocolim}_{\mathcal{C}} (g \circ f) \xrightarrow[\simeq]{} \operatorname{hocolim}_{\mathcal{D}} (g)$$

Def: $\mathcal{C}$ is sifted if

$\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is cofinal.

Explaination

Δ^{op} is sifted

Say $\mathcal{C}$ is sifted.

$F, G : \mathcal{C} \to$ spaces

$$\operatorname{hocolim} F \times G \simeq \operatorname{hocolim} F \times \operatorname{hocolim} G$$

From now on, fiber product = htpy fiber products

$$\operatorname{hocolim} F \times_H G \xrightarrow[\alpha]{} \operatorname{hocolim} F \times_{\operatorname{hocolim} H} \operatorname{hocolim} G$$

<u>Lemma I</u> α is a htpy equivalence if H takes values in pointed connected spaces.

Cor: If $\mathcal{C}$ is sifted

then $\underrightarrow{\mathrm{hocolim}}_{\mathcal{C}}$ and Ω commute.

(for pointed connected spaces)

Cor: Ω^∞ commutes w/ $\underrightarrow{\mathrm{hocolim}}_{\mathcal{C}}$

for connective spectra.

Q: is $U_0(M)$ sifted?

A: No, but almost...

$U_0(M)$ objects $U \subseteq M$

$U \simeq \mathbb{R}^n \amalg \cdots \amalg \mathbb{R}^n$

Morphisms:

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & M & \end{tikzcd}$$

commute up to specified isotopy

"Mapping spaces" space of possibilities for possible isotopy.

$\mathcal{C}$ is a $(\infty, 1)$-catgy

$$\mathcal{U}_0(M) \longrightarrow \mathcal{C}$$

"sifted $(\infty,1)$ cats, etc..."

Lem 2 $\mathcal{U}_0(M) \hookrightarrow \mathcal{C}$ cofinal

Lem 3 $\mathcal{C}$ is sifted

Proof: use fact that since M is a mfld, locally looks like disk

Observation: $\mathcal{F}_X$, $\mathcal{F}_E$ factor through $\mathcal{C}$

$$
\begin{aligned}
\mathcal{F}_X(M) \xleftarrow{\simeq} \Omega^\infty \underset{U \in \mathcal{U}_0(M)}{\underrightarrow{\mathrm{hocolim}}} \mathcal{F}_E(U)
&\cong \Omega^\infty \underset{\mathcal{C}}{\underrightarrow{\mathrm{hocolim}}} \mathcal{F}_E \\
&\cong \underset{\mathcal{C}}{\underrightarrow{\mathrm{hocolim}}} \Omega^\infty \mathcal{F}_E \\
&\cong \underset{\mathcal{U}_0}{\mathrm{hocolim}} \Omega^\infty \mathcal{F}_E(U) \\
&\cong \underset{\mathcal{U}_0}{\mathrm{hocolim}} \mathcal{F}_X(U)
\end{aligned}
$$

Get pf of Thm 1 given ordinary Poincaré duality (spectrum level statement)

Know $X = K(\pi, n)$

More genl X ?

Assume $n \geq 2 \Rightarrow X$ simply connected

Write postnikov tower

$$\begin{array}{c} \vdots \\ \downarrow \\ X_{\leq n+1} \\ \downarrow \\ X_{\leq n} \end{array}$$

prove thm 1 is true

for X_k by induction

$\varprojlim_k$ gives thm for X.

Induction: Obstruction Thy

$$
\begin{tikzcd}
X_{\leq k+1} \arrow[r] \arrow[d] & \ast \arrow[d] \\
X_{\leq k} \arrow[r] & K(\pi_{k+1}, k+2)
\end{tikzcd}
$$

true for this

true for this

true for this

True for X ?

$$\mathcal{F}_{X_{\leq k+1}}(M) = \mathcal{F}_{X_{\leq k}}(M) \times_{\mathcal{F}_K(M)} \ast$$

$$\cong$$

$$\operatorname{hocolim} \mathcal{O} \times_{\operatorname{hocolim} \mathcal{O}} \operatorname{hocolim} \mathcal{O}$$

$\cong$? lemma 2

$$\operatorname{hocolim} \left(\mathcal{O} \times_{\mathcal{O}} \mathcal{O} \right)$$

<u>Rmk</u>: spectrum version is "easy"

does not use Ran space.