

Lurie - Non-abelian Poincare duality

Note Title

10/17/2009

(I got here late!)

$\mathcal{U}(M)$ = cat of open sets $U \subseteq M^n$

$$\mathcal{F}_x : \mathcal{U}(M) \rightarrow \text{Spaces}$$

$$\mathcal{F}_x(U) = \text{Map}_c(U, x)$$

$\mathcal{F}$ is a left co-sheaf

Note! U, V disjoint

$\Rightarrow$ there is a canonical map

$$\mathcal{F}_x(U) \times \mathcal{F}_x(V) \xrightarrow{\sim} \mathcal{F}_x(U \cup V)$$

"addition" new structure

Then in "disjoint unions of disks"

$$\mathcal{U}_o(M) = \text{open sets } U \subseteq M$$

hence to finite unions of disks

Thm 1 $X = (n-1)$ -connected, $M = n$ -mfd

$$\text{Map}_c(M, X) \simeq \text{hocolim}_{U \in \mathcal{U}_0(M)} F_X(U)$$

"cohomology"

↑ "homology"
loses information
of $\pi_i X$

for $i \leq n-1$

Plausibility Argument

Surjectivity on π_0 :

Any cply supported map

$$f: M \rightarrow X$$

can be modified so

$\text{supp}(f)$ is in a finite cover
of disks

pf Triangulate M

$$f|_{\text{sk}^{n-1}M} \text{ null homotopic}$$

✓

e.g., $M = S^1$ $X = \text{pointed connected space}$

$$\begin{array}{ccc} \mathbb{L}X & \xleftarrow{\cong} & \text{cyclic Bar construction} \\ \parallel & & \text{on } \Omega X \\ \text{Map}(S^1, X) & & \end{array}$$

Why is RHS "homology"?

M connected

$$\text{Ran}(M) = \left\{ \begin{array}{l} \text{non-empty finite subsets} \\ S \subseteq M \end{array} \right\}$$

Topology: Basis of opens:

$U_1, \dots, U_n$ disjoint connected
opens in M

$$\begin{aligned} \mathcal{B}(U_1, \dots, U_n) &= \left\{ S \in \text{Ran}(M) \mid \begin{array}{l} S \subseteq \bigcup U_i \\ S \cap U_i \neq \emptyset \end{array} \right\} \\ &= \text{Ran}(U_1) \times \dots \times \text{Ran}(U_n) \end{aligned}$$

Defn

$\mathcal{F}_{\text{Ran}}$: basis of opns $\rightarrow$ spaces
in $\text{Ran}(M)$

$$\mathcal{F}_{\text{Ran}}(B(u_1, \dots, u_n)) = \mathcal{F}_x(u_1) \times \dots \times \mathcal{F}_x(u_n)$$

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"cosheaf"

$$\mathcal{F}_x(u_1 \cup \dots \cup u_n)$$

Thm 2

$\mathcal{F}_{\text{Ran}}$ is a ltry cosheaf
on $\text{Ran}(M)$

Can always recover global section of
cosheaf by applying to basis of
opns

Thm 2

$$\mathcal{F}_{\text{Ran}}(\text{Ran}(M)) \xleftarrow{\text{local}} \mathcal{F}_{\text{Ran}}(B)$$

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$$\mathcal{F}_x(M)$$

$B = B(u_1, \dots, u_n)$
 u_i is a disk

slight diff: $m \geq 1$ above,

before m allowed to be 0

(this is why we assumed M is
connected)

Thm 1 $\Rightarrow$ power duality

$$X = K(A, m) \quad m \geq n$$

"add" $X = \Omega^\infty E$

$$U \subseteq M$$

$$\mathcal{F}_E(U) = \text{Map}_c(U, E)$$

$$\mathcal{F}_E(M) \xleftarrow[\cong]{} \varinjlim_{U \subseteq U_0(M)} \mathcal{F}_E(U)$$

"ordinary power duality"

Apply Ω^∞

$$\mathcal{F}_X(M) \xleftarrow[\simeq]{} \Omega^\infty \xrightarrow{\text{hocolim}} \mathcal{F}_E(U)$$

Want to commute Ω^∞ and $\xrightarrow{\text{hocolim}}$

$$\Omega: \text{Spaces}_X \longrightarrow \text{Spaces}_X$$

does not commute w/ $\xrightarrow{\text{hocolim}}$

but sometimes you get lucky....

Y_0, Z_0 simplicial spaces

$$|Y_0 \times Z_0| \xrightarrow[\simeq]{} |Y_0| \times |Z_0| \quad \text{What's going on?}$$

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$$\text{hocolim}_{[m]} Y_n \times Z_n \longrightarrow \text{hocolim}_{[m], [n]} Y_n \times Z_n$$

Something spectral w/ cat Δ

Def $f: \mathcal{C} \rightarrow \mathcal{D}$

f is cofinal if $\forall g: \mathcal{D} \rightarrow \mathbf{Spec}$

$$\text{locobn}_{\mathcal{C}}(g \circ f) \xrightarrow{\simeq} \text{locobn}_{\mathcal{D}}(g)$$

Def: $\mathcal{C}$ is sifted if

$\mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ is cofinal.

Explanation

Δ^{op} is

sifted

Say C is sifted.

$F, G: C \rightarrow \text{spaces}$

$$\text{hocolim } F \times G \simeq \text{hocolim } F \times \text{hocolim } G$$

From now on, fiber product = hty fiber products

$$\text{hocolim}_{H} F \times G \xrightarrow{\alpha} \text{hocolim } F \times_{\text{hocolim } H} \text{hocolim } G$$

Lemma 1 α is a hty equivalence

if H takes values in pointed
connected spaces.

Cor! If $\mathcal{C}$ is sifted

then $\frac{\text{holim}}{c}$ and Ω commute.

(for pointed connected spaces)

Cor!

Ω^∞ commutes w/ $\frac{\text{holim}}{c}$

for connective spectra.

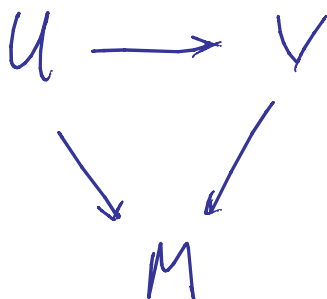
Q! is $\mathcal{U}_0(M)$ sifted?

A! No, but almost...

$\mathcal{U}_0(M)$ objects $U \in M$

$$U \approx \mathbb{R}^n \sqcup \dots \sqcup \mathbb{R}^n$$

Morphisms:



commute up to specified isotopy

"Mapping spaces"

space of possibilities
for possible
isotopy.

$\mathcal{C}$ is a $(\infty, 1)$ -catgy

$$\mathcal{U}_0(M) \rightarrow \mathcal{C}$$

"sifted $(\infty, 1)$ cats, etc. —"

Lem 2 $\mathcal{U}_0(M) \hookrightarrow \mathcal{C}$ cofinal

Lem 3 $\mathcal{C}$ is sifted

Proof's use fact that since M is a manifold,
locally looks like disk

Observation: $\mathcal{F}_X, \mathcal{F}_E$ factor through $\mathcal{C}$

$$\mathcal{F}_X(n) \xleftarrow{\sim} \varinjlim_{u \in U_0(n)} \mathcal{F}_E(u)$$

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$$\varinjlim \mathcal{F}_E$$

$\mathcal{C}$

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$$\xrightarrow{\mathcal{C}} \varinjlim \mathcal{F}_E$$

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$$\varinjlim_{U_0} \mathcal{F}_E(u)$$

U_0

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$$\varinjlim_{U_0} \mathcal{F}_X(u)$$

U_0

Get pf of Thm 1 given
ordinary Reduce duality (spectral level)
statement

Know $X = K(\pi, m)$

More general X ?

Assume $n \geq 2 \Rightarrow X$ simply connected

Write postulator for

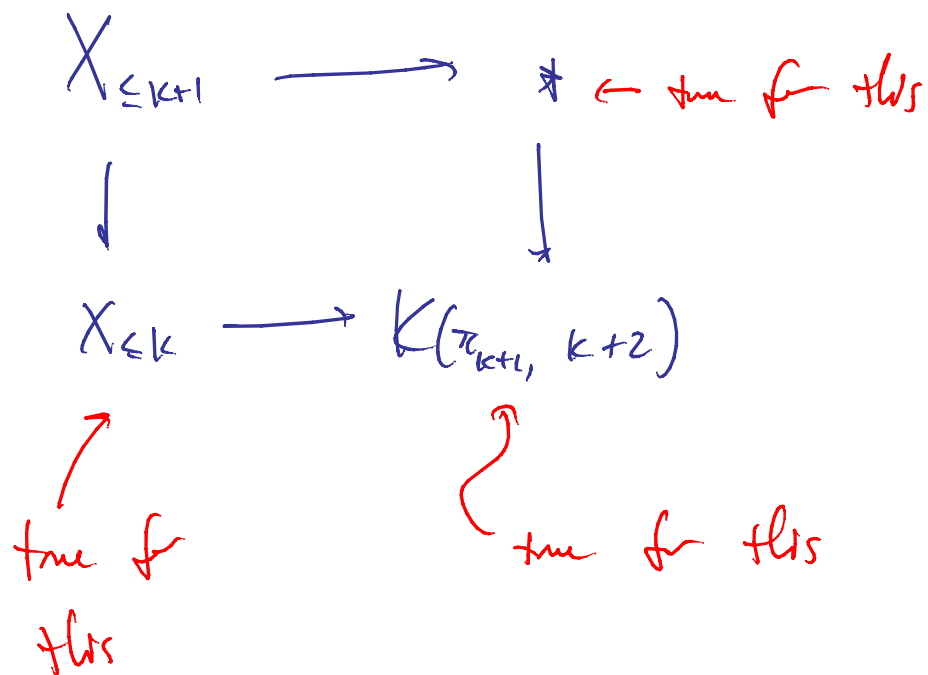
$$\begin{array}{c} \vdots \\ \downarrow \\ X_{\leq n+1} \\ \downarrow \\ X_{\leq n} \end{array}$$

prove that 1 is true

for X_n by induction

$\varprojlim_K$ gives the answer for X .

Induction: Obstruction Thy



True for X ?

$$\mathcal{F}_{X_{\leq k+1}}(M) = \mathcal{F}_{X_{\leq k}}(M) \times \mathcal{F}_K(M)$$

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hook 0 x hook 0

hook 0

12 ? lemma 2

$$\text{hook} \left(\begin{array}{cc} 0 & \times & 0 \\ & 0 & \end{array} \right)$$

Rank spectrum norm is "easy"

does not via Rank Space.