Mayday 2009 @ U of Chicago

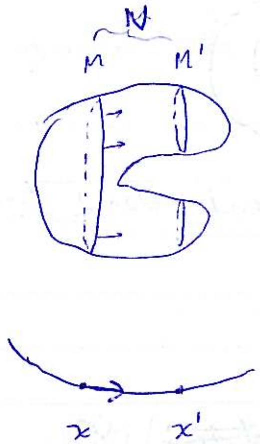
10/16

Kervaire invariant (Mike Hopkins)

Th (Hill-Hopkins-Ravenel): If M is sm stably framed mnf w/ Kerv inv 1, then dim $M = 2, 6, 14, 30, 62$ or 126.

History of the problem

1930s Pontryagin → Study $S^{n+k} \xrightarrow{f} S^n$ in terms of $f^{-1}(x)$ ← reg val



$f^{-1}(x) = M$ is a sm mnf of dim k

+ basis of normal bdl to $M \subset S^{n+k}$

(aka framed mnf)

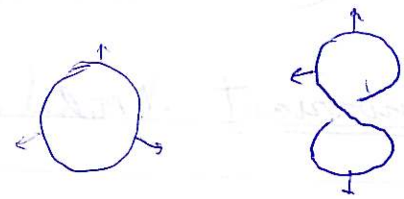
Want result not to depend on choice of x, so identify $M \sim M'$ if $\exists$ N s.t. $\partial N = M \sqcup M'$ (+compatibility...)

Then we say M is cobordant to M', via the cobordism N.

Pontryagin showed $\pi_{n+k} S^n \xrightarrow{\cong}$ cobord gp of framed k-mnf

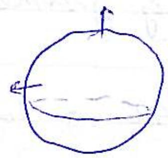
$k = 0$ $\quad \pi_n S^n \cong \mathbb{Z}$, $\quad$ framed 0-mnf ⇝ count pts

$k = 1$ Two possibilities



and $\pi_{n+1} S^n \cong \mathbb{Z}/2$

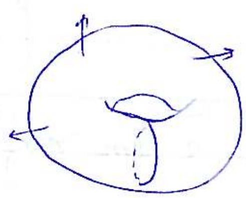
$k = 2$

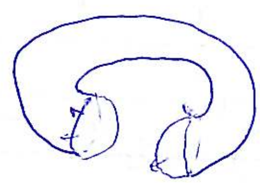


Surf Σ

1) If genus $\Sigma = 0$, $\Sigma \leftrightarrow 0$ in $\pi_{n+2} S^n$

2) Try to reduce the genus





Framing must extend to a disk

$\phi : H_1(M; \mathbb{Z}/2) \to \mathbb{Z}/2$

arc $\longmapsto$ its class as framed 1-mnf

$\phi(x+y) - \phi(x) - \phi(y) = x \cdot y$ inters form (bilin), so ϕ is quadratic

Arf $\phi \in \mathbb{Z}/2$

And in fact $\pi_{n+2} S^n \cong \mathbb{Z}/2$.

Q: In which dim's is every framed mnf cobordant to a (topological) sphere?

A: All dimensions except possibly 2, 6, 14, 30, 62, 126.

1956 Milnor $\rightarrow$ M homeo to S^7 but NOT diffeo to it i.e. exotic sm struc on S^7

1958 & 1963 Kerv-Milnor $\rightarrow$ Θ_n = gp of (h-cobord classes of) n-mnf's M^n homeo to S^n, under connected sum

Computed Θ_n in terms of π_n^S up to a factor of 2.

1960 Kerv $\rightarrow$ M^{2n} $(n-1)$-conn mnf

$$\phi : H_n(M; \mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \quad \text{quadratic}$$

$$\Phi(M) := \mathrm{Arf}\,\phi \in \mathbb{Z}/2 \quad \text{the Kerv invar}$$

constructed a PL mnf M^{10} s.t. if N sm and $N \underset{PL}{\sim} M^{10}$, then $\Phi(N) = 1$

But showed $\Phi(N^{10}) = 0$ for any sm N, so M^{10} is not smoothable.

Q: In which dim's can $\Phi(M)$ be non-zero?

1969 Browder $\to$ If $\Phi(M) = 1$ then dim $= 2^{k+1}-2$

In dim $2^{k+1}-2$, $\exists$ M of Kerv inv 1 iff $\exists$ $\theta_k \in \pi_{2^{k+1}-2}$ ~~corresponding~~ represented by h_j^2 in the Adams s.s. (i.e. our candidate solution h_j^2 survives all the way).

Th (Barratt-Jones-Mahowald): θ_k exists in dim 2, 6, 14, 30, 62

I. James $\to$ EHP seq

$$\cdots \to \pi_m S^n \xrightarrow{E} \pi_{m+1} S^{n+1} \xrightarrow{H} \pi_{m+1} S^{2n+1} \xrightarrow{P} \pi_{m-1} S^n \to \cdots$$

when $m = 2n$, get

$$\begin{tikzcd}
\to \pi_{2n+1} S^{2n+1} \arrow[r, "P"] & \pi_{2n-1} S^n \arrow[r, "E"] & \pi_{2n} S^{n+1} \to \cdots \\
\iota_{2n+1} \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & {[\iota_n, \iota_n]} \arrow[u, phantom, "\in" sloped] &
\end{tikzcd}$$

Q1: For which k is $[\iota_n, \iota_n]$ in E^k?

Q2: Is $[\iota_n, \iota_n]$ divisible by 2?

$\overset{(\pm 1?)}{\sim}$

Q1 $\Leftrightarrow$ S^n has k lin indep vct fields

Q2 $\Rightarrow$ n even $\leftrightarrow$ Hopf invar 1 problem

n odd $\leftrightarrow$ Kerv invar 1 problem

Outline of pf

- Construct a spc $\tilde{\Omega} \circlearrowleft \mathbb{Z}/8$

In fact $\tilde{\Omega} = D^{-1}(MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}})$, D is some "Bott periodicity class"

$\Omega := \tilde{\Omega}^{h\mathbb{Z}/8} \overset{\text{thm}}{=} \tilde{\Omega}^{\mathbb{Z}/8}$

- Detection thm: If θ_j exists, its image in $\pi_{2^{j+1}-2}\Omega$ is non-zero

- Periodicity thm: $\pi_k \Omega \cong \pi_{k+256}\Omega$.

- Gap thm: $\pi_k \Omega = 0$ for $-4 < k < 0$.

Hence for $k \geq 7$, $\pi_{2^{k+1}-2}\Omega = 0$, so θ_k does NOT exist.

Major tools → Equivariant stable homot th.