Gerhardt - Equivariant htpy and K-theory

Note Title 10/17/2009

Goal: A = ring

Compute $K_q(A)$

e.g.: $K_q\left(\mathbb{Z}[x]/(x^{n+1}),\ (x)\right)$

$K_q\left(\mathbb{Z}[x,y]/(xy),\ (x,y)\right)$

Background: Trace methods

Idea: Approximate $K(A)$ by things that are more computable

- Hoschild homology

$$K_q(A) \longrightarrow HH_q(A)$$ Dennis Trace

too algebraic

Top'l Hochschild Homology $T(A)$

$$A \rightsquigarrow HA$$

$$\otimes \rightsquigarrow \wedge$$

(Bökstedt)

$$K(A) \longrightarrow T(A)$$ top'l Dennis trace

Top'l cyclic homology:

p = prime

$C_p \subseteq S^1$ cyclic gp of order p

$T(A) \circlearrowleft S^1$

$F: T(A)^{C_p} \longrightarrow T(A)$ inclusion of fixed pts

$R: T(A)^{C_p} \longrightarrow T(A)$ "restriction"

"cyclotomic structure of $T(A)$"

$$
\begin{tikzcd}
 & TC(A,p) \arrow[d, dotted] \\
 & T(A)^{C_{p^2}} \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \\
 & T(A)^{C_p} \arrow[d, shift right, "R"'] \arrow[d, shift left, "F"] \\
K(A) \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur, dashed, "\mathrm{trc}"] & T(A)
\end{tikzcd}
$$

<u>Def</u>

$$TC^n(A,p) \to T(A)^{C_{p^{n-1}}} \underset{R}{\overset{F}{\rightrightarrows}} T(A)^{C_{p^{n-2}}}$$

(htpy equalizer)

$$TC(A,p) = \underset{\longleftarrow}{\mathrm{holim}}\ TC^n(A,p)$$

<u>cyclotomic</u> <u>trace</u>:

$$K(A) \longrightarrow TC(A)$$

[Bökstedt-Hsiang-Madsen]

TC is a good approx of K-thy [Dundas-McCarthy]

McCarthy - Goodwillie

$I \subset A$ nilpotent

$$K_q(A, I, \mathbb{Z}_p) \xrightarrow[\cong]{} TC_q(A, I, \mathbb{Z}_p)$$

So suffices to compute $TC_q(A, I, \mathbb{Z}_p)$

$$TC_q(A; p)$$

Goal: Understand $T(A)^{C_{p^{n-1}}} =: TR^n(A;p)$

and R and F

$$TR^n_q(A;p) := \pi_q TR^n(A;p)$$

$T(A)$ is a genuine S^1-spectrum

[Lewis-May] Constructed a good stable htpy category of G-spectra.

$G =$ cpt Lie

T, T' two genuine S^1-spectra

λ = finite dim'l S^1-rep

S^λ = 1 pt cptification of λ

$$[T, T']_{S^1} \xrightarrow[\cong]{} [T \wedge S^\lambda, T' \wedge S^\lambda]_{S^1}$$

$T(A)$ = S^1-spectrum (gen)

$T(A)^{C_{p^{n-1}}}$ = fixed point spectrum.

More approachable goal:
htpy fixed points.

T = gen G-spectrum

$$T^{hG} = F(EG_+, T)^G$$

<u>Projection</u>: $EG_+ \longrightarrow S^0$

$$T^G \xrightarrow{\Gamma} T^{hG}$$

$$\Gamma : TR^n(A,p) \longrightarrow T(A)^{hC_{p^{n-1}}}$$

Modern method:

Understand $T(A)^{hC_{p^n}}$

Understand Γ

Recover TR

[Greenlees-May] Put Γ in the fund diagram:

Construction: special case

Cofiber sequence:

$$ES^1_+ \longrightarrow S^0 \longrightarrow \tilde{E}S^1$$

Smash w/ $T(A)$

$$\begin{tikzcd}
(ES^1_+ \wedge T(A))^{C_{p^{n-1}}} \arrow[r] \arrow[d, "S^1\text{-equivalence}"'] & T(A)^{C_{p^{n-1}}} \arrow[r] \arrow[d] & (\hat{E}S^1 \wedge T(A))^{C_{p^{n-1}}} \arrow[d] \\
(ES^1_+ \wedge F(ES^1_+, T(A)))^{C_{p^{n-1}}} \arrow[r] & (F(ES^1_+, T(A)))^{C_{p^{n-1}}} \arrow[r] & (\tilde{E}S^1 \wedge F(ES^1_+, T(A)))^{C_{p^{n-1}}}
\end{tikzcd}$$

$T(A)_{hC_{p^{n-1}}}$ [Adams, Lewis/May] get:

$$\begin{tikzcd}
T(A)_{hC_{p^{n-1}}} \arrow[r] \arrow[d, equal] & TR^n(A;p) \arrow[r, "R"] \arrow[d, "\Gamma"] & TR^{n-1}(A;p) \arrow[d, "\hat{\Gamma}"] \\
T(A)_{hC_{p^{n-1}}} \arrow[r] & T(A)^{hC_{p^{n-1}}} \arrow[r] & T(A)^{tC_{p^{n-1}}}
\end{tikzcd}$$

SSs

$$\hat{H}^{-s}(C_{p^{n-1}}, \pi_t T(A)) \Rightarrow \pi_{s+t} T(A)^{tC_{p^{n-1}}}$$

$$H^{-s}(C_{p^{n-1}}, \pi_t T(A)) \Rightarrow \pi_{s+t} T(A)^{hC_{p^{n-1}}}$$

$$H_s(C_{p^{n-1}}, \pi_t T(A)) \Rightarrow \pi_{s+t} T(A)_{hC_{p^{n-1}}}$$

$T(A)$ cyclotomic $\Longrightarrow$ $\left(\tilde{E}S^1 \wedge T(A)\right)^{C_{p^{n-1}}} \simeq T(A)^{C_{p^{n-2}}}$

Hesselholt–Madsen

Method

- Compute bottom row (use spectral sequences)
- Understand I, $\hat{I}$
  [Tsalidis's thm]
- induct

Hesselholt - Madsen.

Pointed Monoid Alg's (Hesselholt - Madsen)

$$K(A[\Pi])$$

e.g. $\Pi_m = \{0, 1, x, \dots, x^{m-1}\}$ $\quad x^m = 0$

$$A[\Pi] = A[x]/(x^m)$$

e.g. $\Pi_{x,y} = \{0, 1, x, x^2, \dots, y, y^2, \dots\}$

$$A[\Pi_{x,y}] = A[x,y]/(xy)$$

$$T(A[\pi]) \simeq T(A) \wedge N^{cyc}(\pi)$$

$$TR^n_q(A[\pi], p) = \pi_q\, T(A[\pi])^{C_{p^{n-1}}}$$

$$= [S^q \wedge S^1/C_{p^{n-1}+}, T(A[\pi])]_{S^1}$$

$$= [S^q \wedge S^1/C_{p^{n-1}+}, T(A) \wedge N^{cy}(\pi)]_{S^1}$$

need to undrstd

Need to understand S^1-equiv. htpy type of $N^{cy}(\pi)$

Examples: Hesselholt-Madsen

Specify how to build htpy type $N^{cy}(\Pi_m)$ out of representation spheres.

Reduces the computation to

$$[S^q \wedge S^1/C_{p^{n-1}+}, T(A) \wedge S^\lambda]_{S^1}$$

$$\|$$

$$\|$$

$$\pi_{q-\lambda}\left(T(A)^{C_{p^{n-1}}}\right)$$

$$\|$$

$$TR^n_{q-\lambda}(A,p)$$

$RO(S^1)$-graded equivariant htpy gps

[Lewis - Mandell]

New

Goal: Compute $TR^n_{q-\lambda}(A,p)$

$TR^n_{\alpha}(A;p)$

$$\begin{tikzcd}
\pi_\alpha T(A)_{hC_{p^{n-1}}} \arrow[r] \arrow[d, equal] & TR^n_\alpha(A;p) \arrow[r, "R"] \arrow[d, "\Gamma"] & TR^{n-1}_{\alpha'}(A;p) \arrow[d, "\hat{\Gamma}"] \\
\pi_\alpha T(A)_{hC_{p^{n-1}}} \arrow[r] & \pi_\alpha T(A)^{hC_{p^{n-1}}} \arrow[r] & \pi_\alpha T(A)^{tC_{p^{n-1}}}
\end{tikzcd}$$

$$\alpha' = \alpha^{C_{p^{n-1}}}$$

Example

$TR^n_\alpha(\mathbb{F}_p, p)$ [Hesselholt–Madsen]

Angeltveit–G

E.g. $K(\mathbb{Z}[x]/(x^m), (x))$

Need: $TR^n_\alpha(\mathbb{Z}, p)$

or $TR^n_{q-\lambda}(\mathbb{Z}, p)$

First goal: $TR^n_\alpha(\mathbb{Z}, p; \mathbb{Z}/p)$

Construct SS:

$$E^1_{s,t} = \pi_{\alpha^{(n-s)}+t}\left(T(\mathbb{Z})_{hC_{p^s}}, \mathbb{Z}/p\right) \Rightarrow TR^{n+1}_{\alpha+t}(\mathbb{Z}, p, \mathbb{Z}/p)$$

Diffls:
- Bottom row
- Γ, Γ'
- induction.

Can compute $TR^n_\alpha(\mathbb{Z}_p, \mathbb{Z}/p)$

Can compute this for all reps.

Need: Thm (G-V-Hesselholt)

$TR^n_{2i-\lambda}(\mathbb{Z};p)$ is free of a rank we can compute

$TR^n_{2i-1-\lambda}(\mathbb{Z};p)$ has an inductive formula for its order

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(pf) (1) Use LES in 1st row + partial understanding of leftmost term.

Compute rank of $TR^n_{2i-\lambda}(\mathbb{Z};p)$

- use mod p - computations to tell us # summands.

<u>Thm</u> (Angeltveit – Hesselholt – G)

(1) $$K_{2i+1}\left(\mathbb{Z}[x]/(x^m), (x)\right) \cong \mathbb{Z}^{m-1}$$

(2) $$\left|K_{2i}\left(\mathbb{Z}[x]/(x^m), (x)\right)\right| = (mi)!\,(i!)^{m-2}$$

<u>Thm</u> (Ang. – G)

(1) $$K_{2i}\left(\mathbb{Z}[x,y]/(xy), (x,y)\right) \cong \mathbb{Z}$$

(2) $$\left|K_{2i+1}\left(\mathbb{Z}[x,y]/(xy), (x,y)\right)\right| = (i!)^2$$