Ando - Umkehr maps and twisted cohomology

Note Title 10/18/2009

$R = S$-alg

$X \longrightarrow BGL_1 R$

$BGL_1 R$?

$$\begin{array}{ccccc} & & \infty\text{-cat} & & \infty\text{-cat} \\ \mathrm{Sing}(X) & \xrightarrow[\xi]{} & R\text{-lines} & \hookrightarrow & R\text{-mods} \\ \wr & & \wr & & \\ X & & BGL_1 R & & \end{array}$$

subcat of free mods 1's

get

$x \in X \rightsquigarrow \xi_x$

$x \to y \rightsquigarrow \xi_x \xrightarrow[\simeq]{} \xi_y$

$\vdots$

etc...

"family of R-lines parametrized from X"

$\mathrm{Sing}\, X \longrightarrow R\text{-lines} \longrightarrow R\text{-mod}$ "diagram"

$\mathrm{colim}\left(\text{——}\right) =: X^{\xi}$ Generalized "Thom spectrum"

$$R^*(X)_\xi \simeq \pi_{-*} F_R(X^\xi, R)$$

$$S\text{-lines} \simeq BGL_1 S$$

$$X \xrightarrow[V]{} BGL_1 S \qquad \text{"spherical fibrations"}$$

$$S\text{-mod} \xrightarrow[-\wedge R]{} R\text{-mod}$$

$$\begin{tikzcd} & S\text{-line} \arrow[d] \\ X \arrow[ur, dashed, "V"] \arrow[r, "\xi"'] & R\text{-line} \end{tikzcd} \Rightarrow X^\xi \simeq R \wedge X^V$$

So twisted cohomology generalizes coh of thom spectra.

Twisted K-thy

$$H^3(X;\mathbb{Z}) \rightsquigarrow \text{twists of K-thy}$$

ABS orientation

$$K(\mathbb{Z},3) \longrightarrow BGL_1 K$$

$$MSpin^c \xrightarrow[ABS]{} K$$

$$\begin{tikzcd} K(\mathbb{Z},2) \arrow[d] \\ BSpin^c \arrow[d] \\ BSO \arrow[d, "\beta w_2"] \\ K(\mathbb{Z},3) \end{tikzcd}$$

fibration of ∞-loop spaces

$$\begin{tikzcd} \Sigma^\infty_+ K(\mathbb{Z},2) \arrow[r] \arrow[rr, bend right] & MSpin^c \arrow[r] & K \end{tikzcd}$$

E_∞-ring map

adjoint $\quad K(\mathbb{Z},2) \longrightarrow GL_1(K)$

$$\Longrightarrow \quad K(\mathbb{Z},3) \to BK(\mathbb{Z},2) \longrightarrow BGL_1 K$$

$$\begin{tikzcd} K(\mathbb{Z},2) \arrow[r] \arrow[d] & GL_1 K \arrow[d] \\ BSpin^c \arrow[r] \arrow[d] & EGL_1 K \simeq * \arrow[d] \\ BSO \arrow[r] \arrow[d, "\beta w_2"] & BGL_1 K \arrow[d, equal] \\ K(\mathbb{Z},3) \arrow[r, "t"'] & BGL_1 K \end{tikzcd}$$

$$\begin{array}{c} V \\ \downarrow \\ X \end{array} \quad \text{SO-bundle}$$

$$K(X^V) \cong K(X; \beta w_2(V))$$

So this simple cohomological invariant determines K of Thom spectra!

Can also do this for tmf!

$$\Sigma^\infty_+ K(\mathbb{Z},3) \longrightarrow MString \xrightarrow[\text{AHR}]{} tmf$$

$$\begin{array}{c} K(\mathbb{Z},3) \\ \downarrow \\ BSpin \\ \downarrow \\ BString \\ \downarrow \lambda \\ K(\mathbb{Z},4) \end{array}$$

get $K(\mathbb{Z},4) \longrightarrow BGL_1 tmf$

Note: $\pi_3\, tmf$ is $\mathbb{Z}/24$, so NEED

AHR orientation

& spin bundle

So $tmf(X^V) \simeq tmf(X; \lambda(V))$

So H^4 class determines

$tmf(X^V)$ for vector V spin

(More generally all V.B.'s and a

2-stage postnikov tower)

D-branes

$$D \xrightarrow{i} X$$

"subspace of space time where strings ends live"

"Space time"

$\nu = \nu(j)$ Normal bundle

P-T construction

$$X_+ \longrightarrow D^\nu$$

$$\begin{tikzcd}
j_! : K(D) \arrow[r, "\cong", "\nu\ \mathrm{spin}_c"'] & \tilde{K}(D^\nu) \arrow[r] & K(X) \\
V \arrow[u, phantom, "\in" sloped] \arrow[rr, mapsto] & & j_!(V) \arrow[u, phantom, "\in" sloped]
\end{tikzcd}$$

"D-brane charge"

What if you don't have a spin_c structure?

$$\begin{tikzcd}
D \arrow[r, "\nu"'] \arrow[d, "j"'] & BSO \arrow[d, "\beta w_2"] \\
X \arrow[r, dashed, "H"'] & K(\mathbb{Z},3)
\end{tikzcd}$$

Still get! $K(D^\nu; -jH) \longrightarrow K(X; -H)$

$K(D^\nu; -jH) \cong K(D)$

wrt

$j_!(V) \in K(X; -H)$

So D-brane charge lives in

Twisted K-thy.

[Cf. Freed-Witten]

$$F \longrightarrow Y \xrightarrow{f} X$$

cpt mflds

$$Y \hookrightarrow X \times \mathbb{R}^N$$

$$\Sigma^N(X_+) \longrightarrow Y^\nu$$

$$X_+ \longrightarrow Y^{-Tf}$$

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e.g.

$$F \longrightarrow *$$

$$D*_+ \longrightarrow DF_+ \simeq F^{-TF}$$

$$\shortparallel$$

$$S^0$$

$$F \longrightarrow Y \to X$$

$$\rightsquigarrow \quad \mathrm{Sing}\, X \underset{Y}{\overset{*}{\rightrightarrows}} \text{Spaces} \xrightarrow[D(-)_+]{} \text{Spectra}$$

get $\mathrm{Sing}\, X \underset{b}{\overset{a}{\rightrightarrows}}$ Spectra

$$(b \to a) \in \mathrm{Func}(\mathrm{Sing}\, X, \text{Spectra})$$

So get:

$$\begin{array}{ccc} \mathrm{colim}\ b & \longrightarrow & \mathrm{colim}\ a \\ \wr\wr & & \wr\wr \\ \Sigma^{\infty}_{+} X & & Y^{-Tf} \end{array}$$

So get Umkehr map

w/ using P-T collapse map!

$$X^{\xi} = \mathrm{colim}\ \xi \otimes b \longrightarrow \mathrm{colim}\ \xi \otimes a = Y^{\xi \wedge (-T_f)}$$

Umkehr map on twisted R-coh.

"M-branes"

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\nu"] & BSpin \arrow[d, "\lambda"] \\ X \arrow[r, dashed, "J"'] & K(\mathbb{Z},4) \end{tikzcd}$$

get an anchor map in twisted tmf.

Equivariant tmf?

Cannot construct $B\,GL_1\,tmf_{S^1}$ yet

S^1-equivariant

S^1-equivariant Elliptic coh E

C = elliptic curve

$$\begin{tikzcd} V \circlearrowleft S^1 \arrow[d, "\text{SU-bundle}"] \\ X \circlearrowleft S^1 \end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{L}(v)\otimes I(v) \arrow[r] \arrow[d] & \mathcal{L}(c_2)\otimes I(\sigma) \arrow[d] \\
X_{E_{S^1}(X)} \arrow[u, bend left, dashed, "\sigma(v)"] \arrow[r] & X_{E_{S^1}(X^\vee)}
\end{tikzcd}$$

derived scheme

$$X_{E_{S^1}(X^\vee)} \cong T^\vee \otimes_{\mathbb{Z}} C \,/\, W$$

$\mathbb{Z}^n$

Weyl gp

$\mathcal{L}(V) \rightsquigarrow$ twists of

$E_{S^1}(X)$ by $c_2^{S^1}(V)$

$$\sigma(V) : I(V) \cong \mathcal{L}(V)^{-1}$$

$$\Gamma I(V) \cong E_{S^1}(X^\vee)$$

Ando-Greenlees

$c_2^{s'} = 0 \implies$ get equivariant σ-orientation