EXERCISES ON HOMOTOPY COLIMITS

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1. Categorical homotopy theory

Let **Cat** denote the category of small categories and **S** the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{Set})$ of simplicial sets. Let's recall some useful notation. We write $[n]$ for the poset

$$[n] = \{0 < 1 < \ldots < n\}.$$

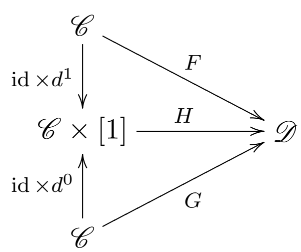
We may regard $[n]$ as a category with objects $0, 1, \ldots, n$ and a unique morphism $a \to b$ iff $a \leq b$. The category Δ of ordered nonempty finite sets can then be realized as the full subcategory of **Cat** with objects $[n]$, $n \geq 0$. The *nerve* of a category $\mathscr{C}$ is the simplicial set $\mathrm{N}\,\mathscr{C}$ with n-simplices the functors $[n] \to \mathscr{C}$. Write $\Delta[n]$ for the representable functor $\Delta(-, [n])$. Since $\Delta[0]$ is the terminal simplicial set, we'll sometimes write it as $*$.

Exercise 1.1. Show that the nerve functor $\mathrm{N} : \mathbf{Cat} \to \mathbf{S}$ is fully faithful.

Exercise 1.2. Show that the natural map $\mathrm{N}(\mathscr{C} \times \mathscr{D}) \to \mathrm{N}\,\mathscr{C} \times \mathrm{N}\,\mathscr{D}$ is an isomorphism. (Here, $\times$ denotes the categorical product in **Cat** and **S**, respectively.)

Exercise 1.3. Suppose $\mathscr{C}$ and $\mathscr{D}$ are small categories.

(1) Show that a natural transformation H between functors $F, G : \mathscr{C} \to \mathscr{D}$ is the same as a functor H filling in the diagram



(2) Suppose that F and G are functors $\mathscr{C} \to \mathscr{D}$ and that $H : F \to G$ is a natural transformation. Show that $\mathrm{N}\,F$ and $\mathrm{N}\,G$ induce homotopic maps $\mathrm{N}\,\mathscr{C} \to \mathrm{N}\,\mathscr{D}$.

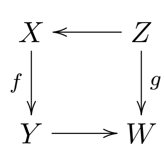
Exercise 1.4. (1) Suppose

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : G$$

is an adjoint pair. Show that $\mathrm{N}\,\mathscr{C}$ and $\mathrm{N}\,\mathscr{D}$ are weakly equivalent simplicial sets via the maps $\mathrm{N}\,F$ and $\mathrm{N}\,G$.

(2) Show that if $\mathscr{C}$ has an initial or terminal object, then $\mathrm{N}\,\mathscr{C}$ is weakly equivalent to a point.

Suppose $\mathscr{C}$ is a small category. The *twisted arrow category* $a\mathscr{C}$ of $\mathscr{C}$ is a category with objects the arrows of $\mathscr{C}$. The maps $f \to g$ are factorizations of g through f, i.e., diagrams



Note that source induces a functor $s : a\mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ and target induces a functor $t : a\mathscr{C} \to \mathscr{C}$.

Exercise 1.5. Show that the functors $s : a\mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ and $t : a\mathscr{C} \to \mathscr{C}$ both have adjoints. Conclude that there is a zig-zag of weak equivalences joining $\mathrm{N}\,\mathscr{C}$ and $\mathrm{N}\,\mathscr{C}^{\mathrm{op}}$.

2. Geometric properties of nerves

For details on the material in this section, see [GJ99]. Recall that $\Delta[n]$ is the representable presheaf $[m] \mapsto \Delta([m],[n])$, i.e., the standard n-simplex. For an arbitrary simplicial set X, the Yoneda lemma gives a natural bijective correspondence between the set X_n and maps $\Delta[n] \to X$. Let's define the *n-skeleton* of X to be the sub-simplicial set $\mathrm{sk}_n X$ with k-simplices given by

$$(\mathrm{sk}_n X)_k = \big\{ g : \Delta[k] \to X \;\big|\; g \text{ factors as } \Delta[k] \to \Delta[\ell] \to X \text{ for some } \ell \le n \big\}.$$

Note that sk_n is a functor equipped with a natural monomorphism $\mathrm{sk}_n \to \mathrm{id}$. We define the n-coskeleton of X to be the simplicial set $\mathrm{ck}_n X$ with k-simplices

$$(\mathrm{ck}_n X)_k = \mathbf{S}(\mathrm{sk}_n \Delta[k], X).$$

Note that there is a natural map $X \to \mathrm{ck}_n X$ induced by the maps $\mathrm{sk}_n \Delta[k] \to \Delta[k]$.

Exercise 2.1. Suppose that $\mathscr{C}$ is a small category. Check that $\mathrm{N}\,\mathscr{C}$ is 2-*coskeletal*, i.e., that the natural map $\mathrm{N}\,\mathscr{C} \to \mathrm{ck}_2 \mathrm{N}\,\mathscr{C}$ is an isomorphism of simplicial sets.

Let's fix some more notation. For $n \ge 1$, let $\partial\Delta[n] = \mathrm{sk}_{n-1} \Delta[n]$. This is the boundary of the standard n-simplex. For $0 \le i \le n$, we define $\Lambda^i[n]$ to be the horn

$$(\Lambda^i[n])_k = \big\{ g : \Delta[k] \to \Delta[n] \;\big|\; g \text{ factors as } \Delta[k] \to \Delta[n-1] \xrightarrow{d^j} \Delta[n] \text{ for some } j \ne i \big\}.$$

Here $d^j : [n-1] \to [n]$ is the unique monomorphism omitting j in the image. The simplicial set $\Lambda^i[n]$ is the union of the $n-1$-faces of $\Delta[n]$, omitting the ith face. Recall that a map $p : X \to Y$ of simplicial sets is a *Kan fibration* if in every diagram

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & X \\ \downarrow & \overset{\ell}{\nearrow} & \downarrow p \\ \Delta[n] & \longrightarrow & Y, \end{array}$$

a lift ℓ exists (not necessarily unique). A simplicial set X is a *Kan complex* if $X \to *$ is a Kan fibration.

Exercise 2.2. Suppose $\mathscr{C}$ is a small category.

(1) Prove that in all diagrams of the shape

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & \mathrm{N}\,\mathscr{C} \\ \downarrow & \overset{\ell}{\nearrow} & \\ \Delta[n] & & \end{array} \tag{2.1}$$

with $0 < i < n$, a lift ℓ exists. (Thus $\mathrm{N}\,\mathscr{C}$ is a *quasicategory*—see [Lur06, Joy06]).

(2) If, furthermore, $\mathscr{C}$ is a groupoid, show that $\mathrm{N}\,\mathscr{C}$ is a Kan complex (i.e., a lift ℓ exists in (2.1) for $0 \le i \le n$).

For a pointed Kan complex (X, x_0), the nth homotopy group $\pi_n(X, x_0)$ is the collection of $\Delta[1]$-homotopy classes of maps $(\Delta[n], \partial\Delta[n]) \to (X, x_0)$. Geometric realization induces an isomorphism $\pi_n(X, x_0) \to \pi_n(|X|, |x_0|)$.

Exercise 2.3. Suppose $\mathscr{G}$ is a small groupoid. Show that $\pi_k \mathscr{G} = 0$ if $k > 1$.

Exercise 2.3 fails spectacularly if $\mathscr{G}$ is not a groupoid: in Section 3, we'll show (modulo a technical lemma) that every homotopy type in $\mathbf{S}$ contains the nerve of a category.

Exercise 2.4. Give an example of a category $\mathscr{C}$ so that $\pi_k \left| \mathrm{N}\mathscr{C} \right| \neq 0$ for some $k > 1$.

3. The bar resolution and homotopy colimits

Let's write $\mathbf{S}^{(2)}$ for the category of *bisimplicial sets*, i.e., the category $\mathrm{Fun}(\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}, \mathbf{Set})$. We define a *diagonal* functor $\mathrm{diag} : \mathbf{S}^{(2)} \to \mathbf{S}$ given by restriction along the diagonal $\Delta \to \Delta \times \Delta$. By adjunction, we may view bisimplicial sets as simplicial objects in the category $\mathbf{S}$, i.e., as functors $\Delta^{\mathrm{op}} \to \mathbf{S}$. We'll usually take this point of view.

Exercise 3.1. Show that there is a natural isomorphism

$$\mathrm{diag}\, X_{\cdot\cdot} \cong \int^{n \in \Delta} X_n \times \Delta[n]$$

for $X_{\cdot\cdot} \in \mathbf{S}^{(2)}$. Here, $\int^{n\in\Delta}$ denotes the *coend:* it is the disjoint union $\amalg_{n,n} X_n \times \Delta[n]$ modulo the relation $(f^*x, y) \sim (x, f_*y)$ for $x \in X_m$, $y \in \Delta[n]$, and $f : [n] \to [m]$ an arrow in Δ (see [ML98]).

We'll recall the following result without proof:

Theorem 3.2. *Suppose X and Y are bisimplicial sets and $f : X \to Y$ a map which induces weak equivalences $X_n \to Y_n$ for all $[n] \in \Delta$. (As above, let's view X and Y as simplicial objects in $\mathbf{S}$). Then* $\mathrm{diag}\, f$ *induces a weak equivalence* $\mathrm{diag}\, X \to \mathrm{diag}\, Y$.

Suppose $\mathscr{I}$ is a small category and $F : \mathscr{I}^{\mathrm{op}} \to \mathbf{S}$ and $X : \mathscr{I} \to \mathbf{S}$ are diagrams. The *bar resolution* $B.(F, \mathscr{I}, X)$ is the bisimplicial set whose simplicial set of n-simplices is

$$\coprod_{i:[n]\to\mathscr{I}} F(i(n)) \times X(i(0)).$$

This coproduct is taken over all functors $i : [n] \to \mathscr{I}$. The simplicial operators are given by their action on the domain of i together with the functoriality of F and X. For example, recall that $d^1 : [0] \to [1]$ is the map sending 0 to 0 (omitting 1 in the image). Given a functor $i : [1] \to \mathscr{I}$, we send the i summand $F(i(1)) \times X(i(0))$ to the inclusion of the summand $i \circ d^1 : [0] \to \mathscr{I}$ with value $F(i(0)) \times X(i(0))$—since F is covariant, the map $i(0) \to i(1)$ induces the required map $F(i(1)) \to F(i(0))$. The *homotopy colimit* of X is the simplicial set

$$\operatorname*{hocolim}_{\mathscr{I}} X = \mathrm{diag}\, B.(*, \mathscr{I}, X).$$

Here $*$ is the constant diagram on the simplicial set $*$.

Exercise 3.3. Show that $\mathrm{hocolim}_{\mathscr{I}} *$ is weakly equivalent to the nerve $\mathrm{N}\,\mathscr{I}$.

Exercise 3.4. Show that there is a natural augmentation $\mathrm{hocolim}_{\mathscr{I}} X \to \mathrm{colim}_{\mathscr{I}} X$.

Exercise 3.5. Suppose that $f : X \to X'$ is a natural transformation of diagrams $\mathscr{I} \to \mathbf{S}$ so that $f_i : X(i) \to X'(i)$ is a weak equivalence for all $i \in \mathrm{ob}\,\mathscr{I}$. Show that the induced map on homotopy colimits $\mathrm{hocolim}\, f : \mathrm{hocolim}\, X \to \mathrm{hocolim}\, X'$ is a weak equivalence.

We can view homotopy colimit as the derived functors of colimit. In fact, the realization of the bar resolution for X is the colimit of a canonical "cofibrant" resolution of X. There is an analogous story for homotopy limits using cosimplicial spaces. See, for example, [BK72] or [Hir03, DHKS04, Shu06] for more modern treatments. An important conceptual result is the following alternative description of the diagonal of a bisimplicial set, which I will cite without proof:

Theorem 3.6. *Suppose $X : \Delta^{\mathrm{op}} \to \mathbf{S}$ is a bisimplicial set. Then* $\mathrm{diag}\, X$ *and* $\mathrm{hocolim}_{\Delta^{\mathrm{op}}} X$ *are weakly equivalent.*

As promised, we also have the following result, which says that categories model all homotopy types in $\mathbf{S}$. This is part of a beautiful story linking the homotopy theory of spaces with all abstract homotopy theories ("model categories"). Cisinski's dissertation [Cis06] along with [Mal05] are wonderful references.

Exercise 3.7. Suppose X is a simplicial set. The natural map

$$\operatorname*{hocolim}_{\Delta^n \to X} \Delta^n \to \operatorname*{colim}_{\Delta^n \to X} \Delta^n$$

is a weak equivalence (this is proved in, e.g., [Hir03] in the section on Reedy categories). Conclude that X is weakly equivalent to $\mathrm{N}(\Delta \downarrow X)$ by a zig-zag of weak equivalences.

4. Homotopy left Kan extensions and homotopy colimits

Suppose $F : \mathscr{I} \to \mathscr{J}$ is a functor between small categories and $X : \mathscr{I} \to \mathbf{S}$ is a $\mathscr{I}$-diagram of simplicial sets. The functor F induces an adjunction

$$F_! : \mathbf{S}^{\mathscr{I}} \rightleftarrows \mathbf{S}^{\mathscr{J}} : F^*$$

between categories of diagrams, where the right adjoint F^* is given by restriction along F and the left adjoint $F_!$ is left Kan extension [ML98]. When $\mathscr{I}$ and $\mathscr{J}$ are groups, this is simply induction. We can compute $F_!$ as follows. Given $j \in \mathscr{J}$, we let $F \downarrow j$ be the comma category with objects pairs $i \in \mathscr{I}$, $\varphi : Fi \to j$ and morphisms $(i, \varphi) \to (i', \varphi')$ given by arrows $h : i \to i'$ making

$$\begin{array}{ccc} Fi & \xrightarrow{Fh} & Fi' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & j & \end{array} \tag{4.1}$$

commute. There is a projection functor $\pi : F \downarrow j \to \mathscr{I}$ forgetting the map to j. Then

$$(F_! X)(j) \cong \operatorname*{colim}_{F \downarrow j} \pi^* X.$$

Let's define a homotopy-invariant version of $F_!$. Note that for all $j \in \mathscr{J}$, there is an $\mathscr{I}^{\mathrm{op}}$-diagram of sets sending i to $\mathscr{J}(Fi, j)$. We may regard this as a diagram of constant simplicial sets. The *homotopy left Kan extension of X along F* is the $\mathscr{J}$-diagram

$$(\mathbf{L}F_! X)(j) = \operatorname{diag} B.\big(\mathscr{J}(F-, j), \mathscr{I}, X\big).$$

Note that if $\mathscr{J}$ is the terminal category, then $\mathbf{L}F_!$ is simply $\operatorname{hocolim}_{\mathscr{I}}$. At the other extreme, if F is the identity functor $\mathscr{I} \to \mathscr{I}$, there is a natural augmentation

$$\operatorname{diag} B.(\mathscr{I}(-, i), \mathscr{I}, F) \to F(i)$$

given by iterated composition; it induces a weak equivalence $\mathbf{L}\,\mathrm{id}_! X \to X$.

Exercise 4.1. Show that $\mathbf{L}F_!$ is homotopy-invariant, i.e., that if $f : X \to X'$ induces a weak equivalence $f_i : X(i) \to X'(i)$ for all $i \in \mathscr{I}$, then $(\mathbf{L}F_! f)(j)$ is a weak equivalence for all $j \in \mathscr{J}$.

Exercise 4.2. Note that $\pi : F \downarrow j \to \mathscr{I}$ induces a homotopy functor $\mathbf{S}^{\mathscr{I}} \to \mathbf{S}^{F \downarrow j}$. Check that the derived version of 4.1 holds, i.e., that there is a weak equivalence

$$(\mathbf{L}F_! X)(j) \simeq \operatorname*{hocolim}_{F \downarrow j} \pi^* X$$

(compare [Cis03]).

5. Thomason's theorem

In this section, we will prove a generalization of Thomason's theorem for homotopy colimits in **Cat** [Tho79]. We'll take the following two results as black boxes. See [Hir03] for a reference or ask me. In Section 6 below, we'll prove a special case of Theorem 5.1.

Theorem 5.1 ([Hir03, Theorem 19.6.7 (a)])**.** *Suppose $F : \mathscr{I} \to \mathscr{J}$ is a* homotopy right cofinal functor*, i.e., that $N(j \downarrow F)$ is (weakly) contractible for all $j \in \mathscr{J}$. If X is a diagram $\mathscr{J} \to \mathbf{S}$, then F induces a weak equivalence*

$$\operatorname*{hocolim}_{\mathscr{I}} F^* X \to \operatorname*{hocolim}_{\mathscr{J}} X$$

of homotopy colimits.

Theorem 5.2. *Suppose*

$$\mathscr{I} \xrightarrow{F} \mathscr{J} \xrightarrow{G} \mathscr{K}$$

are functors between small categories. There is a weak equivalence $\mathbf{L}G_!\mathbf{L}F_!X \to \mathbf{L}(GF)_!X$ natural in diagrams $X \in \mathbf{S}^{\mathscr{I}}$.

The weak equivalence in Theorem 5.2 has a brief description: there is a natural augmentation

$$\operatorname{diag} B.\big(\mathscr{K}(-,k), \mathscr{J}, \operatorname{diag} B.(\mathscr{J}(-,-), \mathscr{I}, X)\big) \to \operatorname{diag} B.(\mathscr{K}(-,k), \mathscr{I}, X).$$

This map realizes $\mathbf{L}G_!\mathbf{L}F_!X \to G_!\mathbf{L}F_!X$—the latter functor is isomorphic to $\mathbf{L}(GF)_!X$, and the map is a weak equivalence.

Exercise 5.3. Suppose that $G : \mathscr{I} \to \mathscr{J}$ is a right adjoint. Show that G is homotopy right cofinal.

Suppose $F : \mathscr{I} \to \mathbf{Cat}$ is a functor. The *Grothendieck construction of F* is a category $\mathscr{I} \int F$ whose objects are pairs (i, x) with $i \in \mathscr{I}$ and $x \in F(i)$. Maps $(i, x) \to (i', x')$ are pairs of maps $f : i \to i'$ and $\varphi : F(f)(x) \to x'$. (The latter is an arrow in $F(i')$.) Composition is forced upon us; see [Tho79] for the details. Note that there is a projection functor $\Pi : \mathscr{I} \int F \to \mathscr{I}$ given by forgetting x. Think of Π as a sort of fibration displaying Fi as the fiber over $\mathscr{I}$ (our terminology here is somewhat backwards). In the following exercise, we'll make use of the comma category $\Pi \downarrow j$. We'll abuse notation a bit and regard the objects of $\Pi \downarrow j$ as pairs $i \to j$, $x \in F(i)$.

Exercise 5.4. Suppose that $j \in \mathscr{I}$.

1. There is a functor $h : \Pi \downarrow j \to F(j)$ sending the data $(f : i \to j, x \in F(i))$ to $F(f)(x)$. Show how to define h on maps to actually make it a functor.
2. We can define a functor $\ell : F(j) \to \Pi \downarrow j$ sending $x \in F(j)$ to (id_j, x). Check that ℓ is left adjoint to h. Conclude that h is homotopy right cofinal.

Exercise 5.5 (Thomason's theorem)**.** Suppose $X : \mathscr{I} \int F \to \mathbf{S}$ is a diagram of simplicial sets.

1. Show that $\operatorname{hocolim}_{\mathscr{I} \int F} X \simeq \operatorname{hocolim}_{\mathscr{I}} \mathbf{L}\Pi_! X$.
2. Show that there is a natural weak equivalence $(\mathbf{L}\Pi_! X)(i) \simeq \operatorname{hocolim}_{F(i)} X$. Note that we may restrict X to a diagram on $F(i)$ by the functor $F(i) \to \mathscr{I} \int F$ sending $x \in F(i)$ to (i, x).
3. Combine these two results to show that $\operatorname{hocolim}_{\mathscr{I} \int F} X \simeq \operatorname{hocolim}_{i \in \mathscr{I}} \operatorname{hocolim}_{F(i)} X$.
4. Show that $\mathrm{N}(\mathscr{I} \int F) \simeq \operatorname{hocolim}_{i \in \mathscr{I}} \mathrm{N}\, F(i)$.

In Exercise 5.5, part 4 is what's usually known as Thomason's theorem for homotopy colimits in **Cat**. The generalization in part 3 is found in, e.g., [CS02].

6. Quillen's Theorem A

In this section we'll prove the following theorem.

Theorem 6.1 ([Qui73, Theorem A])**.** *Suppose $F : \mathscr{C} \to \mathscr{D}$ is a homotopy right cofinal functor. That is, for all $d \in \mathscr{D}$, the simplicial set $d \downarrow F$ is weakly contractible. Then $\mathrm{N}\, F : \mathrm{N}\,\mathscr{C} \to \mathrm{N}\,\mathscr{D}$ is a weak equivalence.*

I am unable to improve on Quillen's excellent exposition in [Qui73]. *Our proof will follow his paper exactly.* Of course, we could apply Theorem 5.1 to the constant $\mathscr{D}$-diagram on $*$ to obtain Theorem 6.1. Actually, to obtain part 4 of Exercise 5.5, Quillen's Theorem A is sufficient. Recall our definition of the Grothendieck construction in Section 5. The construction of the comma category $d \downarrow F$ is functorial in $d \in \mathscr{D}$, i.e., there is a functor $\widetilde{F} : \mathscr{D}^{\mathrm{op}} \to \mathbf{Cat}$ sending $d \in \operatorname{ob} \mathscr{D}$ to $d \downarrow F$. Given a map $j : d' \to d$, we define

$$\widetilde{F}(j)(c, \varphi : d \to Fc) = (c, \varphi \circ j).$$

Let $S(F) = \mathscr{D}^{\mathrm{op}} \int F$.

Exercise 6.2. Verify the following description of $S(F)$: objects are triplets (c, d, φ) with $c \in \mathscr{C}$, $d \in \mathscr{D}$, and $\varphi : d \to Fc$. Arrows $(c, d, \varphi) \to (c', d', \varphi')$ are pairs of arrows $j : d' \to d$, $i : c \to c'$ so that $F(i) \circ \varphi \circ j = \varphi'$.

Note that $S(F)$ is equipped with functors $\pi_{\mathscr{D}} : S(F) \to \mathscr{D}^{\mathrm{op}}$ (because it is a Grothendieck construction) and $\pi_{\mathscr{C}} : S(F) \to \mathscr{C}$ (sending (c, d, φ) to c). Define a bisimplicial set $T(F)$ with

$$T(F)_{p,q} = \big\{(\alpha, \beta, f) \bigm| \alpha : [p] \to \mathscr{D}^{\mathrm{op}}, \beta : [q] \to \mathscr{C}, f : \alpha(0) \to F(\beta(0))\big\}.$$

In the following exercise, we'll compute the homotopy type of $NT(F)$ in three ways: by computing its diagonal directly and then by viewing it as a simplicial object in $\mathbf{S}$ in two ways. Recall that realization and the diagonal functor coincide (Exercise 3.1 and Theorem 3.2).

Exercise 6.3.
(1) Check that $\operatorname{diag} T(F) \cong \mathrm{N}\, S(F)$.
(2) Check that for fixed p, $T(F)_p$ is the simplicial set

$$\coprod_{\alpha : [p] \to \mathscr{D}^{\mathrm{op}}} \mathrm{N}(\alpha(0) \downarrow F).$$

Thus $T(F)$ is the bar resolution $p \mapsto B_p(*, \mathscr{D}^{\mathrm{op}}, \mathrm{N}(- \downarrow F))$. Use the fact that F is right homotopy cofinal to show that realization in the p-direction (i.e., the diagonal) induces a weak equivalence $N\pi_{\mathscr{D}} : \mathrm{N}\, S(F) \to \mathrm{N}\, \mathscr{D}^{\mathrm{op}}$.
(3) Check that for fixed q, $T(F)_q$ is the simplicial set

$$\coprod_{\beta : [q] \to \mathscr{C}} \mathrm{N}(\mathscr{D} \downarrow F\beta(0)).$$

Show that realization in the q-direction (again, the diagonal) induces a weak equivalence $\mathrm{N}\, \pi_{\mathscr{C}} : \mathrm{N}\, S(F) \to \mathrm{N}\, \mathscr{C}$.

Note that we may define a functor $\widetilde{\mathrm{id}_{\mathscr{D}}} : \mathscr{D}^{\mathrm{op}} \to \mathbf{Cat}$ sending d to $d \downarrow \mathscr{D}$. The functor F induces a natural transformation $\widetilde{F} \to \widetilde{\mathrm{id}_{\mathscr{D}}}$ and hence a functor $F' : S(F) \to S(\mathrm{id}_{\mathscr{D}})$.

Exercise 6.4.
(1) Show that relative to the description of $S(F)$ and $S(\mathrm{id})$ in Exercise 6.2, the functor F' sends (c, d, φ) to the triplet (Fc, d, φ).
(2) Show that $S(\mathrm{id}_{\mathscr{D}})$ is the twisted arrow category $a\mathscr{D}$.

(3) Show that the diagram

$$\begin{array}{ccccc}
\mathscr{D}^{\mathrm{op}} & \xleftarrow{\pi_{\mathscr{D}}} & S(F) & \xrightarrow{\pi_{\mathscr{C}}} & \mathscr{C} \\
\| & & \downarrow{\scriptstyle F'} & & \downarrow{\scriptstyle F} \\
\mathscr{D}^{\mathrm{op}} & \xleftarrow{\pi_{\mathscr{D}}} & S(\mathrm{id}_{\mathscr{D}}) & \xrightarrow{\pi_{\mathscr{C}}} & \mathscr{D}
\end{array}$$

commutes. Conclude that F induces a weak equivalence on nerves.

References

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations.* Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin, 1972.

[Cis03] Denis-Charles Cisinski. Images directes cohomologiques dans les catégories de modèles. *Ann. Math. Blaise Pascal*, 10(2):195–244, 2003.

[Cis06] Denis-Charles Cisinski. Les préfaisceaux comme modèles des types d'homotopie. *Astérisque*, (308):xxiv+390, 2006.

[CS02] Wojciech Chachólski and Jérôme Scherer. Homotopy theory of diagrams. *Mem. Amer. Math. Soc.*, 155(736):x+90, 2002.

[DHKS04] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. *Homotopy limit functors on model categories and homotopical categories*, volume 113 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2004.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics.* Birkhäuser Verlag, Basel, 1999.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2003.

[Jar06] J. F. Jardine. Categorical homotopy theory. *Homology, Homotopy Appl.*, 8(1):71–144 (electronic), 2006.

[Joy06] André Joyal. The theory of quasi-categories I. in preparation, 2006.

[Lur06] Jacob Lurie. Higher Topos Theory. *arXiv:math/0608040v4 [math.CT]*, 2006.

[Mal05] Georges Maltsiniotis. La théorie de l'homotopie de Grothendieck. *Astérisque*, (301):vi+140, 2005.

[ML98] Saunders Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1998.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[Shu06] Michael Shulman. Homotopy limits and enriched homotopy theory. *arXiv:math/06101942v2 [math.AT]*, 2006.

[Tho79] R. W. Thomason. Homotopy colimits in the category of small categories. *Math. Proc. Cambridge Philos. Soc.*, 85(1):91–109, 1979.