8-Derived functors and Homotopy (co)limits

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Derived functors

$\mathcal{C} \supset \mathcal{W}$ subcat of "weak equivalences" (satisfies 2 out of 3)

$Ho(\mathcal{C}) := \mathcal{C}[\mathcal{W}^{-1}]$ (if it exists)

$F : \mathcal{C} \longrightarrow \mathcal{D}$ ($\mathcal{D}$ is typically the homotopy category of some other category)

Right derived functor

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\iota"'] & \mathcal{D} \\ Ho(\mathcal{C}) \arrow[ur, dashed, "RF"'] \arrow[u, phantom, ""] & \end{tikzcd}$$

$F \xrightarrow{\eta} RF \circ \iota$

Given any other G, $\mu : F \to G \circ \iota$

$\exists\, \delta : RF \to G$

s.t.

$$\begin{tikzcd} & F \arrow[dl, "\eta"'] \arrow[dr, "\mu"] & \\ RF \circ \iota \arrow[rr, "\delta \circ \iota"'] & & G \circ \iota \end{tikzcd}$$

Left derived functor

... $LF \to F$

Typical implementation:

Find subcat $\mathcal{C}'$ of $\mathcal{C}$

$\mathcal{C} \xrightarrow{\rho} \mathcal{C}'$

$Id \longrightarrow \rho$

$F(c)$

$$\begin{tikzcd} F(x) \arrow[d, "\eta"] \\ RF(x) = F(\rho(X)) \end{tikzcd}$$

$\mathrm{Id} \longrightarrow \rho$

levelwise w.e.

$F(\rho(f))$ is an iso $\forall f \in W$

(e.g. if $F|_{\mathcal{C}'}$ sends w.e. to isos)

Clearly

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \\ Ho(\mathcal{C}) \arrow[ur, "RF"'] & \end{tikzcd}$$

$$\begin{tikzcd} & F(x) \arrow[dl] \arrow[dr, "\mu_x"] & \\ F(\rho(x)) \arrow[rr, bend right] \arrow[dr, "\mu_{\rho(x)}"'] & & G(L(x)) \arrow[dl, "\cong"] \\ & G(L(\rho x)) & \end{tikzcd}$$

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e.g.

$$\begin{tikzcd} Sp \arrow[r, "\Omega^\infty"] \arrow[d] & Ho(Top.) \\ Ho(Sp) \arrow[ur, dashed, "R\Omega^\infty"'] & \end{tikzcd}$$

$$R\Omega^\infty X = \Omega^\infty(\omega X)$$

(Convention
Ω^∞, F, $\wedge$
always derived)

<u>Homotopy limits</u>

w.e. = levelwise w.e..

$$\begin{tikzcd} \mathcal{C}^I \arrow[r, "\lim"] \arrow[d] & \mathcal{C} \arrow[r] & Ho(\mathcal{C}) \\ Ho(\mathcal{C}^I) \arrow[urr, dashed] & & \end{tikzcd}$$

$$R\varprojlim = \operatorname{holim}$$

e.g. <u>Telescope</u>

$$\bigvee X_i \longrightarrow \bigvee X_i \longrightarrow \mathrm{Tel}\, X_i$$

$$\begin{tikzcd} \bigvee X_i \arrow[r] & \bigvee X_j \arrow[r] \arrow[d] & \mathrm{Tel}\, X_i \arrow[dl, dashed] \\ & \varinjlim X_i & \end{tikzcd}$$

If $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \cdots$ sequence of CW inclusions, $\mathrm{Tel}(X_i) \simeq \varinjlim X_i$

Since every $\{X_j\}$ admits such a CW replacement

$\Longrightarrow$ $\mathrm{Tel}(X_i) \simeq \underrightarrow{\mathrm{holim}}\, X_i$ in general.

Note:

$$0 \to \varprojlim{}^1 [\Sigma X_i, Z] \to [\underrightarrow{\mathrm{holim}}\, X_j, Z] \to \varprojlim [X_i, Z] \to 0$$

Similarly for $\underleftarrow{\mathrm{holim}}$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & P \end{tikzcd}$$

$\left(A \to X \vee Y \xrightarrow{\text{cofiber}} P \right)$

"Mayer-Vietoris squares"

$* \circlearrowright G$ $\quad Sp^G = Sp_G$

$\underline{G}$

$$R(-)^G =: (\;)^{hG}$$

Lemma $X \xrightarrow{f} Y$ $\underline{\underline{G\text{-equivariant map}}}$ underlying w.e. $\Longrightarrow$ $\underline{\mathrm{Map}}_G(K_+, X) \to \underline{\mathrm{Map}}_G(K_+, Y)$ $\quad K =$ free G-CW cx equivalence.

(pf)

$$\underline{\mathrm{Map}}_G(\bigvee G_+ \wedge S^i, X) \leftarrow \underline{\mathrm{Map}}_G(K_+^{[i]}, \ ) \longleftarrow \underline{\mathrm{Map}}_G(K_+^{[i-1]} \ \ )$$

$\square$

Cor $\quad X \xrightarrow{\simeq} X^{EG_+} \qquad X^{hG} = (X^{EG_+})^G$

Similarly $\quad X_{hG} \simeq (X \wedge EG_+)_{hG} \qquad (X \wedge EG_+ \xrightarrow{\simeq} X)$

General Construction

Special case $\quad I = \ 0 \leftleftarrows 1 \ \cdots \ 2 \ \cdots \ - - - -$

$= \Delta_i^{op} =$

$$\Delta_i^{op} \to Sp$$
$$\parallel$$

$X_\bullet$ = semi-simplicial spectrum.

$$|X_\bullet| = \varinjlim |X_\bullet|^{[k]}$$

$$\begin{tikzcd}
X_k \wedge \partial\Delta_+^k \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & {|X_\bullet|^{[k-1]}} \arrow[d] \\
X_k \wedge \Delta_+^k \arrow[r] & {|X_\bullet|^{[k]}}
\end{tikzcd}$$

Note: $X_\bullet$ levelwise CW spectra

$\Longrightarrow \quad |X_\bullet| = \underset{k}{\underrightarrow{\mathrm{hocolim}}}\, X_k$

implicitly assume $|X_\bullet|$ "derived"

$X_\bullet \rightsquigarrow \tilde{X}_\bullet$

check that

$X_\bullet \longrightarrow Y_\bullet \quad$ levelwise weq. of CW spectra

Tot $X_\bullet$ ' - - - similar.

$\Rightarrow |X_\bullet| \xrightarrow{\simeq} |Y_\bullet|$

use of CW spaces

Intuition! X_i a diagram

$$\varinjlim X_i \leftarrow \bigvee_{i_0 \in I} X_{i_0} \leftleftarrows \bigvee_{\substack{i_0 \to i_1 \\ \text{in } I}} X_{i_0}$$

$$X_\bullet = \bigvee_{i_0 \in I} X_{i_0} \leftleftarrows \bigvee_{\substack{i_0 \to i_1 \\ \text{in } I}} X_{i_0} \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \bigvee_{i_0 \to i_1 \to i_2} X_{i_0} \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \cdots$$

Thm (B-K)

$$\operatorname{hocolim}_{\longrightarrow} X_\bullet \simeq |X_\bullet|$$

Similar for $\operatorname{holim}_{\longleftarrow}$ - - -