

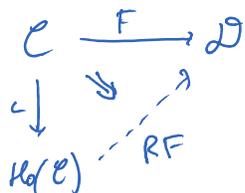
Derived functors

$\mathcal{C} \supset \mathcal{W}$ subset of "weak equivalences" (satisfy 2 out of 3)

$H_0(\mathcal{C}) := \mathcal{C}[\mathcal{W}^{-1}]$ (if it exists)

$F: \mathcal{C} \rightarrow \mathcal{D}$ is typically the homotopy category of some other category

Right derived functor

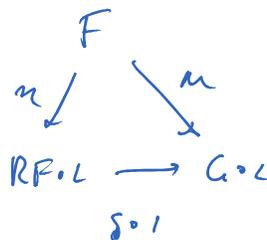


$F \xrightarrow{\eta} RF \circ L$

Given any other $G, \mu: F \rightarrow G \circ L$

$\exists \delta: RF \rightarrow G$

s.t.



Left derived functor

$LF \rightarrow F$

Typical implementation:

Find subcategory \mathcal{C}' of \mathcal{C}

$\mathcal{C} \xrightarrow{p} \mathcal{C}'$

$Id \xrightarrow{\quad} p$
 \uparrow

$$F(x) \downarrow \alpha$$

$$RF(x) = F(\rho(x))$$

$$Id \rightarrow \rho$$

↙ homotopy v.e.

$F(\rho(f))$ is an iso $\forall f \in W$
 (e.g. if $F|_{C'}$ puts v.e. to isos)

Clearly

$$C \rightarrow D$$

$$\downarrow \quad \nearrow RF$$

$$Ho(C) \rightarrow Ho(D)$$

$$F(x) \begin{matrix} \swarrow \\ \downarrow \\ \searrow \end{matrix} \begin{matrix} F(\rho(x)) \\ \\ G(L(\rho(x))) \end{matrix}$$

$$\begin{matrix} \downarrow \mu_x \\ G(L(x)) \end{matrix}$$

$$\begin{matrix} \swarrow \mu_{\rho(x)} \\ G(L(x)) \end{matrix}$$

$$\cong$$

e.g.

$$S_p \xrightarrow{\Omega^\infty} Ho(Top.)$$

$$\downarrow \quad \nearrow R\Omega^\infty$$

$$Ho(S_p) \rightarrow Ho(Top.)$$

$$R\Omega^\infty X = \Omega^\infty(\omega X)$$

Homotopy limits

M.e. = homotopy v.e.

$$C^I \xrightarrow{\lim} C \rightarrow Ho(C)$$

$$\downarrow \quad \nearrow$$

$$Ho(C^I) \xrightarrow{R\lim} Ho(C) = \text{holim}$$

(Counter
 Ω^∞, F, \wedge
 always derived)

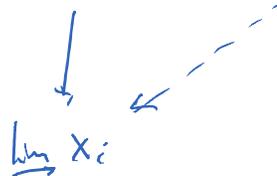
e.g.

Telescope

$$\bigvee X_i \rightarrow \bigvee X_0 \rightarrow \text{Tel } X_i$$

multiple

$$\bigvee X_i \longrightarrow \bigvee X_0 \longrightarrow \text{Tel } X_i$$



If $X_0 \hookrightarrow X_1 \hookrightarrow X_2 \dots$
 some of CW inclusions

$$\text{Tel}(X_i) = \varinjlim X_i$$

Show any $\{X_i\}$ admits such a CW replacement

$$\Rightarrow \text{Tel}(X_i) = \varinjlim X_i$$

in general,

Note

$$0 \longrightarrow \varprojlim [\Sigma X_i, Z] \longrightarrow [\varinjlim X_i, Z] \longrightarrow \varprojlim [X_i, Z] \longrightarrow 0$$

Similarly for \varprojlim

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & P \end{array} \quad \left(\begin{array}{c} \text{coker} \\ A \rightarrow X \vee Y \rightarrow P \end{array} \right)$$

"Mayer-Vietoris squares"

$$\ast \hookrightarrow G \quad Sp \stackrel{G}{=} Sp_G$$

G

$$R(-)^G =: (-)^{hG}$$

Lemma $X \rightarrow Y$
 f

G-constant map
 satisfying etc.

$k =$ free G -CW cpx

$$\Rightarrow \text{Map}_G(k_+, X) \longrightarrow \text{Map}_G(k_+, Y)$$

equivalence.

(pf)

$$\text{Map}_k(U_{G_+ \cap S^i}, X) \leftarrow \text{Map}_G(K_+^{[i]}, X) \leftarrow \text{Map}_G(K_+^{(i-1)}, X) \quad \square$$

Cor! $X \rightarrow X^{EG_+} \quad X^{hG} = (X^{EG_+})^G$

Stably $X_{hG} = (X \wedge EG_+)^{hG} \quad (X \wedge EG_+ \xrightarrow{\simeq} X)$

Good Constructors

Spectral are

$$I = 0 \leq 1 \leq 2 \leq \dots$$

" Δ_i^{op} "

$$\Delta_i^{op} \rightarrow Sp$$

X_0 = semi-simplicial spectra.

$$|X_0| = \varinjlim |X_0|^{[k]}$$

$$\begin{array}{ccc} X_k \wedge 2\Delta_+^k & \longrightarrow & |X_0|^{(k-1)} \\ \downarrow \lrcorner & & \downarrow \\ X_k \wedge \Delta_+^k & \longrightarrow & |X_0|^{[k]} \end{array}$$

Note! X_0 levelwise CW spectra

$$\Rightarrow |X_0| = \varinjlim_k \text{hocolim}_k X_k$$

implicitly assume $|X_0|$ "closed"

$$X_0 \rightsquigarrow \tilde{X}_0$$

check that

$$X_0 \longrightarrow \gamma_0$$

levelwise use of CW spectra

Let X_0 ... sub.

use
of the
spectra

$$\Rightarrow |X_0| \cong |Y_0|$$

Induction! X_i a diagram

$$\text{holim}_{\rightarrow} X_i \leftarrow \bigvee_{i \in I} X_{i_0} \rightleftarrows \bigvee_{\substack{i_0 \rightarrow i_1 \\ \in I}} X_{i_0}$$

$$X_0 = \bigvee_{i \in I} X_{i_0} \rightleftarrows \bigvee_{\substack{i_0 \rightarrow i_1 \\ \in I}} X_{i_0} \rightleftarrows \bigvee_{\substack{i_0 \rightarrow i_1 \rightarrow i_2 \\ \in I}} X_{i_0} \rightleftarrows \dots$$

Thus
(B-K) $\text{holim}_{\rightarrow} X_0 = |X_0|$

Spectra for holim ...