HOMEWORK 5

tues, 3/9/10

DUE: [redacted]

1. (Slightly modified version of Hatcher, Sec. 4.2, problem 31 [redacted]) For a fibration $F \to E \xrightarrow{p} B$ ($F = p^{-1}(*)$) such that the inclusion $F \to E$ is homotopic to a constant map, show that the long exact sequence of homotopy groups breaks up into split short exact sequences giving isomorphisms $\pi_n(B) \cong \pi_n(E) \oplus \pi_{n-1}(F)$. In particular, for the Hopf bundles $S^3 \to S^7 \to S^4$ and $S^7 \to S^{15} \to S^8$ this yields isomorphisms

$$\pi_n(S^4) \cong \pi_n(S^7) \oplus \pi_{n-1}(S^3)$$
$$\pi_n(S^8) \cong \pi_n(S^{15}) \oplus \pi_{n-1}(S^7)$$

Thus $\pi_7(S^4)$ and $\pi_{15}(S^8)$. contain $\mathbb{Z}$ summands.

2. (Hatcher, Sec. 4.2, problem 32 [redacted]) Show that if $S^k \to S^m \to S^n$ is a fiber bundle, then $k = n - 1$ and $m = 2n - 1$.

3. Show that there are fiber bundles

$$O(n-1) \to O(n) \to S^{n-1}$$
$$U(n-1) \to U(n) \to S^{2n-1}.$$

where $O(n)$ is the orthogonal group and $U(n)$ is the unitary group. Deduce that for fixed k the sequences

$$\pi_k(U(1)) \to \pi_k(U(2)) \to \pi_k(U(3)) \to \pi_k(U(4)) \to \cdots$$
$$\pi_k(O(1)) \to \pi_k(O(2)) \to \pi_k(O(3)) \to \pi_k(O(4)) \to \cdots$$

eventually stabilize. (The stable values of these homotopy groups is the subject of the celebrated "Bott periodicity theorem".)

4. (Hatcher, Sec. 4.2, problem 15 [redacted]) Show that a closed simply-connected 3-manifold is homotopy equivalent to S^3. [Use Poincaré duality, and also the fact that closed manifolds are homotopy equivalent to CW-complexes, from Corollary A.12 in the Appendix of Hatcher. The stronger statement that a closed simply-connected 3 manifold is homeomorphic to S^3 is [redacted] proven. This is the Poincaré conjecture.]