4 - Computations of homotopy groups

$\mathbb{R} \to S^1 \Rightarrow \pi_{>1} S^1 = 0$.

More generally, any space w/ a contractible universal cover has: $\pi_{>1} X = 0$

e.g. $\pi_k(T^2) = \begin{cases} \mathbb{Z} \times \mathbb{Z}, & k = 1 \\ 0, & o/w \end{cases}$

Alternatively use $T^2 = S^1 \times S^1$ and:

Prop

$$\pi_k(X_1 \times X_2) = \pi_k(X_1) \times \pi_k(X_2)$$

(pf) universal property of $- \times - \Rightarrow$

$$\{\alpha : S^n \to X_1 \times X_2\} \longleftrightarrow \{\alpha_i : S^n \to X_i\}$$

$$\{H : S^n \wedge I_+ \to X_1 \times X_2\} \longleftrightarrow \{H_i : S^n \wedge I_+ \to X_i\}$$

$\square$

In general

$\pi_4(S^4) = \mathbb{Z}$ (we will see this) Eventually

But $\pi_{>4}(S^4)$ is a mess.

(Table: p 339 of Hatcher)

π_* typically more difficult or sometimes impossible to compute than H_*

but a more powerful invariant

($\pi_* - \text{iso} \Longrightarrow H_* - \text{iso}$) we will prove

Easy exercise (in ANY category)

$X \in \mathcal{C}^I$

$$\mathrm{Map}_{\mathcal{C}}(\varinjlim X, Z) \cong \varprojlim \mathrm{Map}_{\mathcal{C}}(X, Z)$$

$$\mathrm{Map}_{\mathcal{C}}(Z, \varprojlim X) \cong \varprojlim \mathrm{Map}_{\mathcal{C}}(Z, X)$$

However $\varinjlim \mathrm{Map}_{\mathcal{C}}(Z, X_i) \longrightarrow \mathrm{Map}_{\mathcal{C}}(Z, \varinjlim X)$

is typically not an isomorphism

Lem Let $X_0 \to X_1 \to \cdots$

be a sequence of closed inclusions in Top

suppose K is cpt

$$\Longrightarrow \quad \varinjlim \mathrm{Map}(K, X_i) \longrightarrow \mathrm{Map}(K, \underbrace{\varinjlim X_i}_{\overset{\Vert}{\bigcup X_i}})$$

is an isomorphism.

Clearly an injection.

closed inclusion $\Rightarrow$ weak top.
(automatically w. Haus)

just need to check

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr, dashed] & \bigcup_i X_i \subset X \\ & X_j \arrow[u] \end{tikzcd} \quad , \quad \exists\, j$$

Now:

Suppose not. i_k increasing

$\Longrightarrow$ we can pick $x_{i_k} \in f(K) \cap \left(X_{i_k} - X_{i_k - 1}\right)$

Consider $C = \{x_{i_1}, x_{i_2}, \ldots\}$

$C \cap X_{i_k} = \{x_{i_1}, \ldots, x_{i_k}\}$ closed

$\Longrightarrow$ C closed

Consider $U_k = X - \{x_{i_{k+1}}, x_{i_{k+2}}, \ldots\}$

$\uparrow$ open

U_k covers $f(K)$, no finite subcover does.

$\to\leftarrow$ $\square$

Cor: $X_0 \to X_1 \to \cdots$ seq. closed inclusions

$$\varinjlim \pi_k(X_i) = \pi_k\left(\varinjlim X_i\right)$$

(pf) S^k, $S^k \times I$ cpt.

$\square$

Cor: $S^\infty = \varinjlim (S^1 \hookrightarrow S^2 \hookrightarrow S^3 \hookrightarrow \cdots)$

$\pi_k(S^\infty) = 0$

(pf) $\pi_k(\varinjlim_n S^n) = \varinjlim_n (\pi_k S^n)$

zero for $n > k$ $\square$

Cor: $\pi_k(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z}/2, & k = 1 \\ 0, & k \neq 1 \end{cases}$

(pf) $S^\infty \longrightarrow \mathbb{R}P^\infty$ universal 2-fold cover

$(S^n \longrightarrow \mathbb{R}P^n)$

$\Rightarrow$ $\pi_1 \mathbb{R}P^\infty = \mathbb{Z}/2$

$\pi_k \mathbb{R}P^\infty = \pi_k S^\infty = 0 \quad k \geq 2$

Rmk: $X \in \mathrm{Top}_*$ satisfies

$$\pi_k(X) = \begin{cases} \pi, & k = n \\ 0, & k \neq 1 \end{cases}$$

is called Eilenberg-Maclane "$K(\pi, n)$". We shall see these are unique up to homotopy. (always exist)

Action of fundamental gpoid

<u>Recall</u> $G = $ gp $\rightsquigarrow$ $\underline{G}$ cat

one object $*$

$\mathrm{Mor}(*,*) = G$

$$\{\text{gps}\} = \{\text{categories w/ one object, all morphisms are isomorphisms}\}$$

$$\cap| \qquad\qquad \cap|$$

$$\{\text{groupoids}\} = \{\text{categories where all morph are isos}\}$$

<u>Def:</u> $X \in \mathrm{Top}$: the <u>fundamental groupoid</u> $\pi_{\le 1} X$ is defined to be the groupoid

objects = points of X

$$\mathrm{Map}_{\pi_{\le 1} X}(x, y) = \left\{ \begin{array}{l} \text{paths } \gamma : I \to X \\ \gamma(0) = x \\ \gamma(1) = y \end{array} \right\} \Big/ \text{htpy rel } \partial I$$

Composition = path composition

$$\begin{tikzcd} x \arrow[r, "{[\gamma]}"'] \arrow[rr, bend left, "{[\gamma'\gamma]}"] & y \arrow[r, "{[\gamma']}"'] & z \end{tikzcd}$$

identity morph = const paths

Note: $\pi_{oid}(x,x) \cong \pi_1(X)$

$$\begin{array}{c} \pi_{oid}(x,x) \\ \psi \\ [\gamma : x \mapsto x] \end{array}$$

Rmk: depends on convention!

for me $\gamma_1, \gamma_2 \in \pi_1$

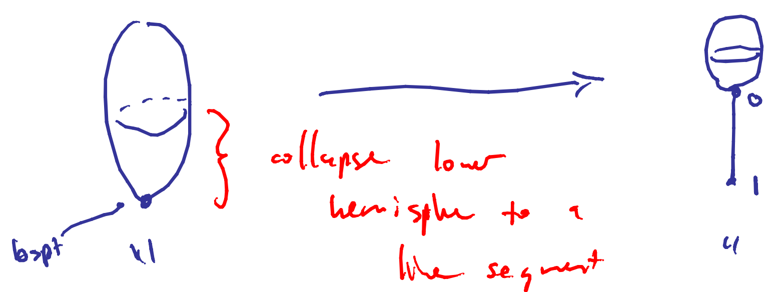
$\gamma_1 \gamma_2$ means

go around γ_2, then γ_1

$\gamma : x \longrightarrow y$ in X

induces: $\gamma_* : \pi_k(X,x) \longrightarrow \pi_k(X,y)$

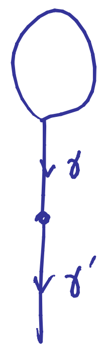
Defn



$S^n \longrightarrow S^n \vee I$

$\alpha : (S^n, *) \longrightarrow (X, x)$

set $\gamma_* \alpha : S^n \longrightarrow S^n \vee I \xrightarrow[\alpha \cup \gamma]{} X$



$$(\gamma'\gamma)_*[\alpha] = \gamma'_* \, \gamma_*[\alpha]$$

$\Rightarrow$ $X \in \mathrm{Top}$

π_k gives a functor

$$\pi_{oid}(X) \longrightarrow Gp$$

$$x \longmapsto \pi_k(X,x)$$

In particular,

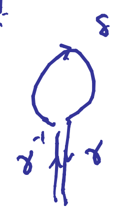
$\pi_1(X,x)$ <u>acts</u> on $\pi_k(X,x)$

$k \geq 1$

<u>Note</u> <u>$k=1$</u>

$\delta \in \pi_1(X,x)$

$\gamma \in \pi_1(X,x)$



$$\Rightarrow \gamma_*[\delta] = \gamma\delta\gamma^{-1}$$

action is by <u>conjugation</u>.

<u>Also</u>: since π_{oid} is a groupoid, $\gamma : x \to y$

$$\gamma_* : \pi_k(X,x) \xrightarrow{\cong} \pi_k(X,y)$$

is an iso.

$\Rightarrow$ iso class of $\pi_k(X,x)$ only depends on path comp. of x.

Weak equivalences

Def $X, Y \in \mathrm{Top}$. A map $f : X \to Y$ is said to be a weak (htpy) equivalence if

(1) $f_* : \pi_0 X \longrightarrow \pi_0 Y$ is a bijection

(2) $\forall\ x \in X$

$$f_* : \pi_k(X, x) \longrightarrow \pi_k(Y, f(x)) \text{ is iso}$$

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prop: $\left(f : X \to Y \text{ htpy equiv.} \right) \Longrightarrow \left(f \text{ is a w.e.} \right)$

(pf) Let $g : Y \longrightarrow X$ be htpy inverse

$$H : gf \simeq 1_X$$

$$H' : fg \simeq 1_Y$$

Pick $x \in X$

$$\pi_k(X, x) \xrightarrow[f]{} \pi_k(Y, f(x)) \xrightarrow[g]{} \pi_k(X, gf(x))$$

htpy H gives a commutative diagram

$$\begin{tikzcd}
 & \pi_k(Y, f(x)) \arrow[r, "g_*"] & \pi_k(X, gf(x)) \arrow[d, "\gamma_*", "\cong"'] \\
\pi_k(X, x) \arrow[ur, "f_*"] \arrow[rr, "\cong", "(1_X)_*"'] & & \pi_k(X, x)
\end{tikzcd}$$

$$\gamma = H|_{\{x\}} : I \longrightarrow X$$

$\Longrightarrow$ $g_* f_*$ isomorphism $\Rightarrow$ f_* monic

similarly show $f_* g_*$ iso $\Longrightarrow$ f_* epic

$\square$