HOMEWORK 2

DUE *THURSDAY*, FEB 20, 2014, SINCE TUESDAY IS AN MIT MONDAY

Turn in the following for credit:

1. Verify the following isomorphisms:

(a) For $X, Y \in \mathrm{Top}_*$, show that the adjunction

$$\mathrm{Map}_*(\Sigma X, Y) \cong \mathrm{Map}_*(X, \Omega Y)$$

descends to an adjunction of homotopy classes of maps:

$$[\Sigma X, Y]_* \cong [X, \Omega Y]_*.$$

Deduce there is an isomorphism

$$\pi_{n-1}(\Omega X) \cong \pi_n(X).$$

(b) Let X be an unbased space. The unreduced suspension $\mathrm{Susp}(X)$ is the space obtained from $X \times I$ by identifying all of the points in $X \times \{0\}$ and all of the points in $X \times \{1\}$. We do not identify points in $X \times \{0\}$ with points in $X \times \{1\}$. Show that there is an isomorphism

$$H_{n+1}(\mathrm{Susp}X) \cong H_n(X)$$

for n greater than or equal to 1.

2. *Hopf fibration.* The purpose of this problem is to verify that there exists a non-trivial element of $\pi_3(S^2)$. The Hopf fibration is a map $\eta : S^3 \to S^2$. It is defined by viewing S^2 as $\mathbb{C}P^1$, and S^3 as the unit sphere in $\mathbb{C}^2$. The map η is then defined by

$$\eta(x, y) = [x : y].$$

(Here, $[x : y]$ denotes the complex line in $\mathbb{C}^2$ spanned by the vector (x, y).)

(a) Let X be the CW complex given by attaching a 4-disk along η.

$$\begin{array}{ccc} S^3 & \xrightarrow{\eta} & \mathbb{C}P^1 \\ \downarrow & & \downarrow \\ D^4 & \longrightarrow & X. \end{array}$$

Show that X is homeomorphic to $\mathbb{C}P^2$.

(b) Show that if η is null homotopic, then X is homotopy equivalent to $S^2 \vee S^4$.

(c) Deduce that η cannot be null homotopic by computing the cup product structure on $H^*(X)$.

Do the following problems - but they do not need to be turned in.

3. (problem 3 on p358 of Hatcher) For an H-space (X, x_0) with multiplication $\mu : X \times X \to X$, show that the group operation in $\pi_n(X, x_0)$ can also be defined by the rule $(f + g)(x) = \mu(f(x), g(x))$. (For the notion of an H-space, consult section 3.C of Hatcher, p281.)

4. (problem 1 on p69 of May) Show that, if $n \geq 2$, then $\pi_n(X \vee Y)$ is isomorphic to $\pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y)$.

5. For a CW pair (X, A), is there an isomorphism $\pi_*(X, A) \cong \pi_*(X/A)$? Justify your answer.

6. Compute the homotopy groups of the quasi-circle (the "circle" containing $\sin(1/x)$ defined on p79, problem 7 of section 1.3 of Hatcher). Deduce that the inclusion of a point on the quasicircle is a weak equivalence. Show the inclusion is not a homotopy equivalence.