6 - Puppe Sequences - fibrations

Note Title 2/18/2010

Cofiber sequences

$A \xrightarrow{f} X$ map of unpointed spaces

$Z \in Top_*$

pointed

$$\begin{tikzcd} A \arrow[r, "f"] & X \arrow[r] \arrow[d, "g"'] & \mathrm{Cone}(f) \arrow[dl, dashed, "\tilde{g}"] \\ & Z & \end{tikzcd}$$

<u>Lemma</u>

there is a bijective correspondence

$$\{\text{pointed factorizations } \tilde{g}\}$$

$$\updownarrow$$

$$\{\text{null htpies } gf \simeq *\}$$

(pf)

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \arrow[dr, bend left, "g"] & \\ C(A) \arrow[r] \arrow[rr, dashed, bend right, "H"'] & \mathrm{Cone}(f) \arrow[r, dashed, "\tilde{g}"] & Z \end{tikzcd}$$

$\square$

<u>Cor:</u>

$$A \xrightarrow{f} X \longrightarrow \mathrm{Cone}(f) \qquad , \; Z \in \mathrm{Top}_*$$

induces an exact sequence of pointed sets

$$[\mathrm{Cone}(f), Z]_* \longrightarrow [X, Z] \xrightarrow{f^*} [A, Z]$$

<u>Similarly</u>

$$A, X \in \mathrm{Top}_*$$

$$\begin{tikzcd} A \arrow[r, "f"] & X \arrow[r] \arrow[d, "g"] & C(f) \arrow[dl, dashed, "\tilde{g}"] \\ & Z & \end{tikzcd}$$

$$\left\{ \begin{matrix} \text{factorizations} \\ \tilde{g} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{(pointed) null homotopies} \\ gf \simeq * \end{matrix} \right\}$$

exact sequence

$$[C(f), Z]_* \longrightarrow [X, Z]_* \longrightarrow [A, Z]_*$$

Continuing the cofiber sequence!

$$\begin{tikzcd} A \arrow[r, "f"'] & X \arrow[r, "i"'] & C(f) \arrow[r, "j"'] & C(i) \arrow[r] \arrow[d, "\text{pointed h.e.}"] & C(j) \arrow[d, "\text{pointed h.e.}"] \\ & & & C(f)/X \arrow[d, phantom, "\cong"] & C(i)/C(f) \arrow[d, phantom, "\cong"] \\ & & & \Sigma A \arrow[r, "-\Sigma f"'] & \Sigma X \end{tikzcd}$$

A well pointed

X well pointed

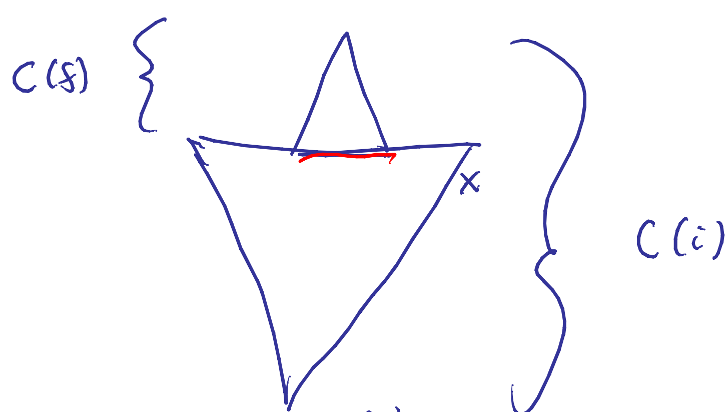
"composite of pushouts is a pushout"

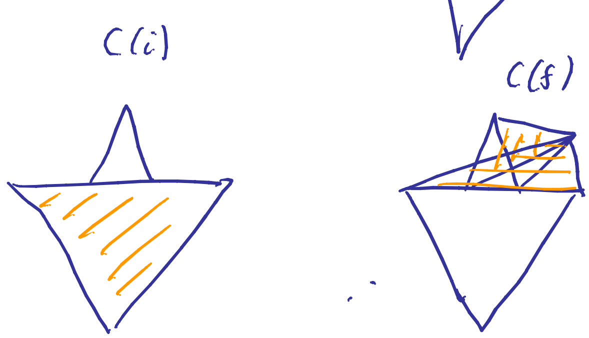
$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & * \arrow[d] \\ A \wedge I \arrow[r] & C(f) \arrow[r] & C(f)/X \end{tikzcd}$$

$$C(f)/X \approx A \wedge I/A \approx \Sigma A$$

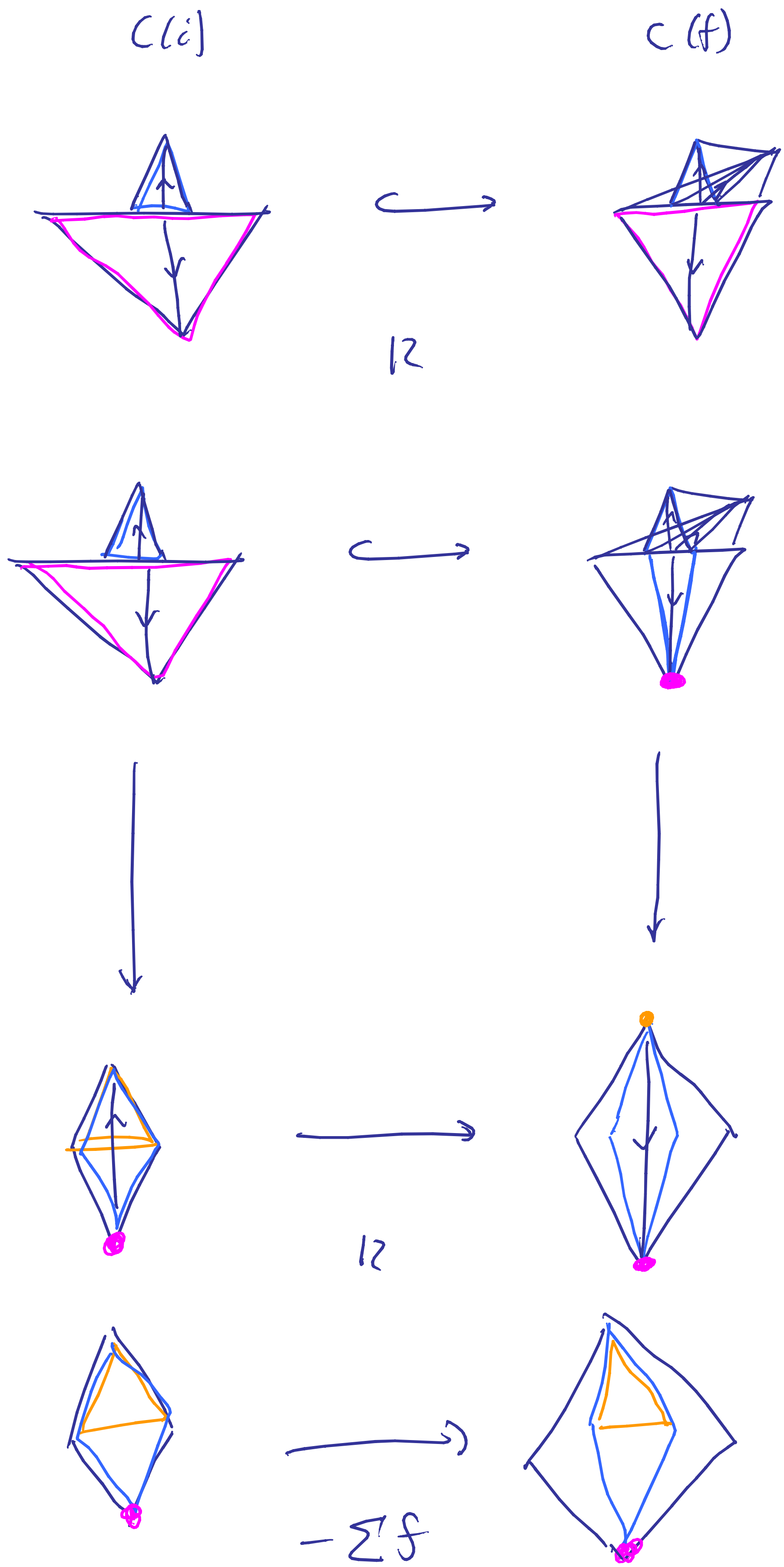
this implies this map is a cofibration

A well pointed $\Rightarrow$ this map is a cofibration





$$\Longrightarrow \quad C(i)/C(f) \cong \Sigma X$$


C(i)
C(f)
≅
≅
−Σf

Aside

$$[X, Z]_* \in \mathrm{Set}_*$$

$$[\Sigma X, Z]_* \cong [S^1, \underline{\mathrm{Map}}_*(X, Z)]$$

$$\pi_1 \underline{\mathrm{Map}}_*(X, Z) \in \mathrm{Gp}$$

$$[\Sigma^2 X, Z]_* \cong \pi_2 \underline{\mathrm{Map}}_*(X, Z) \in \mathrm{Ab}$$

$-\Sigma f$ means (1) $\Sigma A \longrightarrow \Sigma X$

$$t \wedge a \longmapsto (1-t) \wedge f(a)$$

(2) inverse wrt gp structure

Keep going $C(\Sigma f) \cong \Sigma C(f)$

$$A \xrightarrow{f} X \xrightarrow{i} C(f) \xrightarrow{j} \Sigma A \xrightarrow{-f} \Sigma X \xrightarrow{-i} \Sigma C(f) \xrightarrow{-j} \Sigma^2 A \xrightarrow{f} \Sigma^2 X \xrightarrow{i} \Sigma^2 C(f) \xrightarrow{j}$$

each consecutive pair of maps is pointed h.e. to a cofiber sequence

Consequence:

$$[X/A, Z]_* \cong [C(f), Z]_* \qquad \text{If } A \to X \text{ is a cofib}$$

$$[A,Z]_* \leftarrow [X,Z]_* \leftarrow [C(f),Z]_* \leftarrow [\Sigma A, Z]_* \leftarrow [\Sigma X, Z]_* \leftarrow [\Sigma C(f), Z]_* \leftarrow [\Sigma^2 A, Z]_* \leftarrow \cdots$$

$$[\Sigma X/A, Z] \cong [\Sigma C(f), Z]_*$$

exact sequence of pointed sets, grps ...

Rmk: we will show (X a CW cx)

$$K(\pi, n) \quad \pi_k K(\pi,n) = \begin{cases} \pi, & n = k \\ 0, & n \neq k \end{cases}$$

$$[Z, K(\pi,n)] \cong H^n(Z;\pi)$$

$$[Z, K(\pi,n)]_* \cong \tilde{H}^n(Z;\pi)$$

Thus if $A, X \in \mathrm{Top}_*^{wp}$

get:

$$\tilde{H}^n(A) \leftarrow \tilde{H}^n(X) \leftarrow \tilde{H}^n(C(f)) \leftarrow \tilde{H}^n(\Sigma A) \leftarrow \cdots$$

$$\tilde{H}^n(C(f)) \cong H^n(X, A), \qquad \tilde{H}^n(\Sigma A) \cong \tilde{H}^{n-1}(A)$$

LES of a pair...

Dual theory: fibrations

Def: $E \xrightarrow{p} B$ in Top is a fibration

if $\forall$ X

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\exists\, \tilde{H}"] & B
\end{tikzcd}$$

HLP

i.e. $f, g : X \to B$ and $\tilde{f} : X \to E$ is a lift

$H : f \simeq g$ $\Rightarrow \exists$ lift of g, lift of H

$\tilde{H} : \tilde{f} \simeq \tilde{g}$

Check: locally trivial bundles are fibrations.

e.g. covering spaces, principal fiber bundles, etc.

Is $\emptyset \longrightarrow B$ a fibration?

Fibrations are not surjections, but are surjections on path components.

use $X = *$

Id is a fib, fibrations are closed under composition, closed under pullback.

Lemma $E \to *$ is a fibration

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & Y \arrow[d] \\ X \arrow[r] & * \end{tikzcd} \implies X \times Y \longrightarrow X \text{ is a fibration}$$

Homotopy fiber

$X \xrightarrow{f} Y$ any map in Top_*

$$\begin{tikzcd} F(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d, "f"] \\ \underline{\mathrm{Map}}_*(I, Y) \arrow[r, "ev_1"'] & Y \end{tikzcd}$$

$F(f)$ = htpy fiber