18.100A final exam, spring 2007

You may use your book, but nothing else. Cite major theorems that you use.

1. Show that of $\{a_n\}$ is an increasing sequence, and that $a_n \to L$, then

$$\sup\{a_n\} = L.$$

2. Using the definition of integrability, show that the function

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

is integrable on the interval $[-1, 1]$.

3. Which of the following are uniformly continuous?

$$\begin{array}{lll} (a) & f(x) = 1/x, & x \in (1, \infty) \\ (b) & f(x) = 1/x, & x \in (0, 1) \end{array}$$

[For each of the above, if you claim it is uniformly continous, prove it. If you don't think it is, just give me an intuitive explaination.]

4.Prove that the function

$$g(x) = \int_0^x e^{-t^2}\, dt$$

is continuous.

5. Write down the Taylor series for the function $g(x)$ of problem 4. What is the radius of convergence of this power series? [Hint: the Taylor series for e^{-t^2} can be easily deduced by substituting in $y = -t^2$ in the Taylor series for e^y.]

6. Which of the following has uniform convergence? Justify your answers.

(a): the series of functions

$$\sum \frac{\sin(x)}{n^2}$$

(b): the sequence of functions

$$f_n(x) = \sqrt{\frac{x}{n}}$$

7. Suppose that f and g are continuous functions on the interval $[0, 1]$. Show there is a point $x \in [0, 1]$ which minimizes the vertical distance between the graphs of the functions. (In other words, show that there is a point x which minimizes the distance between $f(x)$ and $g(x)$.)

8. Suppose that f is differentiable on $(-\infty, \infty)$, and that $f'(x) > 1$ for all x. Suppose furthermore that $f(0) = 0$. Show that

$$f(x) > x$$

for $x > 0$.