

1. Solve the system of differential equations with given initial conditions

$$\begin{aligned}x'(t) &= 3x(t) + y(t) + 10 & x(0) &= -2 \\y'(t) &= -2x(t) + y(t) - 5 & y(0) &= 3\end{aligned}$$

2. Consider the system of differential equations:

$$\begin{aligned}x' &= 2x - y + z \\y' &= x + 3y - z \\z' &= x + y + z\end{aligned}$$

The eigenvalues of the associated matrix A are 2, 2, 2 and its eigenvectors are given by

$$\vec{u} = k(0, 1, 1)^T$$

where k is an arbitrary constant. Solve the system of equations.

3. Using the relevant information you obtained in Q1, solve the following system of difference equations:

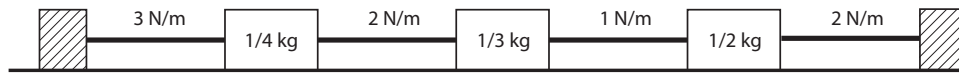
$$\begin{aligned}x(n) &= 3x(n-1) + y(n-1) \\ y(n) &= -2x(n-1) + y(n-1)\end{aligned}$$

4. Using the relevant information you obtained in Q2, solve the following system of difference equations:

$$\begin{aligned}x(n) &= 2x(n-1) - y(n-1) + z(n-1) \\y(n) &= x(n-1) + 3y(n-1) - z(n-1) \\z(n) &= x(n-1) + y(n-1) + z(n-1)\end{aligned}$$

5. Let $p(x) = 2 - 4x^2 + x^5$. Find $P(J)$ if $J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & 0 & -1 & \sqrt{3} \end{pmatrix}$.

6.



Consider a spring mass system above constructed with elastic strings with the given spring constants. Let x meters be the position of the $1/4$ -kg mass from equilibrium, y meters be the position of the $1/3$ -kg mass from equilibrium, and z meters be the position of the $1/2$ -kg mass from equilibrium. Assume that the surface they are sitting on is frictionless.

6a. Write down the equations of motion for the system above using Newton's Law and Hook's Law (Hint: consider all the forces acting on each mass)

6b. Rewrite your equations in Q6(a) as a linear first order system of equations.

6c. Write down the associated matrix in 6(b).

7. Solve the system of differential equations with given initial conditions

$$\begin{aligned}x'(t) &= -4x(t) + 6y(t) + 6 & x(0) &= -1 \\y'(t) &= -3x(t) + 5y(t) + 2 & y(0) &= 1\end{aligned}$$

5. Solve the system of differential equations with given initial conditions

$$\begin{aligned} x'(t) &= 3x(t) + y(t) + 10 & x(0) &= -2 \\ y'(t) &= -2x(t) + y(t) - 5 & y(0) &= 3 \end{aligned}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}}_A \begin{pmatrix} x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} 10 \\ -5 \end{pmatrix}}_{\vec{B}} \Rightarrow \vec{X}' = A\vec{X} + \vec{B} = A(\vec{X} + A^{-1}\vec{B})$$

$$A^{-1} = \frac{1}{3 - (-2)} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \Rightarrow A^{-1}\vec{B} = \frac{1}{5} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{X}' = A(\vec{X} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}). \text{ Let } \vec{Y} = \vec{X} + \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\vec{Y}' = \vec{X}' = A\vec{Y}. \text{ Solve } \vec{Y}' = A\vec{Y}$$

Eigenvectors and Eigenvalues of A:

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda-1)(\lambda-3) + 2 = \lambda^2 - 4\lambda + 5 = 0$$

$$\Rightarrow (\lambda^2 - 4\lambda + 4) + 1 = 0 \Rightarrow (\lambda - 2)^2 = -1 \Rightarrow \lambda = 2 \pm i$$

Eigenvectors for $\lambda = 2 + i$:

$$\begin{pmatrix} 3-(2+i) & 1 \\ -2 & 1-(2+i) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} 1-i & 1 \\ -2 & -1-i \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(1-i)u_1 + u_2 = 0 \Rightarrow u_2 = (-1+i)u_1 \Rightarrow \vec{u} = \begin{bmatrix} u_1 \\ -u_1 + u_1 i \end{bmatrix}$$

Canonical Form:

$$= \begin{pmatrix} u_1 \\ -u_1 \end{pmatrix} + i \begin{pmatrix} 0 \\ u_1 \end{pmatrix}$$

$$P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; \quad J = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\text{Then } A = PJP^{-1}$$

Q5 continue: $\vec{y}' = A\vec{y} \Rightarrow \vec{y}(t) = e^{A \cdot t} \cdot \vec{y}(0)$

$$\vec{y}(t) = P e^{J \cdot t} P^{-1} \vec{y}(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \cos t & e^{2t} \sin t \\ -e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \cos t + c_2 e^{2t} \sin t \\ -c_1 e^{2t} \sin t + c_2 e^{2t} \cos t \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{2t} \cos t + c_2 e^{2t} \sin t \\ -c_1 e^{2t} \cos t - c_2 e^{2t} \sin t - c_1 e^{2t} \sin t + c_2 e^{2t} \cos t \end{bmatrix}$$

$$\vec{y}(t) = \begin{bmatrix} x(t) + 3 \\ y(t) + 1 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \cos t + c_2 e^{2t} \sin t \\ (c_2 - c_1) e^{2t} \cos t - (c_1 + c_2) e^{2t} \sin t \end{bmatrix}$$

$$x(t) = c_1 e^{2t} \cos t + c_2 e^{2t} \sin t - 3$$

$$y(t) = (c_2 - c_1) e^{2t} \cos t - (c_1 + c_2) e^{2t} \sin t - 1$$

$$\begin{cases} x(0) = c_1 - 3 = -2 \\ y(0) = c_2 - c_1 - 1 = 3 \end{cases} \Rightarrow \begin{cases} c_2 = 3 + c_1 + 1 = 3 + 2 = 5 \\ c_1 = 1 \end{cases}$$

$$\begin{cases} x(t) = e^{2t} \cos t + 5e^{2t} \sin t - 3 \\ y(t) = 4e^{2t} \cos t - 6e^{2t} \sin t - 1 \end{cases}$$

9a

6. Consider the system of differential equations:

$$\begin{aligned}x' &= 2x - y + z \\y' &= x + 3y - z \\z' &= x + y + z\end{aligned}$$

The eigenvalues of the associated matrix A are 2, 2, 2 and its eigenvectors are given by

$$\vec{u} = k(0, 1, 1)^T = k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where k is an arbitrary constant. Solve the system of equations.

$$\vec{X}' = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \vec{X} = A \vec{X}$$

We need to write A in canonical form:

We only have 1 representative eigenvector we can pick.

There must be 2 generalized eigenvectors.

1st generalized eigenvector:

$$(A - 2I)\vec{v} = \vec{u} \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix}$$

$$\Rightarrow \begin{cases} -v_2 + v_3 = 0 \\ v_1 + v_2 - v_3 = k \end{cases} \Rightarrow \begin{cases} v_3 = v_2 \\ v_1 = k - v_2 + v_3 = k \end{cases}$$

$$\vec{v} = \begin{bmatrix} k \\ v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

2nd generalized eigenvector:

$$(A - 2I)\vec{w} = \vec{v} \Rightarrow \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} k \\ v_2 \\ v_2 \end{bmatrix}$$

$$-w_2 + w_3 = k \Rightarrow w_3 = k + w_2$$

$$w_1 + w_2 - w_3 = v_2 \Rightarrow w_1 = v_2 - w_2 + w_3 = v_2 + k$$

$$\vec{w} = \begin{bmatrix} v_2 + k \\ w_2 \\ k + w_2 \end{bmatrix} = w_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + v_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Canonical Form of A :

$$\text{Pick } \vec{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; (k=1, r_2=0=w_2)$$

$$\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \vec{u} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} ; J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\vec{X}(t) = P e^{J \cdot t} P^{-1} \vec{X}(0)$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & te^{2t} & t^2 e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t} \\ c_2 e^{2t} + c_3 t e^{2t} \\ c_3 e^{2t} \end{bmatrix}$$

$$x(t) = (c_2 + c_3) e^{2t} + c_3 t e^{2t}$$

$$y(t) = c_1 e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t}$$

$$z(t) = (c_1 + c_3) e^{2t} + c_2 t e^{2t} + c_3 t^2 e^{2t}$$

10a

7. Using the relevant information you obtained in Q5, solve the following system of difference equations:

$$x(n) = 3x(n-1) + y(n-1)$$

$$y(n) = -2x(n-1) + y(n-1)$$

$$\begin{pmatrix} x(n) \\ y(n) \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x(n-1) \\ y(n-1) \end{pmatrix} \Rightarrow \vec{X}(n) = A \vec{X}(n-1) = A^n \vec{X}(0)$$

$$A = P J P^{-1} \text{ where } P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; J = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

$$\vec{X}(n) = P J^n P^{-1} \vec{X}(0) \quad \begin{matrix} \nearrow \\ (C_1 \\ C_2) \text{ later} \end{matrix}$$

$$\text{Write } J = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} R \cos \theta & R \sin \theta \\ -R \sin \theta & R \cos \theta \end{bmatrix}$$

$$\left. \begin{array}{l} R \cos \theta = 2 \text{ (1)} \\ R \sin \theta = 1 \text{ (2)} \end{array} \right\} \Rightarrow \textcircled{2}/\textcircled{1} : \tan \theta = 1/2 \Rightarrow \theta = \arctan(1/2) = 0.46$$

$$\textcircled{1}^2 + \textcircled{2}^2 : R^2 \sin^2 \theta + R^2 \cos^2 \theta = 1 + 4 \Rightarrow R^2 = 5 \Rightarrow R = \sqrt{5}$$

$$J = \begin{bmatrix} \sqrt{5} \cos(0.46) & \sqrt{5} \sin(0.46) \\ -\sqrt{5} \sin(0.46) & \sqrt{5} \cos(0.46) \end{bmatrix}$$

$$\vec{X}(0) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5^{n/2} \cos(0.46n) & 5^{n/2} \sin(0.46n) \\ -5^{n/2} \sin(0.46n) & 5^{n/2} \cos(0.46n) \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C_1 \cdot 5^{n/2} \cos(0.46n) + C_2 \cdot 5^{n/2} \sin(0.46n) \\ -C_1 \cdot 5^{n/2} \sin(0.46n) + C_2 \cdot 5^{n/2} \cos(0.46n) \end{bmatrix}$$

$$x(n) = C_1 \cdot 5^{n/2} \cos(0.46n) + C_2 \cdot 5^{n/2} \sin(0.46n)$$

$$y(n) = (-C_1 + C_2) \cdot 5^{n/2} \cos(0.46n) - (C_1 + C_2) \cdot 5^{n/2} \sin(0.46n)$$

8. Using the relevant information you obtained in Q6, solve the following system of difference equations:

$$x(n) = 2x(n-1) - y(n-1) + z(n-1)$$

$$y(n) = x(n-1) + 3y(n-1) - z(n-1)$$

$$z(n) = x(n-1) + y(n-1) + z(n-1)$$

$$\vec{X}(n) = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix} \vec{X}(n-1) \Rightarrow \vec{X}(n) = A \vec{X}(n-1) = A^n \vec{X}(0)$$

$$\vec{X}(n) = P J^n P^{-1} \vec{X}(0) ; P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} ; J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\vec{X}(n) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^n & n2^{n-1} & \binom{n}{2}2 \\ 0 & 2^n & n2^{n-1} \\ 0 & 0 & 2^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\text{Here } \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{(n-1)n}{2}$$

$$\vec{X}(n) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \cdot 2^n + c_2 n 2^{n-1} + c_3 \cdot \frac{n(n-1)}{2} \cdot 2^{n-2} \\ c_2 \cdot 2^n + c_3 \cdot n 2^{n-1} \\ c_3 \cdot 2^n \end{bmatrix}$$

$$x(n) = (c_2 + c_3) \cdot 2^n + c_3 \cdot n 2^{n-1}$$

$$y(n) = c_1 \cdot 2^n + c_2 \cdot n 2^{n-1} + c_3 \cdot n(n-1) \cdot 2^{n-2}$$

$$z(n) = (c_1 + c_3) \cdot 2^n + c_2 \cdot n 2^{n-1} + c_3 \cdot n(n-1) 2^{n-2}$$

9. Let $p(x) = 2 - 4x^2 + x^5$. Find $P(J)$ if $J = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 1 \\ 0 & 0 & 0 & -1 & \sqrt{3} \end{pmatrix}$.

$$P(J) = 2 \cdot I - 4J^2 + J^5$$

$$\begin{pmatrix} -2 & 3 \\ 0 & -2 \end{pmatrix}^5 = \left[3 \begin{pmatrix} -2/3 & 1 \\ 0 & -2/3 \end{pmatrix} \right]^5 = 3^5 \begin{bmatrix} (-2/3)^5 & 5(-2/3)^4 \\ 0 & (-2/3)^5 \end{bmatrix}$$

$$= \begin{bmatrix} (-2)^5 & 5(-2)^4 \cdot 3 \\ 0 & (-2)^5 \end{bmatrix} = \begin{bmatrix} -32 & 240 \\ 0 & -32 \end{bmatrix}$$

$$\begin{pmatrix} -2 & 3 \\ 0 & -2 \end{pmatrix}^2 = \begin{bmatrix} (-2)^2 & 2(-2) \cdot 3 \\ 0 & (-2)^2 \end{bmatrix} = \begin{bmatrix} 4 & -12 \\ 0 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} R \cos \theta & R \sin \theta \\ -R \sin \theta & R \cos \theta \end{bmatrix} \quad \begin{matrix} R \cos \theta = \sqrt{3} \\ R \sin \theta = 1 \end{matrix} \Rightarrow \begin{matrix} R^2 = 3+1 \Rightarrow R=2 \\ \tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \end{matrix}$$

$$\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}^5 = \begin{bmatrix} 2 \cos \pi/6 & 2 \sin \pi/6 \\ -2 \sin \pi/6 & 2 \cos \pi/6 \end{bmatrix}^5 = \begin{bmatrix} 2^5 \cos 5\pi/6 & 2^5 \sin 5\pi/6 \\ -2^5 \sin 5\pi/6 & 2^5 \cos 5\pi/6 \end{bmatrix}$$

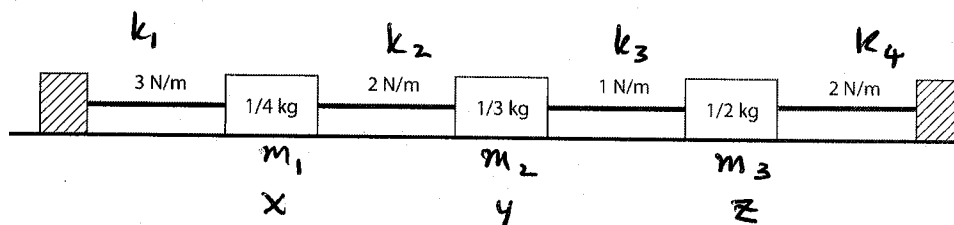
$$= \begin{bmatrix} 32(-\sqrt{3}/2) & 32(1/2) \\ -32(1/2) & 32(-\sqrt{3}/2) \end{bmatrix} = \begin{bmatrix} -16\sqrt{3} & 16 \\ -16 & -16\sqrt{3} \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{bmatrix}^2 = \begin{bmatrix} 2^2 \cos \pi/3 & 2^2 \sin \pi/3 \\ -2^2 \sin \pi/3 & 2^2 \cos \pi/3 \end{bmatrix} = \begin{bmatrix} 2 & 2\sqrt{3} \\ -2\sqrt{3} & 2 \end{bmatrix}$$

$$P(J) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 16 & 0 & 0 & 0 & 0 \\ 0 & 16 & -48 & 0 & 0 \\ 0 & 0 & 16 & 0 & 0 \\ 0 & 0 & 0 & 8 & 8\sqrt{3} \\ 0 & 0 & 0 & -8\sqrt{3} & 8 \end{bmatrix} + \begin{bmatrix} 32 & 0 & 0 & 0 & 0 \\ 0 & -32 & 240 & 0 & 0 \\ 0 & 0 & -32 & 0 & 0 \\ 0 & 0 & 0 & -16\sqrt{3} & 16 \\ 0 & 0 & 0 & 16 & -16\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 0 & 0 & 0 & 0 \\ 0 & -46 & 288 & 0 & 0 \\ 0 & 0 & -46 & 0 & 0 \\ 0 & 0 & 0 & -6-16\sqrt{3} & 16-8\sqrt{3} \\ 0 & 0 & 0 & -16+8\sqrt{3} & -6-16\sqrt{3} \end{bmatrix}$$

10.



Consider a spring mass system above constructed with elastic strings with the given spring constants. Let x meters be the position of the $1/4$ -kg mass from equilibrium, y meters be the position of the $1/3$ -kg mass from equilibrium, and z meters be the position of the $1/2$ -kg mass from equilibrium. Assume that the surface they are sitting on is frictionless.

10a. Write down the equations of motion for the system above using Newton's Law and Hook's Law (Hint: consider all the forces acting on each mass)

$$m_1 \ddot{x} = -k_1 x - k_2 x + k_2 y \Rightarrow \frac{1}{4} \ddot{x} = -3x - 2x + 2y = -5x + 2y$$

$$m_2 \ddot{y} = k_2 x - k_2 y - k_3 y + k_3 z \Rightarrow \frac{1}{3} \ddot{y} = 2x - 2y - y + z = 2x - 3y + z$$

$$m_3 \ddot{z} = k_3 y - k_3 z - k_4 z \Rightarrow \frac{1}{2} \ddot{z} = y - z - 2z = y - 3z$$

10b. Rewrite your equations in Q10(a) as a linear first order system of equations.

$$u = \dot{x} \Rightarrow \dot{u} = \ddot{x} = -20x + 8y$$

$$v = \dot{y} \Rightarrow \dot{v} = \ddot{y} = 6x - 9y + 3z$$

$$w = \dot{z} \Rightarrow \dot{w} = \ddot{z} = 2y - 6z$$

$$\dot{x} = u$$

$$\dot{y} = v$$

$$\dot{z} = w$$

$$\dot{u} = -20x + 8y$$

$$\dot{v} = 6x - 9y + 3z$$

$$\dot{w} = 2y - 6z$$

10c. Write down the associated matrix in ~~20~~ 10(b)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} u \\ v \\ w \\ -20x + 8y \\ 6x - 9y + 3z \\ 2y - 6z \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -20 & 8 & 0 & 0 & 0 & 0 \\ 6 & -9 & 3 & 0 & 0 & 0 \\ 0 & 2 & -6 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ u \\ v \\ w \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -20 & 8 & 0 & 0 & 0 & 0 \\ 6 & -9 & 3 & 0 & 0 & 0 \\ 0 & 2 & -6 & 0 & 0 & 0 \end{bmatrix}$$

11. Solve the system of differential equations with given initial conditions

$$x'(t) = -4x(t) + 6y(t) + 6 \quad x(0) = -1$$

$$y'(t) = -3x(t) + 5y(t) + 2 \quad y(0) = 1$$

$$\vec{X}' = \underbrace{\begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}}_A \vec{X} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = A \left[\vec{X} + A^{-1} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \right] \quad \begin{matrix} -15+6 \\ -9+4 \end{matrix}$$

$$A^{-1} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \frac{1}{-20+18} \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -9 \\ -5 \end{bmatrix}$$

$$\vec{Y} = \vec{X} + \begin{bmatrix} -9 \\ -5 \end{bmatrix} \quad \text{so} \quad \vec{Y}' = A \vec{Y} \Rightarrow \vec{Y}(t) = e^{At} \vec{Y}(0).$$

Eigenvalues of A: $\begin{vmatrix} -4-\lambda & 6 \\ -3 & 5-\lambda \end{vmatrix} = -(4+\lambda)(5-\lambda) + 18$

$$= -(20 + 5\lambda - 4\lambda - \lambda^2) + 18 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = -1, +2$$

$\lambda = -1$: $\begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow -3u_1 + 6u_2 = 0 \Rightarrow u_1 = 2u_2$

$$\vec{u} = \begin{pmatrix} 2u_2 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} u_2$$

$\lambda = 2$: $\begin{pmatrix} -6 & 6 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Rightarrow -v_1 + v_2 = 0 \Rightarrow v_2 = v_1$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1$$

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad ; \quad J = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{aligned} \vec{Y}(t) &= P e^{Jt} P^{-1} \vec{Y}(0) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{2t} \end{pmatrix} = \begin{pmatrix} 2c_1 e^{-t} + c_2 e^{2t} \\ c_1 e^{-t} + c_2 e^{2t} \end{pmatrix} \end{aligned}$$

$$\vec{X} = \vec{Y} - \begin{pmatrix} -9 \\ -5 \end{pmatrix} = \vec{Y} + \begin{pmatrix} 9 \\ 5 \end{pmatrix}$$

$$x(t) = 2c_1 e^{-t} + c_2 e^{2t} + 9$$

$$y(t) = c_1 e^{-t} + c_2 e^{2t} + 5$$

$$x(0) = -1 = 2c_1 + c_2 + 9 \Rightarrow 2c_1 + c_2 = -10 \quad \textcircled{1}$$

$$y(0) = 1 = c_1 + c_2 + 5 \Rightarrow c_1 + c_2 = -4 \quad \textcircled{2}$$

$$\textcircled{1} - \textcircled{2}: c_1 = -10 + 4 = -6$$

$$c_2 = -4 - c_1 = -4 + 6 = 2$$

$$x(t) = -12e^{-t} + 2e^{2t} + 9$$

$$y(t) = -6e^{-t} + 2e^{2t} + 5.$$

16 a

Math 20480: Intro to Dynamical Sys.
Sample Final Exam
May 10, 2052

Name: _____

Class Time: _____

- The Honor Code is in effect for this examination. All work is to be your own.
- Calculators may be use for the examination.
- The exam lasts for two hours.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 14 pages of the test.

Sign the pledge. “On my honor, I have neither given nor received unauthorized aid on this Exam”:

Good Luck!

Please do NOT write in this box.

1. _____

2. _____

3. _____

4. _____

5. _____

6. _____

7. _____

8. _____

9. _____

10. _____

Total _____

Name: _____

Class Time: _____

Partial Credit

You must show your work on the partial credit problems to receive credit!

1.(10 pts.) **1a.** Find the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$.

1b. (10 pts.) Solve the following system of differential equations:

$$\begin{aligned} x' &= 5x + 5y - 5; & x(0) &= -1 \\ y' &= -2x - y + 5; & y(0) &= 2 \end{aligned}$$

Name: _____

Class Time: _____

1b. Continue ...

Name: _____

Class Time: _____

2.(10 pts.) Solve the following second order linear differential equations with constant coefficients.

2a. $4y'' + 3y' - y = 0$

2b. $y'' - 4y' + 4y = e^{2t} - 2e^t$

Name: _____

Class Time: _____

3.(10 pts.) Using row reduction find the inverse of the following 3×3 matrix.

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

Name: _____

Class Time: _____

4.(10 pts.)



(Part A). Consider a spring mass system above constructed with springs with the given spring constants. Let x meters be the position of the $1/2$ -kg mass from equilibrium, and y meters be the position of the 1 -kg mass from equilibrium. Assume that the surface they are sitting on is frictionless.

Write down the equations of motion for the system above.

(Part B). Rewrite the equations of motion above into a linear **first** order system of equations. Use the variables u , v , w , and p in your answer.

Name: _____

Class Time: _____

5.(10 pts.) **(Part A)**. If $T(x)$ is the polynomial $T(x) = 2 + x^4$ and J is the matrix

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Find the matrix $T(J)$.

(Part B. Not Related to Above). Find the following matrix exponential:

$$\text{Exp} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \stackrel{?}{=}$$

Name: _____

Class Time: _____

6.(10 pts.) Consider the system of difference equations:

$$\begin{aligned}x(n) &= 3x(n-1) - y(n-1) + z(n-1) \\y(n) &= x(n-1) + 4y(n-1) - 2z(n-1) \\z(n) &= x(n-1) + y(n-1) + z(n-1)\end{aligned}$$

Let Q be the matrix associated to the systems of difference equations above. The eigenvalues and their associated eigenvectors of matrix Q are given below:

$$\lambda_1 = 2 : \quad \vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix}; \quad \lambda_2 = 3 : \quad \vec{v} = \begin{pmatrix} s \\ s \\ s \end{pmatrix}$$

(Part A). Write down the matrix Q , the canonical form J of Q , and find a matrix P such that $Q = PJP^{-1}$.

Name: _____

Class Time: _____

(Q6 continue ... Part B). (10 pts.) Solve the given system of difference equations:

$$x(n) = 3x(n-1) - y(n-1) + z(n-1)$$

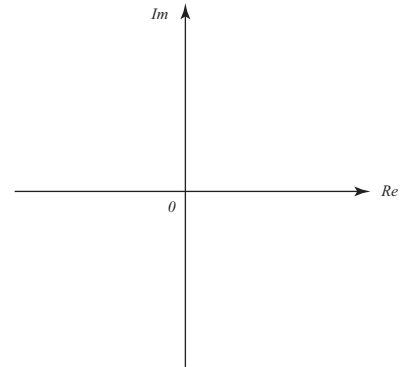
$$y(n) = x(n-1) + 4y(n-1) - 2z(n-1)$$

$$z(n) = x(n-1) + y(n-1) + z(n-1)$$

Name: _____

Class Time: _____

7.(10 pts.) Write the complex number $z = \sqrt{3} - i$ in the form $re^{i\theta}$ where $r > 0$ and $0 \leq \theta \leq 2\pi$. Give EXACT values of r and θ . You may find the Argand plane helpful.



Using your answer above, find the following:

(i) The conjugate $\bar{z}$ in the form $Re^{i\alpha}$ where $R > 0$ and $0 \leq \alpha \leq 2\pi$.

(ii) z^{53} in the form $a + bi$ where a and b are numbers to be determined. Give also the argument of z^{53} between 0 and 2π .

Name: _____

Class Time: _____

8.(10 pts.) Evaluate $\begin{pmatrix} \sqrt{3} & 1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}^6$

Entries of your answers should be completely evaluated with no trigonometric functions.

Name: _____

Class Time: _____

9.(10 pts.) A kind of “live bearing” fish goes through four life stages before dying:

Stage 1: Fry

Stage 3: “Young Adult”

Stage 2: “Juvenile”

Stage 4: “Adult”

The Lefkovitch matrix of a kind of fish that goes through four life stages before dying is given by the matrix L below:

$$L = \begin{pmatrix} 0 & 0 & 5 & 8 \\ 0.01 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{pmatrix}$$

9a. Draw the lifecycle graph of the fish. Label the vertices 1, 2, 3, and 4 according to stages and weight the edges.

9b. Using the MatLab information in the next page, find the following information:

(i). Estimate the **continuous growth rate** after a long time. Is the population expected to increase or decrease after a long time?

(ii). The stable population vector. Round to three decimal places for each entry.

Name: _____

Class Time: _____

10.(10 pts.) The MatLab information for a matrix A is given on the next page. Answer the following question about A

10a. Write the canonical form J of A with all real entries.

10b. Find a real matrix P with all integer entries such that $A = PJP^{-1}$. Your answer should have no unnecessary common factors.

Name: _____

Class Time: _____

10c. Write down all eigenspaces of matrix A . For generalized eigenspaces, set the parameters so that Jordan chain relations are indicated.

Lefkovitch Matrix MatLab Data Sheet

```
>> L=[0 0 5 8; 0.01 0.4 0 0; 0 0.6 0 0; 0 0 0.6 0]
```

L =

0	0	5.0000	8.0000
0.0100	0.4000	0	0
0	0.6000	0	0
0	0	0.6000	0

```
>> [V,E]=eig(L)
```

V =

Columns 1 through 3

0.9967	0.9984	0.9984
0.0478	-0.0127 - 0.0141i	-0.0127 + 0.0141i
0.0471	-0.0236 + 0.0165i	-0.0236 - 0.0165i
0.0464	0.0207 + 0.0386i	0.0207 - 0.0386i

Column 4

-0.9980
0.0142
-0.0280
0.0554

E =

Columns 1 through 3

0.6087	0	0
0	0.0475 + 0.3919i	0
0	0	0.0475 - 0.3919i
0	0	0

Column 4

0
0
0
-0.3036

Matrix A MatLab Information

```
>> [V,E]=jordan(A)
```

V =

0 - 0.5000i	0 + 0.5000i	-1.0000	0
0.5000	0.5000	0	0
0.5000 - 0.5000i	0.5000 + 0.5000i	0	-0.5000
0.5000	0.5000	0	-0.5000

E =

2.0000 - 1.0000i	0	0	0
0	2.0000 + 1.0000i	0	0
0	0	-1.0000	1.0000
0	0	0	-1.0000

Name: _____

Class Time: _____

Partial Credit

You must show your work on the partial credit problems to receive credit!

1. (10 pts.) 1a. Find the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}$.

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 5 \\ -2 & -1-\lambda \end{vmatrix} = (5-\lambda)(-1-\lambda) + 10$$

$$= -5 + \lambda - 5\lambda + \lambda^2 + 10 = \lambda^2 - 4\lambda + 5 = (\lambda - 2)^2 + 1 = 0$$

$$\Rightarrow \lambda = 2 \pm i$$

Eigenvector of $2+i$: $\begin{pmatrix} 5-(2+i) & 5 \\ -2 & -1-(2+i) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 3-i & 5 \\ -2 & -3-i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow (3-i)u_1 + 5u_2 = 0$$

$$\Rightarrow u_2 = \left(-\frac{3}{5} + \frac{1}{5}i\right)u_1 \Rightarrow \vec{u} = \begin{pmatrix} u_1 \\ \left(-\frac{3}{5} + \frac{1}{5}i\right)u_1 \end{pmatrix}$$

Eigenvectors of $2-i$: $\vec{v} = \begin{pmatrix} u_1 \\ \left(-\frac{3}{5} - \frac{1}{5}i\right)u_1 \end{pmatrix}$

1b. (10 pts.) Solve the following system of differential equations:

$$x' = 5x + 5y - 5; \quad x(0) = -1$$

$$y' = -2x - y + 5; \quad y(0) = 2$$

$$\vec{X}' = \underbrace{\begin{pmatrix} 5 & 5 \\ -2 & -1 \end{pmatrix}}_A \vec{X} + \underbrace{\begin{pmatrix} -5 \\ 5 \end{pmatrix}}_B = A(\vec{X} + A^{-1}\vec{B})$$

$$A^{-1}\vec{B} = \frac{1}{5} \begin{bmatrix} -1 & -5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

Name: _____

Class Time: _____

1b. Continue ...

$$\text{Set } \vec{y} = \vec{x} + \begin{pmatrix} -4 \\ 3 \end{pmatrix} \Rightarrow \vec{y}' = \vec{x}'$$

$$\vec{y}' = A\vec{y} \Rightarrow \vec{y}(t) = e^{A \cdot t} \cdot \vec{y}(0)$$

$$P = \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix} ; J = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \text{ so } A = PJP^{-1}$$

$$\vec{y} = P e^{J \cdot t} \cdot P^{-1} \vec{y}(0) = P \begin{bmatrix} e^{2t} \cos t & e^{2t} \sin t \\ -e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} \underbrace{P^{-1} \vec{y}(0)}_{\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

$$\vec{x}(t) = P \begin{bmatrix} e^{2t} \cos t & e^{2t} \sin t \\ -e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix} \Rightarrow P \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} 5c_1 = -5 \Rightarrow c_1 = -1 \\ -3c_1 + c_2 = 5 \Rightarrow c_2 = 5 + 3c_1 = 2 \end{cases}$$

$$\vec{x}(t) = \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} \cos t & e^{2t} \sin t \\ -e^{2t} \sin t & e^{2t} \cos t \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -e^{2t} \cos t + 2e^{2t} \sin t \\ e^{2t} \sin t + 2e^{2t} \cos t \end{bmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

$$\begin{cases} x(t) = -5e^{2t} \cos t + 10e^{2t} \sin t + 4 \\ y(t) = 8e^{2t} \cos t - 5e^{2t} \sin t - 3 \end{cases}$$

Name: _____

Class Time: _____

2.(10 pts.) Solve the following second order linear differential equations with constant coefficients.

2a. $4y'' + 3y' - y = 0$

Aux eqn: $4\lambda^2 + 3\lambda - 1 = 0 \Rightarrow (4\lambda - 1)(\lambda + 1) = 0$
 $\Rightarrow \lambda = -1, \frac{1}{4}$

$$y(t) = C_1 e^{-t} + C_2 e^{\frac{1}{4}t}$$

2b. $y'' - 4y' + 4y = e^{2t} - 2e^t$

Aux eqn: $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0$
 $\Rightarrow \lambda = 2, 2$

hom. solution: $y_h = C_1 e^{2t} + C_2 t e^{2t}$

Particular solution: $y_p = A t^2 e^{2t} + B e^t$

$$y_p' = 2A t e^{2t} + 2A t e^{2t} + B e^t$$

↑
both e^{2t} , $t e^{2t}$ are hom. solution

$$y_p'' = 4A t e^{2t} + 4A t e^{2t}$$

$$+ 2A e^{2t} + 4A t e^{2t} + B e^t$$

$$= 4A t^2 e^{2t} + 8A t e^{2t} + 2A e^{2t} + B e^t$$

$$y_p'' - 4y_p' + 4y_p = \begin{matrix} 4A t^2 e^{2t} + 8A t e^{2t} + 2A e^{2t} + B e^t \\ - 8A t^2 e^{2t} - 8A t e^{2t} \\ 4A t^2 e^{2t} - 4B e^t + 4B e^t \end{matrix}$$

$$= 2A e^{2t} + B e^t = e^{2t} - 2e^t \Rightarrow A = \frac{1}{2}, B = -2$$

$$y(t) = y_h + y_p = C_1 e^{2t} + C_2 t e^{2t} + \frac{1}{2} t^2 e^{2t} - 2e^t$$

④

Name: _____

Class Time: _____

3.(10 pts.) Using row reduction find the inverse of the following 3×3 matrix.

$$Q = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$[Q | I] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{r_3 - r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array} \right] \xrightarrow{r_2 + r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array} \right]$$

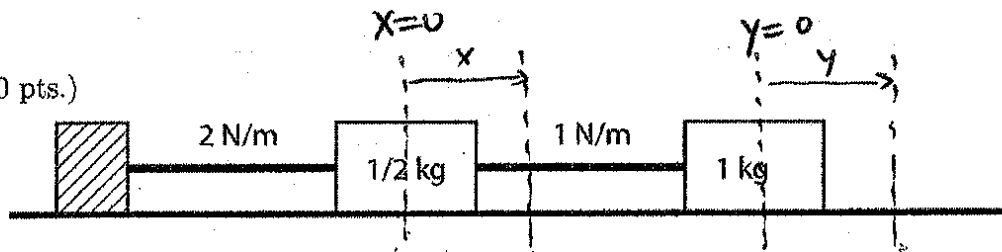
$$Q^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

You ~~can~~ can check : $Q \cdot Q^{-1} = I$.

Name: _____

Class Time: _____

4. (10 pts.)



(Part A). Consider a spring mass system above constructed with springs with the given spring constants. Let x meters be the position of the $1/2$ -kg mass from equilibrium, and y meters be the position of the 1 -kg mass from equilibrium. Assume that the surface they are sitting on is frictionless.

Write down the equations of motion for the system above.

$$\begin{aligned} \frac{1}{2} \ddot{x} &= -2x - x + y \\ \ddot{y} &= x - y \end{aligned} \Rightarrow \begin{aligned} \ddot{x} &= -6x + 2y \\ \ddot{y} &= x - y \end{aligned}$$

(Part B). Rewrite the equations of motion above into a linear first order system of equations. (Use the variables u , v , w , and p in your answer.) $\leftarrow$ No need.

$$\begin{aligned} u &= \dot{x} \Rightarrow \dot{u} = \ddot{x} = -6x + 2y \\ v &= \dot{y} \Rightarrow \dot{v} = \ddot{y} = x - y \end{aligned}$$

$$\begin{cases} \dot{x} = u \\ \dot{y} = v \\ \dot{u} = -6x + 2y \\ \dot{v} = x - y \end{cases}$$

Name: _____

Class Time: _____

5.(10 pts.) (Part A). If $T(x)$ is the polynomial $T(x) = 2 + x^4$ and J is the matrix

$$J = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{4(4-1)}{2!} = 6$$

Find the matrix $T(J)$.

$$T(J) = 2I_3 + J^4$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} (-1)^4 & 4(-1)^3 & \begin{pmatrix} 4 \\ 2 \end{pmatrix}(-1)^2 \\ 0 & (-1)^4 & 4(-1)^3 \\ 0 & 0 & (-1)^4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -4 & 6 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 6 \\ 0 & 3 & -4 \\ 0 & 0 & 3 \end{bmatrix}$$

(Part B. Not Related to Above). Find the following matrix exponential:

$$\text{Exp} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 3 \end{pmatrix} = ?$$

$$\text{Exp} \begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}$$

$$\begin{matrix} \lambda \swarrow \\ \begin{pmatrix} -3/2 & 1 \\ 0 & -3/2 \end{pmatrix} \cdot (-2) \end{matrix} \quad \begin{matrix} \nwarrow t \\ \end{matrix}$$

$$= \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$$

$$= \begin{bmatrix} e & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^3 & -2e^3 \\ 0 & 0 & 0 & e^3 \end{bmatrix}$$

Name: _____

Class Time: _____

6. (10 pts.) Consider the system of difference equations:

$$x(n) = 3x(n-1) - y(n-1) + z(n-1)$$

$$y(n) = x(n-1) + 4y(n-1) - 2z(n-1)$$

$$z(n) = x(n-1) + y(n-1) + z(n-1)$$

Let Q be the matrix associated to the systems of difference equations above. The eigenvalues and their associated eigenvectors of matrix Q are given below:

$$\lambda_1 = 2: \quad \vec{u} = \begin{pmatrix} 0 \\ r \\ r \end{pmatrix}; \quad \lambda_2 = 3: \quad \vec{v} = \begin{pmatrix} s \\ s \\ s \end{pmatrix}$$

(Part A). Write down the matrix Q , the canonical form J of Q , and find a matrix P such that $Q = PJP^{-1}$.

$$\vec{X}(n) = \underbrace{\begin{pmatrix} 3 & -1 & 1 \\ 1 & 4 & -2 \\ 1 & 1 & 1 \end{pmatrix}}_Q \vec{X}(n-1)$$

$$|Q - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 1 \\ 1 & 4-\lambda & -2 \\ 1 & 1 & 1-\lambda \end{vmatrix}$$

$$(3-\lambda) \begin{vmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} 1 & 4-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (3-\lambda) \left[(4-\lambda)(1-\lambda) + 2 \right] + \underbrace{(1-\lambda+2)}_{3-\lambda} + \underbrace{(1-4+\lambda)}_{-3+\lambda}$$

$$= (3-\lambda) (\lambda^2 - 5\lambda + 4 + 2 + 1 - 1) = (3-\lambda) (\lambda^2 - 5\lambda + 6)$$

$$= (3-\lambda) (\lambda-3)(\lambda-2) = 0 \Rightarrow \lambda = 2, 3, 3 \leftarrow \text{repeated } 3$$

Generalized Eigenvector for $\lambda = 3$:

$$(Q - 3I) \vec{w} = \vec{v} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} s \\ s \\ s \end{pmatrix} \Rightarrow \begin{cases} -w_2 + w_3 = s \\ w_1 + w_2 - 2w_3 = s \end{cases}$$

$$w_3 = s + w_2 \Rightarrow w_1 = s - w_2 + 2w_3 = 3s + w_2$$

$$\vec{w} = \begin{pmatrix} 3s + w_2 \\ w_2 \\ s + w_2 \end{pmatrix} = s \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} + w_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Name: _____

Class Time: _____

(Q6 continue ... Part B). (10 pts.) Solve the given system of difference equations:

$$x(n) = 3x(n-1) - y(n-1) + z(n-1)$$

$$y(n) = x(n-1) + 4y(n-1) - 2z(n-1)$$

$$z(n) = x(n-1) + y(n-1) + z(n-1)$$

$$P = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \leftarrow \begin{matrix} r=1 \\ s=1, w_2=0 \end{matrix}; \quad J = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\vec{X}(n) = Q^n \vec{X}(0) = P J^n P^{-1} \underbrace{\vec{X}(0)}_{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}$$

$$\begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 & 0 \\ 0 & 3^n & n3^{n-1} \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 3 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 2^n \\ c_2 3^n + c_3 n 3^{n-1} \\ c_3 3^n \end{bmatrix}$$

$$x(n) = (c_2 + 3c_3)3^n + c_3 n 3^{n-1}$$

$$y(n) = c_1 2^n + c_2 3^n + c_3 n 3^{n-1}$$

$$z(n) = c_1 2^n + (c_2 + c_3)3^n + c_3 n 3^{n-1}$$

Name: _____

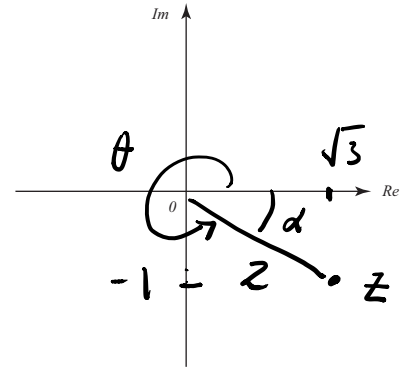
Class Time: _____

7.(10 pts.) Write the complex number $z = \sqrt{3} - i$ in the form $re^{i\theta}$ where $r > 0$ and $0 \leq \theta \leq 2\pi$. Give EXACT values of r and θ . You may find the Argand plane helpful.

$$\left. \begin{aligned} r \cos \theta &= \sqrt{3} \\ r \sin \theta &= -1 \end{aligned} \right\} \Rightarrow r^2 = 3 + 1 \Rightarrow r = 2$$

$$\tan \theta = \frac{-1}{\sqrt{3}} \quad \alpha = \text{reference angle}$$

$$\tan \alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6} \quad \left| \quad \theta = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \right. \\ z = 2e^{\frac{11\pi}{6}i}$$



Using your answer above, find the following:

(i) The conjugate $\bar{z}$ in the form $Re^{i\alpha}$ where $R > 0$ and $0 \leq \alpha \leq 2\pi$.

$$\bar{z} = \sqrt{3} + i = 2e^{\frac{\pi}{6}i}$$

(ii) z^{53} in the form $a + bi$ where a and b are numbers to be determined. Give also the argument of z^{53} between 0 and 2π .

$$\begin{aligned} z^{53} &= \left(2e^{\frac{11\pi}{6}i} \right)^{53} = 2^{53} e^{\frac{583\pi}{6}i} \\ &= 2^{53} e^{\frac{583\pi}{6}i} \quad \frac{583\pi}{6} = \left(97 + \frac{1}{6} \right) \pi \\ &\quad \sim \left(1 + \frac{1}{6} \right) \pi = \frac{7\pi}{6} \\ &= 2^{53} e^{\frac{7\pi}{6}i} = 2^{53} \left(\cos \frac{7\pi}{6} + i \sin \frac{7\pi}{6} \right) \\ &= 2^{53} \left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = -2^{52} \sqrt{3} - 2^{52}i \end{aligned}$$

$$\begin{array}{r} 97 \\ 6 \overline{) 583} \\ \underline{54} \\ 43 \\ \underline{42} \\ 1 \end{array}$$

Name: _____

Class Time: _____

8.(10 pts.) Evaluate $\begin{pmatrix} \sqrt{3} & 1 & 0 & 0 \\ -1 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}^6$

Entries of your answers should be completely evaluated with no trigonometric functions.

$$\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix}^6 = \begin{pmatrix} 2 \cos \frac{\pi}{6} & 2 \sin \frac{\pi}{6} \\ -2 \sin \frac{\pi}{6} & 2 \cos \frac{\pi}{6} \end{pmatrix}^6$$

$$= \begin{pmatrix} 2^6 \cos \pi & 2^6 \sin \pi \\ -2^6 \sin \pi & 2^6 \cos \pi \end{pmatrix} = \begin{pmatrix} -64 & 0 \\ 0 & -64 \end{pmatrix}$$

$$\begin{aligned} r \cos \theta &= \sqrt{3} \\ r \sin \theta &= 1 \\ \Rightarrow \begin{cases} r = 2 \\ \tan \theta = \frac{1}{\sqrt{3}} \end{cases} \\ \theta &= \pi/6 \end{aligned}$$

$$\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix}^6 = \left[2 \cdot \begin{pmatrix} 3/2 & 1 \\ 0 & 3/2 \end{pmatrix} \right]^6 = 2^6 \begin{pmatrix} \frac{3^6}{2^6} & 6 \cdot \frac{3^5}{2^5} \\ 0 & \frac{3^6}{2^6} \end{pmatrix}$$

$$= \begin{pmatrix} 3^6 & 6 \cdot 2 \cdot 3^5 \\ 0 & 3^6 \end{pmatrix} = \begin{pmatrix} 3^6 & 4 \cdot 3^6 \\ 0 & 3^6 \end{pmatrix}$$

$$\begin{bmatrix} -64 & 0 & 0 & 0 \\ 0 & -64 & 0 & 0 \\ 0 & 0 & 729 & 2916 \\ 0 & 0 & 0 & 729 \end{bmatrix}$$

$$\begin{aligned} 3^4 \cdot 3^2 \\ &= 81 \times 9 \\ &= 729 \end{aligned}$$

$$\begin{aligned} 4 \cdot 3^6 \\ &= 2800 + 80 + 36 \\ &= 2800 + 116 \end{aligned}$$

Name: _____

Class Time: _____

9. (10 pts.) A kind of "live bearing" fish goes through four life stages before dying:

Stage 1: Fry

Stage 3: "Young Adult"

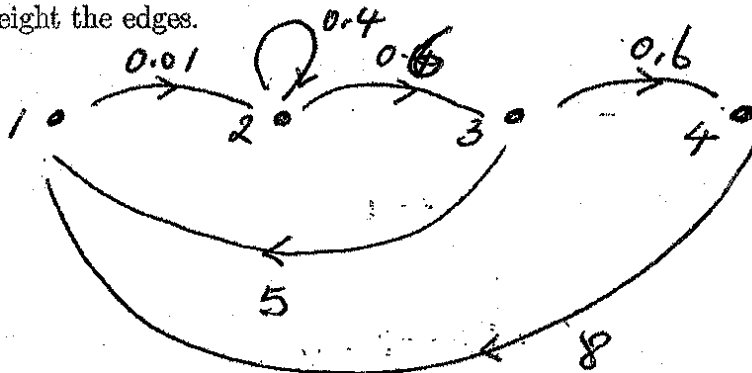
Stage 2: "Juvenile"

Stage 4: "Adult"

The Lefkovitch matrix of a kind of fish that goes through four life stages before dying is given by the matrix L below:

$$L = \begin{pmatrix} 0 & 0 & 5 & 8 \\ 0.01 & 0.4 & 0 & 0 \\ 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 \end{pmatrix}$$

9a. Draw the lifecycle graph of the fish. Label the vertices 1, 2, 3, and 4 according to stages and weight the edges.



9b. Using the MatLab information in the next page, find the following information:

(i). Estimate the continuous growth rate after a long time. Is the population expected to increase or decrease after a long time?

Largest eigenvalue = 0.6087

$e^k = 0.6087$

$\Rightarrow k = \ln(0.6087) < 0 \Rightarrow \text{decrease}$

(ii). The stable population vector. Round to three decimal places for each entry.

rescale $(0.9967 \quad 0.0472 \quad 0.0471 \quad 0.0464)^T$
to have sum of entries = 1, $\text{current sum} =$

Stable population vector =

$(0.876, 0.042, 0.041, 0.041)^T$

Name: _____

Class Time: _____

10.(10 pts.) The MatLab information for a matrix A is given on the next page. Answer the following question about A

10a. Write the canonical form J of A with all real entries.

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

10b. Find a real matrix P with all integer entries such that $A = PJP^{-1}$

$$P = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$

Name: _____

Class Time: _____

10c. Write down all eigenspaces of matrix A. For generalized eigenspaces, set the parameters so that Jordan chain relations are indicated.

$$E_{2+i} = \left\{ \begin{pmatrix} 0 \\ s \\ s \\ s \end{pmatrix} + \begin{pmatrix} s \\ 0 \\ s \\ 0 \end{pmatrix} i ; s \in \mathbb{R} \right\}$$

$$E_{2-i} = \left\{ \begin{pmatrix} 0 \\ s \\ s \\ s \end{pmatrix} - \begin{pmatrix} s \\ 0 \\ s \\ 0 \end{pmatrix} i ; s \in \mathbb{R} \right\}$$

$$E_{-1} = \left\{ r \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; r \in \mathbb{R} \right\}$$

1st generalized eigenspace for $\lambda = -1$

$$\left\{ r \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; r, t \in \mathbb{R} \right\}$$

Lefkovitch Matrix MatLab Data Sheet

```
>> L=[0 0 5 8; 0.01 0.4 0 0; 0 0.6 0 0; 0 0 0.6 0]
```

L =

0	0	5.0000	8.0000
0.0100	0.4000	0	0
0	0.6000	0	0
0	0	0.6000	0

```
>> [V,E]=eig(L)
```

V =

Columns 1 through 3

0.9967	0.9984	0.9984
0.0478	-0.0127 - 0.0141i	-0.0127 + 0.0141i
0.0471	-0.0236 + 0.0165i	-0.0236 - 0.0165i
0.0464	0.0207 + 0.0386i	0.0207 - 0.0386i

Column 4

-0.9980
0.0142
-0.0280
0.0554

E =

Columns 1 through 3

0.6087	0	0
0	0.0475 + 0.3919i	0
0	0	0.0475 - 0.3919i
0	0	0

Column 4

0
0
0
-0.3036

Matrix A MatLab Information

```
>> [V,E]=jordan(A)
```

V =

0 - 0.5000i	0 + 0.5000i	-1.0000	0
0.5000	0.5000	0	0
0.5000 - 0.5000i	0.5000 + 0.5000i	0	-0.5000
0.5000	0.5000	0	-0.5000

E =

2.0000 - 1.0000i	0	0	0
0	2.0000 + 1.0000i	0	0
0	0	-1.0000	1.0000
0	0	0	-1.0000