Math 10360 Review for Exam 3

1. Let $f(x, y) = x^3y + \ln(y - x)$. Find the following limits:

(a) $\lim_{h\to 0} \dfrac{f(1 + h, 2) - f(1, 2)}{h}$ (b) $\lim_{h\to 0} \dfrac{f(1, 2 + h) - f(1, 2)}{h}$ Ans. (a) 5; (b) 2.

2. Let $f(x, y) = \dfrac{x}{1 + y} + e^{xy}$. Find all first and second partial derivatives of f.

$\frac{\partial f}{\partial x} = \frac{1}{y+1} + ye^{xy}$; $\frac{\partial f}{\partial y} = -x(1+y)^{-2} + xe^{xy}$; $\frac{\partial^2 f}{\partial x^2} = y^2e^{xy}$; $\frac{\partial^2 f}{\partial y^2} = 2x(1+y)^{-3} + x^2e^{xy}$; $\frac{\partial^2 f}{\partial x\partial y} = -(1+y)^{-2} + e^{xy} + xye^{xy} = \frac{\partial^2 f}{\partial y\partial x}$

3. Let $z = f(x, y)$ be a function, with $f(1, 2) = 5$, $\frac{\partial f}{\partial x}(1, 2) = 2$, $\frac{\partial f}{\partial y}(1, 2) = -3$. (a) Estimate the value of $f(0.8, 2.1)$, (b) Find the elasticity coefficient of z relative to x at $(1, 2)$ and the elasticity coefficient of z relative to y at $(1, 2)$.

4. Let $f(x, y) = ye^x + 2x - 2y + 5$. Find $f(0, 0)$ and use linear approximation to estimate $f(-0.05, 0.01)$.

Ans. $f(-0.05, 0.01) \approx 4.89$.

5. The population $p(t)$ (in thousands) of cheetah is modeled by the differential equation $\dfrac{dp}{dt} = p\left(1 - \dfrac{p}{4}\right)$. If the initial population is 3 thousand, find $p(t)$. You are required to solve the differential equation.

(Ans: $p(t) = \dfrac{12e^t}{1 + 3e^t} = \dfrac{12}{e^{-t} + 3}$)

6. A cup of warm coffee at 80^oC is cooling off outdoors according to the equation

$$\frac{dy}{dt} = k(y - 20)$$

where $y(t)$ is the temperature (in ^{o}C) of the coffee at time t minutes. If the temperature of the coffee is 60^oC after 10 minutes, what is (i) the value of k, and (ii) the temperature when $t = 15$ minutes?

(Ans: (i) $k = \frac{1}{10}\ln\left(\frac{2}{3}\right)$, (ii) $y(15) = 60\left(\frac{2}{3}\right)^{3/2} + 20$)

7. Solve the following initial value problem: $x^2y' = 2xy - 3$; $y(1) = -1$. (Ans: $y(x) = x^{-1} - 2x^2$)

8. The temperature at a point (x, y) on a table is $T(x, y)$, measured in degree Celsius. A bug crawls so that its position on the table after t seconds is given by $x = \sqrt{1 + t}$, $y = 2 + (t/3)$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds? (Ans: 2 ^{o}C/s)

9. The temperature at a point (x, y) on a flat metal plate is given by

$$T(x, y) = \frac{60}{1 + x^2 + y^2}$$

where T is measured in ^{0}C and x, y in meters. Determine which is bigger: the sensitivity of the temperature T in the x-direction or in the y-direction at the point $(2, 1)$. Discuss the elasticity of T at $(2, 1)$.

10. Find the rate of change of z with respect to y if $10\cos(x + y + z) = xz^2 + x^2y$

11. $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial u}{\partial s}$ where $u = xyz^2$; $x = \dfrac{t}{s}$, $y = t^2 + 2s$ and $z = e^{st}$. Find u_t when $t = 1$ and $s = -1$.

12. A tank contains 1000 L of brine with 15 kg of dissolved salt. Brine that contains 0.03 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept throughly mixed and drains from the tank at the same rate. How much salt is in the tank (a) after t minutes and (b) after 20 minutes? (c) What is the limiting concentration? How would your answers change if the brine is drain away at a rate of (i) 5 L/min, and (ii) 20 L/min?

13. The body weight of an animal which initially weighs 10kg is modeled by:

$$\frac{dw}{dt} = -\ln\left(\frac{w}{5}\right); \qquad w(0) = 10$$

Use Euler's method with $\Delta t = 0.1$ to estimate $w(0.2)$

14. Let $P(t)$ be the immigrant population (in thousands) of a small country, where t is time in months since the beginning of this year. A demographic study has shown that $P(t)$ is a solution to the following initial value problem:

$$\frac{dP}{dt} = 0.4t \cdot P, \quad P(0) = 2$$

Use Euler's Method with 3 steps to estimate $P(1.5)$.

15. (a) Find **all** possible estimates for the value of $\frac{\partial f}{\partial x}(2,1)$ and $\frac{\partial f}{\partial y}(2,1)$ if $f(x,y)$ is given by the table below. (b) How would you estimate $f(2.1, 1.2)$ using the central difference estimates for the partial derivatives of $f(x,y)$ at $(2,1)$?

		x		
	**	**1.5**	**2.0**	**2.5**
y	**0.0**	36	35	34
	1.0	38	37	35
	2.0	44	42	38

16. Find the Nth partial sum of the series:

$$\sum_{n=4}^{\infty} \frac{8}{n^2+n}.$$

What is the sum of the series?

Math 10360 Review for Exam 3 Answers

12a. $\frac{dy}{dt} = 0.3 - \frac{y}{100}$; $y(0) = 15$ so $y(t) = 30 - 15e^{-t/100}$

12b. $y(20) = 30 - 15e^{-0.2}$

12c. Concentration at time t, $c(t) = \frac{y(t)}{1000} = 0.03 - 0.015e^{-t/100}$. Therefore $\lim_{t\to\infty} c(t) = 0.03$

12 (i) a. $\frac{dy}{dt} = 0.3 - \frac{5y}{1000+5t} = 0.3 - \frac{y}{200-t}$; $y(0) = 15$ so $y(t) = 0.15(200+t) - \frac{3000}{(200+t)}$

12 (i) b. $y(20) = 0.15(220) - \frac{3000}{220} = 33 - \frac{150}{11} = \frac{213}{11}$

12 (i) c. Concentration at time t, $c(t) = \frac{y(t)}{1000+5t} = 0.03 - \frac{600}{(200+t)^2}$. Therefore $\lim_{t\to\infty} c(t) = 0.03$

12 (ii) a. $\frac{dy}{dt} = 0.3 - \frac{20y}{1000-10t} = 0.3 - \frac{2y}{100-t}$; $y(0) = 15$ so $y(t) = 0.3(100-t) - \frac{3(100-t)^2}{2000}$

12 (ii) b. $y(20) = 0.3(80) - \frac{3(80)^2}{2000} = 24 - \frac{3(6400)}{2000} = 24 - \frac{48}{5} = 14\frac{2}{5}$

12 (ii) c. Concentration at time t, $c(t) = \dfrac{y(t)}{1000 - 10t} = 0.03 - \dfrac{3(100-t)}{20000}$. The tank dries out at $t = 100$ minutes. Just before it dries out $c = 0.03$ so saltier than at the start. In fact, $c'(t) > 0$ implies the concentration is increasing and reaches 0.03 just before it dries out.

13. $w(0) = 10$; $w(0.1) \approx 9.930685282$; $w(0.2) \approx 9.862066124$.

14. $P(0) = 2$; $P(0.5) = 2$; $P(1) = 2.2$; $P(1.5) = 2.64$.

Math 10360: Calculus B
Exam III
April 21, 2050

Name: ______________________________

Class Time: __________________________

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 10 pages of the test.

Honor pledge. "As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.":

____

Good Luck!

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	a	b	c	d	e
2.	a	b	c	d	e
3.	a	b	c	d	e
4.	a	b	c	d	e
5.	a	b	c	d	e
6.	a	b	c	d	e
7.	a	b	c	d	e
8.	a	b	c	d	e

Please do NOT write in this box.

Multiple Choice	________
9.	________
10.	________
11.	________
12.	________
13.	________
Total	________

Name: ______________________________

Class Time: __________________________

Multiple Choice

1.(5 pts.) Let $P(K, L)$ be the production function for a manufacturing company where K is the amount of capital and L is the amount of labor. Suppose

$$P(20, 30) = 10; \qquad \frac{\partial P}{\partial K}(20, 30) = 2; \qquad \frac{\partial P}{\partial L}(20, 30)(20, 30) = -3$$

Using the linear approximation of $P(K, L)$ at $(20, 30)$, estimate the amount of production when $K = 20.5$ and $L = 29.5$.

(a) 7.5

(b) 12.5

(c) 11.5

(d) 8.5

(e) 10.5

2.(5 pts.) Find the second partial derivative $\dfrac{\partial^2 f}{\partial y \partial x}$ if $f(x, y) = \ln(3x + 2y)$.

(a) $\dfrac{6}{(3x + 2y)^2}$

(b) $\dfrac{-4}{(3x + 2y)^2}$

(c) $\dfrac{4}{(3x + 2y)^2}$

(d) $\dfrac{-9}{(3x + 2y)^2}$

(e) $\dfrac{-6}{(3x + 2y)^2}$

Name: ______________________________

Class Time: __________________________

3.(5 pts.) A 500-gallon tank contains 200 gallons of brine with concentration 1/4 pounds of salt per gallon. Brine containing 2/5 pounds of salt per gallon is pumped into the tank at a rate of 5 gallons per minute. The mixture is pumped out of the tank at a rate of 4 gallons per minute. If $y(t)$ is the amount of salt in the tank at time t, find the differential equation modeling the amount of salt in the tank at time t.

(a) $\frac{dy}{dt} = 2 - \frac{y}{50}$

(b) $\frac{dy}{dt} = 2 - \frac{4y}{200 - 4t}$

(c) $\frac{dy}{dt} = \frac{8}{5} - \frac{5y}{200 + t}$

(d) $\frac{dy}{dt} = 2 - \frac{4y}{200 - t}$

(e) $\frac{dy}{dt} = 2 - \frac{4y}{200 + t}$

4.(5 pts.) For the same scenario above, how long it take before the brine overflow?

(a) 100 minutes

(b) 75 minutes

(c) 300 minutes

(d) 500 minutes

(e) 60 minutes

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Class Time: __________________________

5.(5 pts.) Consider the solution curve of the differential equation below that passing through the point $(-1, 1)$.

$$y' = x^2 + y^2$$

Find the equation of the tangent line to the solution curve at $(-1, 1)$.

(a) $y = 2x - 3$

(b) $y = -2x - 1$

(c) $y = 2x + 3$

(d) $y = 2x$

(e) $y = 1$

6.(5 pts.) Find all values of x for which the series $\sum_{n=1}^{\infty} 3(2-x)^n$ is convergent.

Hint: Geometric series.

(a) $-1 < x < 1$

(b) $-\frac{1}{3} < x < \frac{1}{3}$

(c) $\frac{5}{3} < x < \frac{7}{3}$

(d) $\frac{1}{3} < x < 1$

(e) $1 < x < 3$

Name: ______________________________

Class Time: __________________________

7.(5 pts.) Find the Nth-**partial sum** of the series $\sum_{n=2}^{\infty} \left(\sqrt{n+3} - \sqrt{n+2}\right)$

(a) $\sqrt{N+4} - 2$

(b) $\sqrt{N+3} - \sqrt{N+2}$

(c) $2 - \sqrt{N+2}$

(d) -2

(e) Divergent

8.(5 pts.) Find y in terms of t if

$$\ln(1-y) - \ln(2-y) = t.$$

(a) $y = \dfrac{1+2e^t}{1+e^t}.$

(b) $y = \dfrac{1+e^t}{1+2e^t}.$

(c) $y = \dfrac{2-e^t}{1-e^t}.$

(d) $y = \dfrac{1-2e^t}{1-e^t}.$

(e) $y = \dfrac{1-e^t}{2-e^t}.$

Name: ______________________________

Class Time: __________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

9.(12 pts.) Find the partial derivative of z with respect to y if

$$\frac{x}{(2y+z)^2} = e^{y^2} z - 3z^4$$

Name: ___________________________

Class Time: ________________________

10.(12 pts.) **[Part A].** Consider the initial value problem:

$$y' = (y+t)^2; \qquad y(0) = 1$$

Use Euler's method with two equal steps to estimate $y(1)$.

[**Part B. (Unrelated to Part A)**] Write the following repeated decimal as fraction. Show clear how you apply geometric series

$0.1\overline{02}$

Name: ____________________________

Class Time: ________________________

11.(12 pts.) Solve the following initial value problem:

$$(x^2+1)\left(\frac{dy}{dx}-3\right)=2xy; \qquad y(1)=0$$

Name: ______________________________

Class Time: __________________________

12.(12 pts.) [**Part A.**] Let $u = \sin(x^2 + y^2)$. Find $\dfrac{\partial u}{\partial s}$ if $x = st$ and $y = \dfrac{1}{s + 2t}$. Give your answer in terms of s and t.

[**Part B. (Unrelated to Part A)**] Discuss the elasticity of $z = e^{x^2 + xy^2}$ relative to x and y at $(x, y) = (1, 1)$

Name: ____________________________

Class Time: ________________________

13.(12 pts.) A ball is **projected** from the ground to a height of 10 feet. Each time it drops h feet, it rebounds to a height of $h/3$ feet. Answer each of the questions below:

(a) What is the total distance travelled when the ball hits the ground the third time?

(b) Write down a geometric series that gives the **total** distance travelled by the ball if the motion persists. Give its first term, and common ratio.

First Term = ________________ Common Ratio = ________________

(c) Find the total distance travelled by the ball. You should simplify your answer as far as possible.

Math 10360: Calculus B
Exam III
April 21, 2050

Name: ______________________________

Class Time: ANSWERS

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6.	a	b	c	d	•
7.	•	b	c	d	e
8.	a	b	c	•	e

Please do NOT write in this box.

Multiple Choice	________
9.	________
10.	________
11.	________
12.	________
13.	________
Total	________

Name: ______________________

Class Time: ______________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

11.(12 pts.) Find the partial derivative of z with respect to y if

$$\frac{x}{(2y+z)^2} = e^{y^2}z - 3z^4$$

$$\frac{\partial}{\partial y}\left(x(2y+z(x,y))^{-2}\right) = \frac{\partial}{\partial y}\left(e^{y^2}z(x,y) - 3(z(x,y))^4\right)$$

$$x(2y+z(x,y))^{-3}\cdot(-2)\left(2+\frac{\partial z}{\partial y}\right)$$

$$= e^{y^2}\frac{\partial z}{\partial y} + 2ye^{y^2}z - 3(4)(z(x,y))^3\cdot\frac{\partial z}{\partial y}$$

$$\frac{-2x}{(2y+z)^3}\left(2+\frac{\partial z}{\partial y}\right)$$

$$= e^{y^2}\frac{\partial z}{\partial y} + 2yze^{y^2} - 12z^3\frac{\partial z}{\partial y}$$

$$-\frac{4x}{(2y+z)^3} - \frac{2x}{(2y+z)^3}\frac{\partial z}{\partial y} = e^{y^2}\frac{\partial z}{\partial y} + 2yze^{y^2} - 12z^3\frac{\partial z}{\partial y}$$

$$12z^3\frac{\partial z}{\partial y} - e^{y^2}\frac{\partial z}{\partial y} - \frac{2x}{(2y+z)^3}\frac{\partial z}{\partial y} = 2yze^{y^2} + \frac{4x}{(2y+z)^3}$$

$$\left(12z^3 - e^{y^2} - \frac{2x}{(2y+z)^3}\right)\frac{\partial z}{\partial y} = 2yze^{y^2} + \frac{4x}{(2y+z)^3}$$

$$\frac{\partial z}{\partial y} = \frac{2yze^{y^2} + \frac{4x}{(2y+z)^3}}{12z^3 - e^{y^2} - \frac{2x}{(2y+z)^3}} = \frac{2yze^{y^2}(2y+z)^3 + 4x}{(12z^3 - e^{y^2})(2y+z)^3 - 2x}$$

Name: ______________________________

Class Time: ______________________________

$\Delta t = \frac{1-0}{2} = \frac{1}{2}$

10.(12 pts.) **[Part A].** Consider the initial value problem:

$$y' = (y+t)^2; \qquad y(0) = 1$$

Use Euler's method with two equal steps to estimate $y(1)$.

0 — 0.5 — 1

$y(0.5) \simeq y(0) + y'(0)\cdot\Delta t$

$= 1 + 1\cdot\frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}$

$y'(t) = (y(t) + t)^2$
$y'(0) = (y(0) + 0)^2$
$= 1^2$

$y(1) \simeq y(0.5) + y'(0.5)\cdot\Delta t$

$= \frac{3}{2} + 4\cdot\frac{1}{2}$

$= \frac{3}{2} + 2$

$= \frac{7}{2}$

$y'(0.5) = (y(0.5) + 0.5)^2$
$= \left(\frac{3}{2} + \frac{1}{2}\right)^2$
$= (2)^2 = 4$

[**Part B. (Unrelated to Part A)**] Write the following repeated decimal as fraction. Show clear how you apply geometric series

$0.1\overline{02} = 0.1020202\ldots = 0.1 + 0.0020202\ldots$

$= 0.1 + 0.002 + 0.00002 + 0.0000002 + 0.000000002 + \ldots$

$\times \frac{1}{100} \quad \times \frac{1}{100} \quad \times \frac{1}{100}$

$= 0.1 + \frac{0.002}{1 - \frac{1}{100}} = \frac{1}{10} + \frac{\frac{2}{1000}}{\frac{99}{100}} = \frac{1}{10} + \frac{2}{1000} \times \frac{100}{99}$

$= \frac{1}{10} + \frac{1}{495} = \frac{99 + 2}{990} = \frac{101}{990}.$

$99 \times 5 = 495$

$2\times 5 \qquad 5\times 99$

Name: ____________________

Class Time: ____________________

13.(12 pts.) Solve the following initial value problem:

$$(x^2+1)\left(\frac{dy}{dx}-3\right)=2xy; \qquad y(1)=0$$

$$(x^2+1)\frac{dy}{dx}-3(x^2+1)=2xy$$

$$(x^2+1)\frac{dy}{dx}-2xy=3(x^2+1)$$

$$\frac{dy}{dx}-\frac{2x}{x^2+1}y=3$$

Integrating factor: $e^{\int \frac{-2x}{(x^2+1)}dx}=e^{-\ln(x^2+1)}=e^{\ln(x^2+1)^{-1}}$

$$=\frac{1}{x^2+1}$$

$$\frac{1}{x^2+1}\cdot\frac{dy}{dx}-\frac{2x}{(x^2+1)^2}y=\frac{3}{x^2+1}$$

check:

$$((x^2+1)^{-1})'$$

$$=-(x^2+1)^{-2}\cdot 2x$$

$$=\frac{-2x}{(x^2+1)^{-2}}$$

$$\frac{1}{x^2+1}\cdot\frac{dy}{dx}+\left(\frac{1}{x^2+1}\right)'\cdot y=\frac{3}{x^2+1}$$

$$\left(\frac{1}{x^2+1}y\right)'=\frac{3}{x^2+1}$$

$$\frac{1}{x^2+1}y=\int\frac{3}{x^2+1}dx=3\arctan(x)+C$$

$$y(1)=0 \Rightarrow \frac{1}{2}(0)=3\arctan(1)+C=\frac{3\pi}{4}+C \Rightarrow C=\frac{-3\pi}{4}$$

$$\frac{1}{x^2+1}y=3\arctan(x)-\frac{3\pi}{4} \Rightarrow y=3(x^2+1)\arctan(x)-\frac{3\pi}{4}(x^2+1)$$

Name: ______________________________

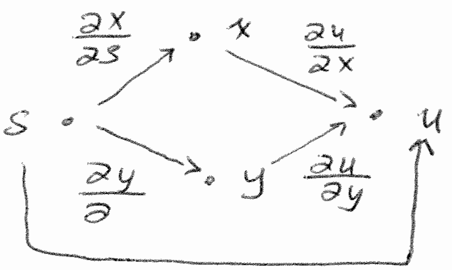
Class Time: ______________________________

14.(12 pts.) [**Part A.**] Let $u = \sin(x^2 + y^2)$. Find $\dfrac{\partial u}{\partial s}$ if $x = st$ and $y = \dfrac{1}{s+2t}$. Give your answer in terms of s and t.

$(s+2t)^{-1}$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= 2x\cos(x^2+y^2)\cdot t + 2y\cos(x^2+y^2)\cdot\left[-(s+2t)^{-2}\cdot 1\right]$$



$$= 2st^2\cos\left[s^2t^2 + \frac{1}{(s+2t)^2}\right]$$

$$+ 2\cdot\frac{1}{s+2t}\cos\left(s^2t^2+\frac{1}{(s+2t)^2}\right)\cdot\frac{-s}{(s+2t)^2}$$

$$x^2+y^2 = s^2t^2 + \frac{1}{(s+2t)^2}$$

$$= \left(2st^2 - \frac{2s}{(s+2t)^3}\right)\cos\left(s^2t^2 + \frac{1}{(s+2t)^2}\right)$$

[**Part B. (Unrelated to Part A)**] Discuss the elasticity of $z = e^{x^2+xy^2}$ relative to x and y at $(x, y) = (1, 1)$

- Find the percentage change in z when x increases by 1% and y does not change.

$$\Delta x = \frac{1}{100}\times 1 = \frac{1}{100}\ ;\ \Delta y = 0.$$

$$\frac{\Delta z}{z(1,1)}\times 100\% \leftarrow ?\ ;\ z(1,1) = e^2$$

$$\Delta z \simeq \frac{\partial z}{\partial x}(1,1)\cdot\Delta x + \underbrace{\frac{\partial z}{\partial y}(1,1)\cdot\Delta y}_{0}$$

$$\frac{\partial z}{\partial x}(1,1) = (2x+y^2)e^{x^2+xy^2}\Big|_{\substack{x=1\\y=1}} = 3e^2$$

$$\Delta z \simeq 3e^2\cdot\frac{1}{100} = \frac{3e^2}{100}$$

$$\frac{\Delta z}{z(1,1)}\times 100\% = \frac{\frac{3e^2}{100}}{e^2}\times 100\% = \boxed{3\%}$$

A similar computation can be done for elasticity relative to y

$Z_y(x,y) = 2xye^{x^2+xy^2}$

$Z_y(1,1) = 2e^2$

Here $\Delta x = 0$ and $\Delta y = 0.01$

So ΔZ is approx $2e^2/100$

So elasticity relative to y is 2%

Name: ____________________

Class Time: ____________________

13.(12 pts.) A ball is **projected** from the ground to a height of 10 feet. Each time it drops h feet, it rebounds to a height of $h/3$ feet. Answer each of the questions below:

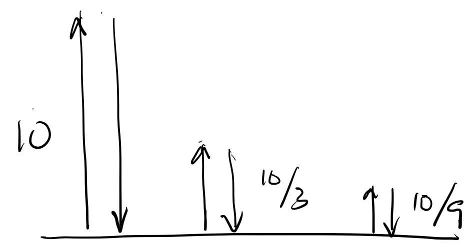
(a) What is the total distance travelled when the ball hits the ground the third time?

$$10 \times 2 + \frac{10}{3} \times 2 + \frac{10}{9} \times 2$$

$$= 20 + \frac{20}{3} + \frac{20}{9}$$

$$= 20\left(1 + \frac{1}{3} + \frac{1}{9}\right) = 20\left(\frac{9+3+1}{9}\right) \quad \to 13$$

$$= 260/9 \text{ ft.}$$



(b) Write down a geometric series that gives the **total** distance travelled by the ball if the motion persists. Give its first term, and common ratio.

$$20 + \frac{20}{3} + \frac{20}{3^2} + \cdots + \frac{20}{3^{n-1}} + \cdots$$

First Term = 20 Common Ratio = $1/3$

(c) Find the total distance travelled by the ball. You should simplify your answer as far as possible.

$$\text{Total distance} = \frac{20}{1 - \frac{1}{3}}$$; Note common ratio $|1/3| < 1$

$$= \frac{20}{\frac{2}{3}} = 20 \times \frac{3}{2} = 30 \text{ ft.}$$

Math 10360: Calculus B
Exam III
November 17, 2050

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Class Time: ________________________

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4.	a	b	c	d	e
5.	a	b	c	d	e
6.	a	b	c	d	e
7.	a	b	c	d	e
8.	a	b	c	d	e

Please do NOT write in this box.

Multiple Choice	______
9.	______
10.	______
11.	______
12.	______
13.	______
Total	______

Name: ______________________________

Class Time: __________________________

Multiple Choice

1.(5 pts.) Let $u = 2x^3 - 3y^2$. If $x = \ln(t^2)$ and $y = t^2 - e^2$ find the rate of change of u with respect to t when $t = e$.

(a) $\dfrac{24}{e}$

(b) $\dfrac{48}{e}$

(c) $\dfrac{24}{e} - 2e$

(d) $\dfrac{48}{e} - 2e$

(e) $\dfrac{48}{e} - 12e^2$

2.(5 pts.) Solve the initial value problem:

$$\frac{dy}{dx} = (2y-1)x; \qquad y(0) = 0$$

(a) $y = \dfrac{1}{2}(e^{x^2} - 1)$

(b) $y = \dfrac{1}{2}(1 - e^{x^2})$

(c) $y = e^{x^2} - 1$

(d) $y = \dfrac{1}{2}(1 - e^{x^2/2})$

(e) $y = \dfrac{1}{2}(e^{x^2/2} - 1)$

Name: ______________________________

Class Time: __________________________

3.(5 pts.) Consider the geometric series whose eighth term is 0.03 and whose ninth term is -0.15. What is the tenth term of this geometric series?

(a) $0.03(5)^{-10}$

(b) $0.03(5)^{10}$

(c) $-0.03(5)^{9}$

(d) -0.75

(e) 0.75

4.(5 pts.) Evaluate the following geometric series

$$2e^{-1} + 4e^{-3} + 8e^{-5} + 16e^{-7} + \cdots$$

where $e = 2.71828...$ is the natural number.

(a) $\dfrac{2e^{-1}}{1 - 2e^{-2}}$

(b) $\dfrac{2e^{-1}}{1 + 2e^{-2}}$

(c) Series is divergent.

(d) $\dfrac{2}{1 + 2e^{-2}}$

(e) $\dfrac{2}{1 - 2e^{-2}}$

Name: ______________________________

Class Time: __________________________

5.(5 pts.) Let $f(x,y) = e^{x^2 y}$. Find the value of the limit

$$\lim_{h \to 0} \frac{f(2,-1+h) - f(2,-1)}{h} \ .$$

(a) $2e^{-2}$

(b) e^{-4}

(c) $-2e^{-2}$

(d) $4e^{-4}$

(e) Does not exist.

6.(5 pts.) Evaluate the following geometric series

$$1 - \frac{\pi}{2} + \frac{\pi^2}{4} - \frac{\pi^3}{8} + \cdots$$

(a) The series is divergent.

(b) $\dfrac{1}{2+\pi}$

(c) $\dfrac{2}{2-\pi}$

(d) $\dfrac{1}{2-\pi}$

(e) $\dfrac{2}{2+\pi}$

Name: ______________________________

Class Time: __________________________

7.(5 pts.) Find the limit of the sequence $\{e^{-2n}\sin(100n)\}_{n=1}^{\infty}$.

(a) e

(b) e^{-1}

(c) 0

(d) 1

(e) Does not exist.

8.(5 pts.) Consider the initial value problem:

$$y' = 3x + y, \qquad y(0) = 1.$$

Using Euler's method with **TWO** steps of equal size estimate $y(0.2)$

(a) 2

(b) 1.424

(c) 1.24

(d) 1.1

(e) 4.3

Name: ______________________________

Class Time: __________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

9.(12 pts.) **[Part A.]** A population of a specie of river dolphins is given, in the thousands, by the function

$$p(t) = \frac{2e^t + 7}{2e^t + 2}$$

where t is time in years. Find the time t for which the population of dolphins is 2 thousand.

[Part B (No relation with the above)]. Find the sum of the first 20 terms of the given series:

$$\sum_{n=0}^{\infty} \frac{3^{2n+1}}{4^n}$$

Name: ___________________________

Class Time: ________________________

10.(12 pts.) Find the Nth partial sum of the series below.

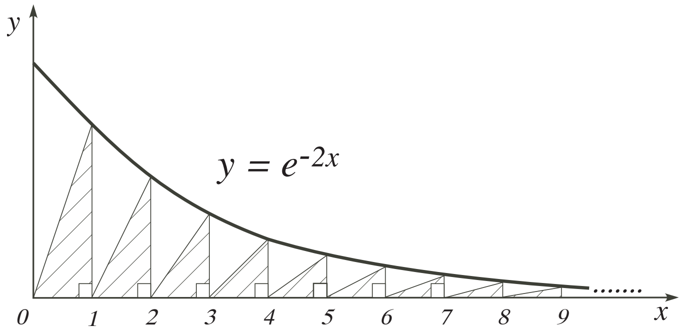
$$\sum_{n=5}^{\infty}\left[\cos\left(\frac{1}{2n-1}\right)-\cos\left(\frac{1}{2n+1}\right)\right]$$

Use your answer for the partial sum to find the sum of the series.

Name: ____________________________

Class Time: ________________________

11.(12 pts.)



An infinite sequence of right angle triangles with heights given by the curve $y = e^{-2x}$ is constructed as show above.

(a) Let T_n be the area of the nth triangle constructed. Write down the first three terms of the sequence $\{T_n\}$ and give also a formula for the general term T_n.

$T_1 =$ ____________; $T_2 =$ ____________; $T_3 =$ ____________

$T_n =$ ____________

(b) Write down the geometric series that gives the total area enclosed by the infinite sequence of triangles. Also give the **first term** and the **common ratio** of your series. **You must give at least the first three terms and the general term of the series in your answer.**

First term of the series = ________________ Common ratio of the series = ________________

(c) Find the total area enclosed by the infinite sequence of triangles.

Name: ____________________________

Class Time: ________________________

12.(12 pts.) Solve for $y(x)$ if it satisfies the following initial value problem:

$$y' = 3x - y\cot(x) \quad \text{and} \quad y(\pi/4) = 2.$$

Name: ______________________________

Class Time: __________________________

13.(12 pts.) Consider the function $f(x, y) = 3x^2 + 2y^3 + 4$.

a. Find the elasticity of $f(x, y)$ relative to y at $(1, 2)$.

b. Use the linear approximation of $f(x, y)$ at $(1, 2)$ to estimate the value of $f(1.1, 1.8)$.

Math 10360: Calculus B
Exam III
November 17, 2050

Name: ______________________________

Class Time: ANSWERS

- The Honor Code is in effect for this examination. All work is to be your own.
- No calculators.
- The exam lasts for one hour and 15 minutes.
- Be sure that your name is on every page in case pages become detached.
- Be sure that you have all 10 pages of the test.

Honor pledge. "As a member of the Notre Dame community, I will not participate in or tolerate academic dishonesty.":

____

Good Luck!

PLEASE MARK YOUR ANSWERS WITH AN X, not a circle!

1.	a	•	c	d	e
2.	a	•	c	d	e
3.	a	b	c	d	•
4.	•	b	c	d	e
5.	a	b	c	•	e
6.	•	b	c	d	e
7.	a	b	•	d	e
8.	a	b	•	d	e

Please do NOT write in this box.

Multiple Choice	________
9.	________
10.	________
11.	________
12.	________
13.	________
Total	________

Name: ____________________________

Class Time: ________________________

Partial Credit

You must show your work on the partial credit problems to receive credit!

9.(12 pts.) **[Part A.]** A population of a specie of river dolphins is given, in the thousands, by the function

$$p(t) = \frac{2e^t + 7}{2e^t + 2}$$

where t is time in years. Find the time t for which the population of dolphins is 2 thousand.

$$\frac{2e^t+7}{2e^t+2} = 2 \Rightarrow 2e^t + 7 = \underbrace{2(2e^t+2)}_{4e^t+4}$$

$$\Rightarrow 7-4 = 4e^t - 2e^t \Rightarrow 2e^t = 3$$

$$\Rightarrow e^t = \frac{3}{2} \Rightarrow t = \ln\left(\frac{3}{2}\right) \text{ years.}$$

[Part B (No relation with the above)]. Find the sum of the first 20 terms of the given series:

$$\sum_{n=0}^{\infty} \underbrace{\frac{3^{2n+1}}{4^n}}_{a_n} = \frac{3(1-(9/4)^{20})}{(1-9/4)} \leftarrow -\frac{5}{4}$$

1st term $(n=0)$

$$c = \frac{3^{0+1}}{4^0} = 3$$

$$= -\frac{4}{5} \times 3\left(1-(9/4)^{20}\right)$$

$$= -\frac{12}{5}\left(1-\left(\frac{9}{4}\right)^{20}\right)$$

$$= \frac{12}{5}\left(\left(\frac{9}{4}\right)^{20}-1\right)$$

Common ratio

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3^{2n+3}}{4^{n+1}}}{\frac{3^{2n+1}}{4^n}} = \frac{3^{2n+3}}{4^{n+1}} \cdot \frac{4^n}{3^{2n+1}}$$

$$= \frac{3^2}{4} = \frac{9}{4}$$

Name: ______________________________

Class Time: __________________________

10.(12 pts.) Find the Nth partial sum of the series below.

$$\sum_{n=5}^{\infty}\left[\cos\left(\frac{1}{2n-1}\right)-\cos\left(\frac{1}{2n+1}\right)\right]$$

Use your answer for the partial sum to find the sum of the series.

$n = 5, 6, 7, \ldots, ?$ (N terms)

$? - 5 = N - 1$

$? = N + 4$

$$S_N = \sum_{n=5}^{N+4}\left[\cos\left(\frac{1}{2n-1}\right)-\cos\left(\frac{1}{2n+1}\right)\right]$$

$= \cos\left(\frac{1}{9}\right) - \cos\left(\frac{1}{11}\right)$ $n = 5$

$\cos\left(\frac{1}{11}\right) - \cos\left(\frac{1}{13}\right)$ $n = 6$

$\cos\left(\frac{1}{13}\right) - \cos\left(\frac{1}{15}\right)$ $n = 7$

$\vdots$

$\cos\left(\frac{1}{2N+3}\right) - \cos\left(\frac{1}{2N+5}\right)$ $n = N+2$

$\cos\left(\frac{1}{2N+5}\right) - \cos\left(\frac{1}{2N+7}\right)$ $n = N+3$

$\cos\left(\frac{1}{2N+7}\right) - \cos\left(\frac{1}{2N+9}\right)$ $n = N+4$

Nth partial sum, $S_N = \cos\left(\frac{1}{9}\right) - \cos\left(\frac{1}{2N+9}\right)$

Sum of the given series $= \lim_{N\to\infty} S_N$

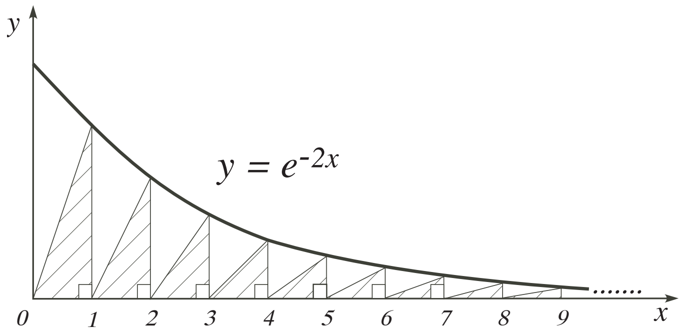
$= \lim_{N\to\infty}\left(\cos\left(\frac{1}{9}\right) - \cos\left(\frac{1}{2N+9}\right)\right) = \cos\left(\frac{1}{9}\right) - 1$

($\frac{1}{2N+9} \to 0$; $1 = \cos(0)$)

Name: ____________________

Class Time: ____________________

11.(12 pts.)



An infinite sequence of right angle triangles with heights given by the curve $y = e^{-2x}$ is constructed as show above.

(a) Let T_n be the area of the nth triangle constructed. Write down the first three terms of the sequence $\{T_n\}$ and give also a formula for the general term T_n.

$T_1 = \frac{1}{2}e^{-2}$; $T_2 = \frac{1}{2}e^{-4}$; $T_3 = \frac{1}{2}e^{-6}$

$T_n = \frac{1}{2}e^{-2n}$

(b) Write down the geometric series that gives the total area enclosed by the infinite sequence of triangles. Also give the **first term** and the **common ratio** of your series. **You must give at least the first three terms and the general term of the series in your answer.**

$$\frac{1}{2}e^{-2} + \frac{1}{2}e^{-4} + \frac{1}{2}e^{-6} + \cdots + \frac{1}{2}e^{-2n} + \cdots$$

$\times e^{-2}$ $\quad \times e^{-2}$

First term of the series = $\frac{1}{2}e^{-2}$ Common ratio of the series = e^{-2}

(c) Find the total area enclosed by the infinite sequence of triangles.

$$= \frac{\frac{1}{2}e^{-2}}{1 - e^{-2}} = \frac{e^{-2}}{2 - 2e^{-2}}$$

common ratio

$|e^{-2}| = \left|\frac{1}{e^{2}}\right| < 1$

OR $\dfrac{1}{2e^{2} - 2}$

Name: ______________________________

Class Time: __________________________

12.(12 pts.) Solve for $y(x)$ if it satisfies the following initial value problem:

$$y' = 3x - y\cot(x) \quad \text{and} \quad y(\pi/4) = 2.$$

$y' + \cot(x) \cdot y = 3x$ ← Linear 1st order

Integrating factor $= e^{\int \cot(x)\,dx} = e^{\int \frac{\cos x}{\sin x}\,dx}$

$u = \sin x$

$du = \cos x\,dx$

$= e^{\int \frac{1}{u}\,du} = e^{\ln u} = u = \sin x$

$\sin x\, y' + \underbrace{\cot x \cdot \sin x}_{\cos x} \cdot y = 3x \sin x$

$(\sin x \cdot y)' = 3x \sin x$

$\sin x \cdot y = \int \underbrace{3x}_{u} \underbrace{\sin x\,dx}_{dv}$

$u = x \Rightarrow du = dx$

$dv = \sin x\,dx$

$v = \int \sin x\,dx = -\cos x$

$\sin x \cdot y = 3x(-\cos x) - 3\int -\cos x\,dx$

$= -3x\cos x + 3\sin x + C$

$(\sin\frac{\pi}{4}) \cdot 2 = -3 \cdot \frac{\pi}{4}\cos\left(\frac{\pi}{4}\right) + 3\sin\frac{\pi}{4} + C$

$C = \frac{3\pi}{4} \cdot \frac{\sqrt{2}}{2} - \sin\frac{\pi}{4} = \frac{3\pi}{4} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}\left(\frac{3\pi}{4} - 1\right)$

divide by $\sin x$ → $y = -3x\cot x + 1 + \dfrac{\frac{\sqrt{2}}{2}\left(\frac{3\pi}{4} - 1\right)}{\sin x}$

Name: ______________________________

Class Time: ___________________________

13.(12 pts.) Consider the function $f(x,y) = 3x^2 + 2y^3 + 4$.

$\Delta x = 0$; $\Delta y = 1\%$ of $2 = \frac{2}{100}$

a. Find the elasticity of $f(x,y)$ relative to y at $(1,2)$.

$$\Delta f \simeq f_x(1,2)\cdot\Delta x + f_y(1,2)\cdot\Delta y = 0 + (0+6y^2)\Big|_{\substack{x=1\\y=2}} \cdot \frac{2}{100}$$

$$= 0 + 24\cdot\frac{2}{100} = \frac{48}{100}$$

$$\text{Required elasticity} = \frac{\Delta f}{f(1,2)} \times 100\% = \frac{\frac{48}{100}}{3+16+4} \times 100\%$$

$$= \frac{48}{23}\ \%$$

b. Use the linear approximation of $f(x,y)$ at $(1,2)$ to estimate the value of $f(1.1, 1.8)$.

$\Delta x = 1.1 - 1 = 0.1$

$\Delta y = 1.8 - 2 = -0.2$

$f_x(1,2) = (6x)\Big|_{\substack{x=1\\y=2}} = 6$

$f_y(1,2) = (6y^2)\Big|_{\substack{x=1\\y=2}} = 24$

$$\Delta f = f(1.1, 1.8) - f(1,2)$$

$$\simeq f_x(1,2)\cdot\Delta x + f_y(1,2)\cdot\Delta y$$

$$= 6(0.1) + 24(-0.2) = 0.6 - 4.8$$

$$= -4.2$$

$$f(1.1, 1.8) \simeq f(1,2) - 4.2 = (3+16+4) - 4.2$$

$$= 23 - 4.2 = 18.8$$