**Math 10350 – Example Set 06A**
**Section 3.7 The Chain Rule**

**Definition 1.** (The Composite Function) A function $h(x)$ is said to be a composite function of $g(x)$ followed by $f(x)$ if $h(x) = f(g(x))$. We may write: $h : x \xmapsto{g}$ ________ $\xmapsto{f}$ ________

**1.** Find functions $f(x)$ and $g(x)$, not equal $x$, such that $h(x) = f(g(x))$:

(a) $h(x) = (x^4 + 2x^2 + 7)^{21}$ $\qquad h : x \xmapsto{g}$ ____________ $\xmapsto{f}$ ____________

**Ans:** $f(x) \stackrel{?}{=}$ ____________ and $g(x) \stackrel{?}{=}$ ____________

(b) $h(x) = \sin(x^2 + 1)$ $\qquad h : x \longmapsto$ ____________ $\longmapsto$ ____________

**Ans:** $f(x) \stackrel{?}{=}$ ____________ and $g(x) \stackrel{?}{=}$ ____________

How can we find the derivative of composite functions? That is how can we evaluate these:

$$\frac{d}{dx}\left((x^4 + 2x^2 + 7)^{21}\right) \stackrel{?}{=} \qquad\qquad \left(\sin(x^2 + 1)\right)' \stackrel{?}{=}$$

In general we want a formula for $\dfrac{d}{dx}[f(g(x))] = [f(g(x))]' \stackrel{?}{=}$

**World Guinness Record Approved Text:** "The razor-toothed piranhas of the genera Serrasalmus and Pygocentrus are the most ferocious freshwater fish in the world. In reality they seldom attack a human."

Think about it: In a competition for the title of "Fastest Text Messager", it is observed that Competitor $A$ inputs text three times faster than $B$, and Competitor $B$ inputs text two times faster than $C$. How much faster is Competitor A than Competitor C? Why?

$$A \;--\; \underset{\text{Rate of } A \text{ relative to } B}{3} \;--\; B \;--\; \underset{\text{Rate of } B \text{ relative to } C}{2} \;--\; C$$

$$\text{---------- ? Rate of } A \text{ relative to } C \text{ ? ----------}$$

**The Chain Rule.** Suppose $y = f(g(x))$. To find a formula for $\frac{dy}{dx} = \frac{d}{dx}[f(g(x))]$, we set $u = g(x)$ then $y = f(u)$.

$$y \;--\; \underset{\text{Rate of } y \text{ relative to } u}{\frac{dy}{du}} \;--\; u \;--\; \underset{\text{Rate of } u \text{ relative to } x}{\frac{du}{dx}} \;--\; x$$

$$\underset{\text{---------- ? Rate of } y \text{ relative to } x \text{ ? ----------}}{\frac{dy}{dx} \stackrel{?}{=}}$$

So we expect $\boxed{\dfrac{dy}{dx} = \qquad\qquad}$

Our guess is in fact correct, and the formula for $\frac{dy}{dx}$ is called the **Chain Rule** (in Leibniz notation).

But $\frac{dy}{dx} = \frac{d}{dx}[f(g(x))] = [f(g(x))]'$, $\frac{dy}{du} = f'(u) = f'(g(x))$ and $\frac{du}{dx} = g'(x)$. Thus we also have:

$$\boxed{\frac{d}{dx}[f(g(x))] = [f(g(x))]' = \qquad\qquad\qquad\qquad}$$

**2.** Find the derivative of the following functions:

**a.** $\sin(x^2 + 1)$

**b.** $(x^4 + 2x^2 + 7)^{21}$

**c.** $f(t) = \sqrt[3]{1 + t^3}$

**d.** $g(x) = e^{5x^2+3}$

**e.** $y = \sec(e^x)$

**f.** $y(r) = \csc^3(\pi r)$

**3.** Find the $x-$coordinates of the points on the curve $y = \frac{1}{2}x^2\sqrt{16 - x^2}$ where the tangent lines are horizontal. Such values of $x$ are also called **stationary points** of the function $y = \frac{1}{2}x^2\sqrt{16 - x^2}$.

$y' = -\frac{1}{2}x^3(16 - x^2)^{-1/2} + x(16 - x^2)^{1/2} = \frac{32x - 3x^3}{2(16 - x^2)^{1/2}}$

**4.** Find the derivative of $w = \left(\dfrac{t+1}{t^2+2}\right)^4$

$\frac{dw}{dt} = \frac{4(t+1)^3(2-2t-t^2)}{(t^2+2)^5}$

## Math 10350 – Example Set 06B
## Section 3.7 The Chain Rule
## Section 3.9 Derivative of the Natural Log

**1.** Consider the functions $f(x) = e^x$ and $g(x) = \ln x$.

**a.** Use the fact that (1) the exponential function and logarithmic function are mutual inverse, (2) $\dfrac{d}{dx}(e^x) = e^x$ and (3) the chain rule, find a formula for $\dfrac{d}{dx}(\ln x)$.

**b.** Using the change of base formula $\log_b x = \dfrac{\ln x}{\ln b}$, show that $\dfrac{d}{dx}(\log_b x) = \dfrac{1}{x \ln b}$.

**2.** Find the equation of the tangent line to the graph of $y = \ln\left(\dfrac{e^x - 1}{e^x + 1}\right)$ when $x = \ln(2)$.

$\left(y = \frac{4}{3}x - \frac{4}{3}\ln 2 - \ln 3\right)$

**3.** Find the derivative of each of the following functions:

**a.** $g(z) = \log_{10}(\ln z)$

**b.** $s(t) = e^{(\ln t)^3}$

**c.** $y = x^e + e^x$

**d.** $y = x^x$

**Math 10350 – Example Set 06C**
**Section 3.8 Implicit Differentiation**
**including Logarithmic Differentiation**

**1.** Find the derivative of the given functions:

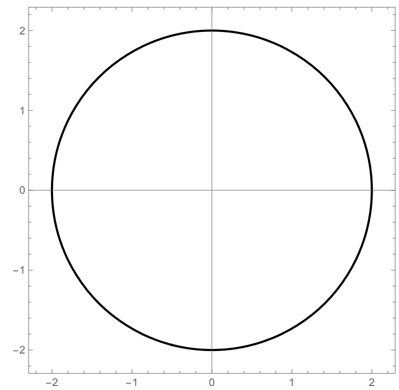
(a) $(2x+1)^{\cos(e)}$ (b) $(2e+1)^{\cos x}$ (a) $(2x+1)^{\cos x}$

**Implicit Differentiation.** We have computed the derivative of a function $y = f(x)$. Here $y$ is explicitly given as a function of $x$. For example $y = x^3 + x + 2$.

Next we want to find the derivative of $y$ with respect to $x$ when $x$ and $y$ are connected in a relation (an expression).

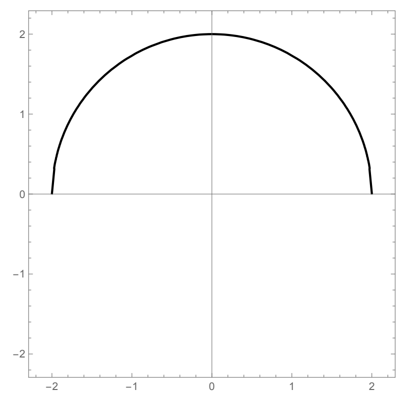
For example the graph of $x^2 + y^2 = 4$ is

a ________________________________________.



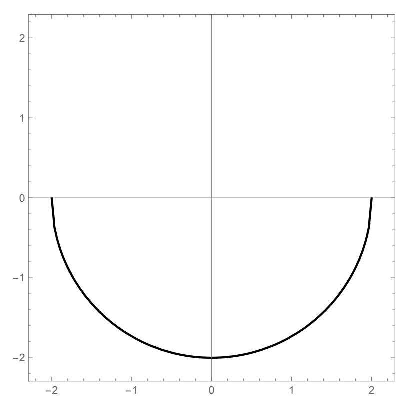
So $y$ is not a function of $x$.

But we interpret $y$ as a function of $x$ by considering "portions" or "sections" of the graph of the relation. For example, we can solve for $y$ in $x^2 + y^2 = 4$ to given two "branches" of $y$ as functions of $x$:



$y =$ ______________________

$\dfrac{dy}{dx} =$ ______________________



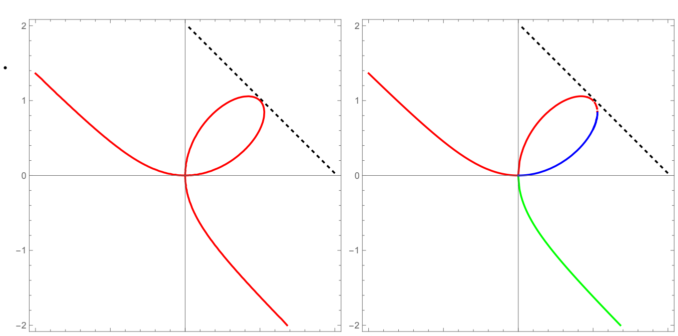
$y =$ ______________________

$\dfrac{dy}{dx} =$ ______________________

**Remark:** For a general relation between $x$ and $y$, it is difficult to write $y$ as a function of $x$. For example, it is hard to find $y$ in terms of $x$ for relations like $x^3 + y^3 = 2xy$.

However we can still interpret $y$ as an implicit function of $x$. To find $\dfrac{dy}{dx}$ in such situations we apply a powerful method

called ________________________________.



**3.** Consider the curve given by the implicit relation $x^3 + y^3 = 2xy$.

(a) Verify that the point $(1, 1)$ is on the curve $x^3 + y^3 = 2xy$, (b) find $\dfrac{dy}{dx}$ and (c) find the equation of the tangent to the curve at $(1, 1)$.

(b) $y' = \frac{2y-3x^2}{3y^2-2x}$; (c) $y = -x + 2$

**Math 10350 – Example Set 06C**
**Power Functions, Exponential Functions, and Mixing them.**

**1.** Determine whether the following functions are of the form $[f(x)]^n$, $a^{g(x)}$, and $[f(x)]^{g(x)}$ where $a$ and $n$ are constants, and $f(x)$ and $g(x)$ are functions of $x$. Find their derivatives.

**a.** $y = (2x^2 + 5)^{e^2+3}$

**b.** $y = (2x^2 + 5)^{e^x+3}$

**c.** $y = (2\pi^2 + 5)^{x^2+3}$

**d.** $y = (\sin(e) + \cos^2(e))^{x^2}$

**e.** $y = (\sin(e) + \cos^2(e))^{\pi^2}$

**f.** $y = (\sin(x) + \cos^2(e))^{e^2}$