## Math 10350 – Example Set 02A
## Basic Exponential Equations: Section 1.6

**Discrete Compound Interest.** If $P$ dollars (principal) is deposited in an account at annual interest rate $r$ (in decimal form) compounded $n$ times a year, then the balance of the account $A$ after $t$ years is given by

$$\boxed{A(t) = P\left(1 + \frac{r}{n}\right)^{nt}}$$

**1.** Find a formula for the balance $A$ for an investment with principle \$1000 earning interest at a rate of 5% compounded (a) quarterly, (b) monthly, (c) weekly, and (d) daily. Compare the balances when the interest are compounded quarterly, monthly, weekly, and daily after 10 years. What do you observe as the number of compounding increases each year?

**Continuous Compound Interest.** If $P$ dollars is deposited in an account at annual interest rate $r$ (in decimal form) compounded **continuously**, then the balance of the account $A$ after $t$ years is given by

$$\boxed{A(t) = Pe^{rt}}$$

**2.** Write down a formula for the balance $A$ for an investment with principle \$1000 earning interest at a rate of 5% compounded continuously after $t$ years. What is the amount after 10 years? Find the doubling time of the account in years and months (to the nearest integer)? ($1000e^{0.05t}$, $1000e^{0.5}$, $20\ln(2) \approx 13$ yrs. 10 mths)

**3.** You like to see your investment triple at the end of 20 years. At what rate should your investment be growing per annum if interest is (a) compounded quarterly and (b) continuous?

(a) $4(3^{1/80} - 1) \approx 0.0553$ ie 5.53% , (b) $\frac{\ln 3}{20} \approx 0.0549$ ie 5.49%

**4.** Rachel wishes to make a one-time deposit into an account that earns interest at an annual rate of 4% compounded daily. If she likes the account to grow to \$20,000 after 10 years, how much must her initial one-time deposit be? Give your answer to the nearest cents. The value of the deposit you found open called the **Present Value** of the future \$20,000 value of the investment.

**The Present Value** of an investment is what the future value of the investment at maturity is worth TODAY.

If the value of an investment earning interest at rate $r$ compounded $n$ times a year after t years is \$ F (Future Value) then the present value (\$ PV) of the investment is

$$\boxed{PV = \frac{F}{\left(1+\frac{r}{n}\right)^{nt}} = F\left(1+\frac{r}{n}\right)^{-nt}}$$

If the value of an investment earning interest at rate $r$ compounded continuously after t years is \$ F (Future Value) then the present value (\$ PV) of the investment is

$$\boxed{PV = \frac{F}{e^{rt}} = Fe^{-rt}}$$

**5.** An endowment fund is to be deposited into an account that earns interest at an annual rate of 6% compounded continuously. If the endowment wishes to paid out \$5,000 each year for the next five years, how much must we deposit into the account to support the pay out? Give your answer to the nearest cents.

$(5000e^{-0.06} + 5000e^{-0.12} + 5000e^{-0.18} + 5000e^{-0.24} + 5000e^{-0.30} \approx 20957.01)$

**Hint:** Compute each of the following amount:
The amount that must be deposited today to give the first \$5000 payout.

The amount that must be deposited today to give the second \$5000 payout.

The amount that must be deposited today to give the third \$5000 payout.

The amount that must be deposited today to give the fourth \$5000 payout.

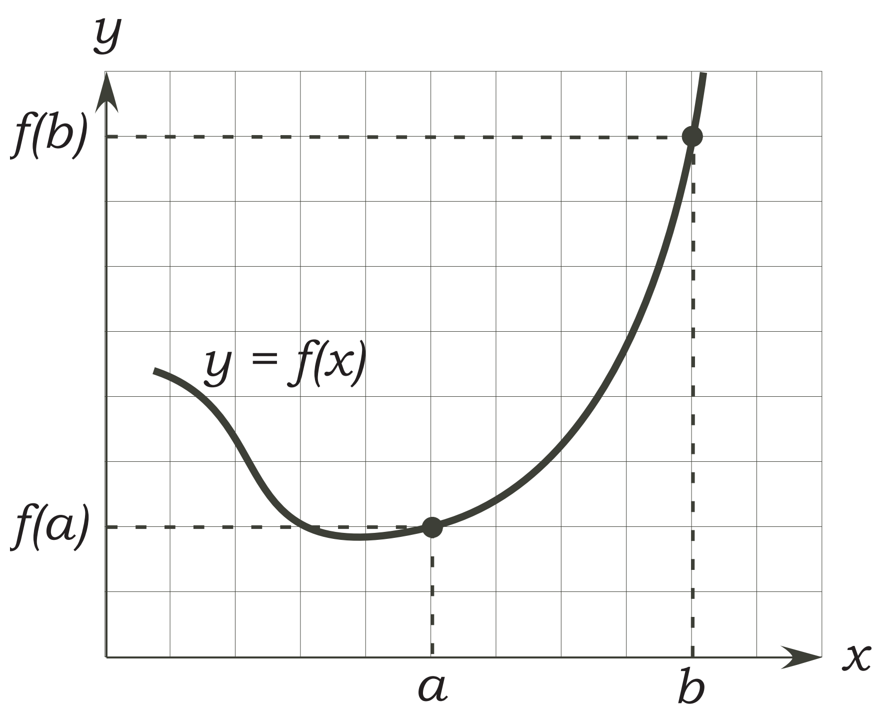
The amount that must be deposited today to give the fifth \$5000 payout.

**Math 10350 – Example Set 02B**
**Sections 2.1, 2.2, 2.3 & 2.4**

**The Average Rate of Change of a function** $f(x)$ over the interval $[a, b]$

is given by $= \frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{\Delta f}{\Delta x} =$ ______________________________.

Sketch in the graph the chord (secant line) whose slope gives this average value.




In the special case when the function is the position $s(t)$ meter of a particle moving on a straight line at time $t$ seconds, the average rate of change of the position over the time interval $a \leq t \leq b$ is also called the

____________________________________ over the time interval $a \leq t \leq b$

$= \frac{\text{Change in position}}{\text{Change in time}} = \frac{\Delta s}{\Delta t} =$ ________________________ m/sec.

**The Average Velocity and Instantaneous Velocity**

**1.** The position (vertical height measured from the ground) of a ball projected vertically up from the ground is given by $s(t) = 30t - 5t^2$ meter at time $t$ second. Find each of the following values and simplifying your answer.

(1a) Average rate of change of the position of the ball over the time interval $1 \leq t \leq 4 =$

Average velocity of the ball over time interval $1 \leq t \leq 4 = \dfrac{\text{Change in position}}{\text{Change in time}} =$

(1b) Average velocity over the time duration between 1 and $t$ (assuming $t \neq 1$) $=$

(1c) Complete the table:

| $t$ | 0.99 | 0.999 | 0.9999 | 1 | 1.0001 | 1.001 | 1.01 |
|---|---|---|---|---|---|---|---|
| $\dfrac{s(t)-s(1)}{t-1}$ | | | | ? | | | |

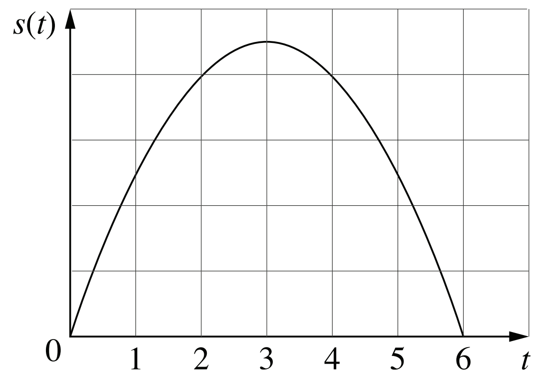
(1d) From the table, what could you observe about $\dfrac{s(t)-s(1)}{t-1}$? Give a physical interpretation for your observation.

(1e) The above observation, we say that instantaneous velocity $v(1)$ of the ball at $t = 1$ second is given by the

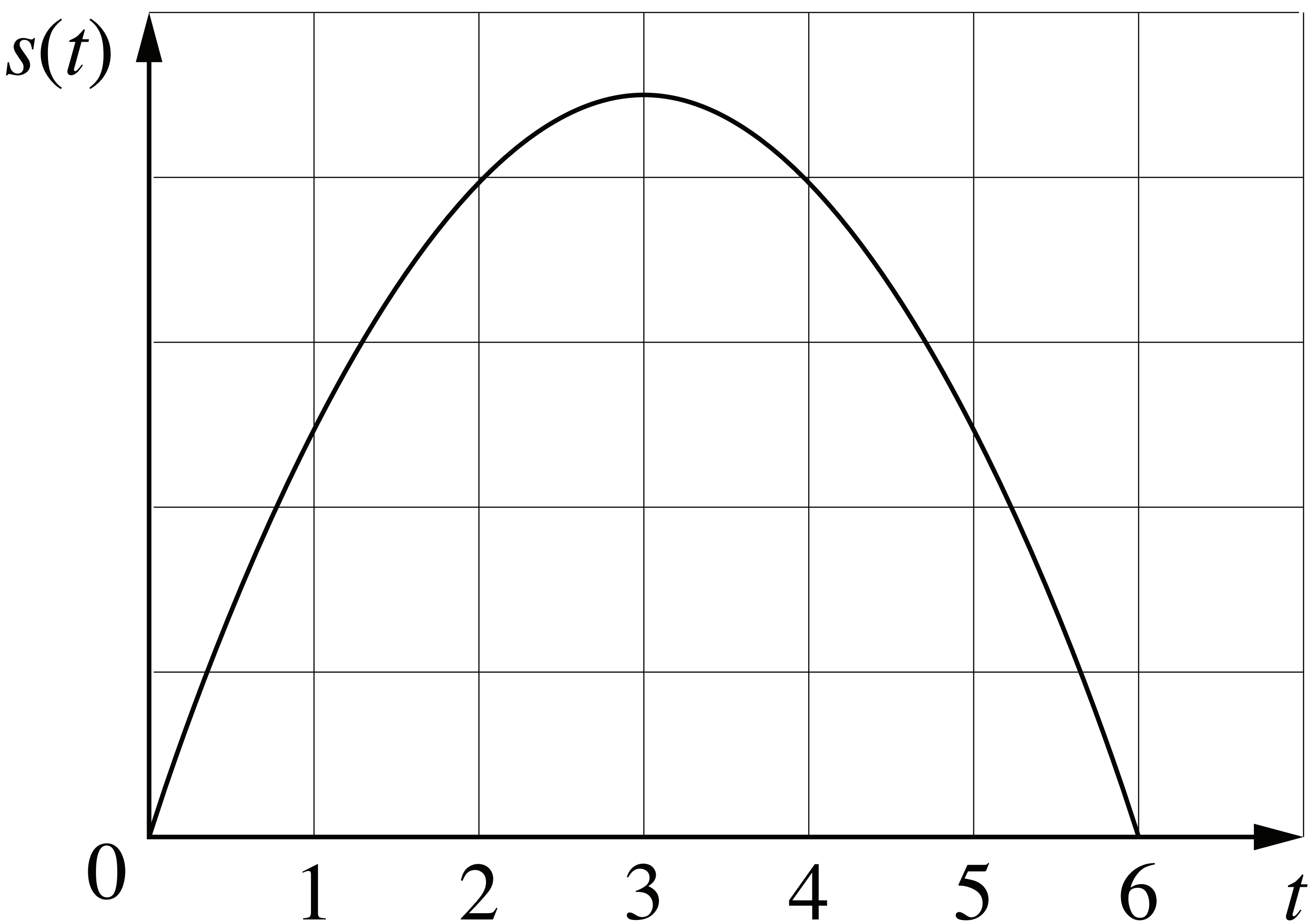
____________________ of the average velocity $\dfrac{s(t)-s(1)}{t-1}$ as time $t$ __________________________.

This denoted by $v(1) = s'(1) =$ __________________________.

(1f) Give a graphical interpretation of the average velocity of the ball over the time interval $1 \le t \le 4$. Of course, we can also interpret the average velocity over the time interval between 1 and any $t (\neq 1)$.




(1g) Give a graphical interpretation of the instantaneous velocity $v(1)$ of the ball at $t = 1$ second.




(1h) Find the equation of the tangent line to the graph of $s(t) = 30t - 5t^2$ at $t = 1$.

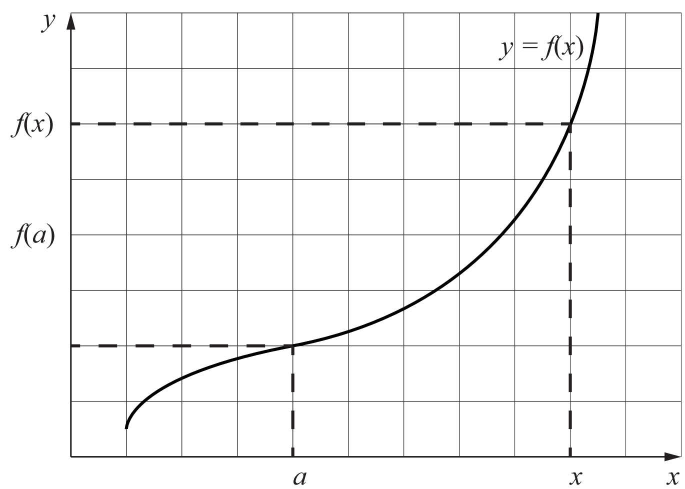
**Summary.** We have computed the instantaneous velocity at time $t = a$ of a particle moving on a straight line with position function $s(t)$. We online the key steps below.

**Step 1:** The average velocity of the particle over the time interval between $t$ and $a$ is ____________________.

**Step 2:** The instantaneous velocity of the particle at $t = a$ is

$$v(a) = s'(a) =$$

The same limiting process above can be applied to many functions besides the position function of a particle. We can mimic the same limiting process to find the **Instantaneous Rate of Change of a function $f(x)$ at a given $x = a$.** Illustrate the process in the graph below.




**Step 1:** The average rate of change of $f(x)$ over the time interval between $x$ and $a$ is ____________________.

**Step 2:** The instantaneous rate of change of $f(x)$ at $x = a$ is

$$f'(a) =$$

$f'(a)$ is also called the ________________________________________

and it gives the ________________________________ to the graph of $f(x)$ at $x = a$.

**Math 10350 – Example Set 02C**
**Sections 2.1, 2.2, 2.3 & 2.4**

**1.** Water is flowing into a tank at a rate such that the volume $V(t)$ (in cubic feet) of water in the tank at time $t \geq 0$ (in minutes) is given by $V(t) = \sqrt{t+4}$. Answer the following questions:

(a) Find the average rate of change of the volume of water over the time duration $[5, 12]$. What is the unit of your answer?

(b) Using limits, find the rate of change of the volume of water at the fifth minute. What is the unit of your answer?

**2.** Let $f(x) = \dfrac{2}{x+4}$. Find the following value or expressions related to $f(x)$.

(a) Find the difference quotient of $f(x)$ at $x = -1$? Give your answer in terms of $h \neq 0$ simplifying as far as possible.

(b) Using limits, find the derivative of $f(x)$ at $x = -1$ or the instantaneous rate of change of $f(x)$ at $x = -1$.

(c) Find the equation of the tangent line to the graph of $f(x)$ at $x = -1$. $\left(y = -\frac{2}{9}x + \frac{4}{9}\right)$