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A law of large numbers concerning the distribution of critical points of random Fourier series

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On the flat torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$ we consider the Gaussian random function $F_{\mathfrak{a}}^R$ defined as a random Fourier series (1.1). The Fourier coefficients are mean zero independent normal variables whose variances depend on the frequencies via an even Schwartz function $\mathfrak{a}$ on $\mathbb{R}$ and large rescaling parameter R. For any open subset U of the torus denote by $Z_R(U)$ the number of critical points of $F_{\mathfrak{a}}^R$ in U. We prove that if U is contained in a geodesic ball, then the variance of $Z_R(U)$ is asymptotic to $const \times R^m vol\big[U\big]$ as $R \to \infty$. We use this to prove that if $m \geq 2$, then as $N \to \infty$, the random measures $N^{-m} Z_N(-)$ converge a.s. to an explicit multiple of the volume measure on the flat torus.

1. Introduction

The main goal of this paper is to describe the distribution of critical points of certain Gaussian random Fourier series in m angular variables, $m \geq 2$

Denote by $\mathbb{T}^m$ the m-dimensional torus $\mathbb{R}^m/\mathbb{Z}^m$ and by g_1 the flat metric of volume 1. In terms of angular coordinates $\vec{\theta} = (\theta^1, \dots, \theta^m)$, $\theta^i \in \mathbb{R} \bmod \mathbb{Z}$, we have $g_1 = (d\theta^1)^2 + \cdots + (d\theta^m)^2$. For $R > 0$, meant to be large, we denote by Δ_R the Laplacian of the metric $g_R = R^2 g_1$ and by vol_R the corresponding volume density. We have

$$vol_R\big[d\vec{\theta}\big] = R^m vol_1\big[d\vec{\theta}\big] = R^m d\theta^1 \cdots d\theta^m, \quad vol_R\big[\mathbb{T}^m\big] = R^m,$$

$$\Delta_R = R^{-2}\Delta_1 = -R^2 \sum_{k=1}^m \partial^2_{\theta^k}.$$

A complete orthonormal system of *complex* eigenfunctions of Δ_1 is given by $\big(e_{\vec{\ell}}\big)_{\vec{\ell} \in \mathbb{Z}^m}$

$$e_{\vec{\ell}}\big(\vec{\theta}\big) = e^{2\pi \boldsymbol{i}\langle \vec{\ell}, \vec{\theta}\rangle}, \quad \langle \vec{\ell}, \vec{\theta}\rangle := \sum_{j=1}^m \ell_j \theta^j.$$

More precisely, $\forall \vec{k} \in \mathbb{Z}^m$,

$$\Delta_1 e_{\vec{k}} = \lambda_{\vec{k}} e_{\vec{k}}, \quad \lambda_{\vec{k}} := \big|2\pi \vec{k}\big|^2 = (2\pi)^2 \sum_{j=1}^m k_j^2.$$

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To describe a complete orthonormal system of *real* eigenfunctions of Δ_R we introduce the lexicografic order on $\mathbb{Z}^m$, $\vec{\ell} \succ 0$ iff $\exists i_0$ such that $\ell_{i_0} > 0$ and $\ell_i = 0$, $\forall i < i_0$. Let $u_0 := 1$. For $\vec{k}, \vec{\ell} \succ 0$, we set

$$u_{\vec{k}}(\vec{\theta}) := \sqrt{2}\cos\big(2\pi\langle\vec{k},\vec{\theta}\rangle\big),\quad v_{\vec{\ell}}(\vec{\theta}) := \sqrt{2}\sin\big(2\pi\langle\vec{\ell},\vec{\theta}\rangle\big),$$
$$u^R_{\vec{k}} = R^{-m/2}u_{\vec{k}},\quad v^R_{\vec{\ell}} = R^{-m/2}v_{\vec{\ell}},\quad \lambda_{\vec{k}}(R) := R^{-2}\lambda_{\vec{k}}.$$

The collection

$$\{u_0\} \cup \big\{u^R_{\vec{k}},\ v^R_{\vec{\ell}};\ \vec{k} \succ 0,\ \ \vec{\ell} \succ 0\big\}$$

is a complete $L^2(M, g_R)$-orthonormal system of real eigenfunctions of Δ_R. Then, $\forall \vec{k} \in \mathbb{Z}^m$,

$$\Delta_R u^R_{\vec{k}} = \lambda_{\vec{k}}(R)u^R_{\vec{k}},\quad \Delta_R v^R_{\vec{\ell}} = \lambda_{\vec{\ell}}(R)v^R_{\vec{\ell}}.$$

Fix an even Schwarz function $\mathfrak{a} \in \mathcal{S}(\mathbb{R})$ such that $\mathfrak{a}(0) = 1$. We will refer to such an $\mathfrak{a}$ as *amplitude*. Fix independent standard normal random variables

$$A_0,\ \ \big\{A_{\vec{k}},\ B_{\vec{\ell}};\ \vec{k} \succ 0,\ \ \vec{\ell} \succ 0\big\}$$

and consider the random Fourier series

$$\begin{aligned} F^R_{\mathfrak{a}}(\vec{\theta}) &= \mathfrak{a}(0)A_0u^R_0(\vec{\theta}) + \sum_{\vec{\ell}\succ 0}\mathfrak{a}\big(\lambda_{\vec{\ell}}(R)^{1/2}\big)\big(A_{\vec{\ell}}u^R_{\vec{\ell}}(\vec{\theta}) + B_{\vec{\ell}}v^R_{\vec{\ell}}(\vec{\theta})\big)\\ &= R^{-m/2}\Big(A_0u_0(\vec{\theta}) + \sum_{\vec{\ell}\succ 0}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)\big(A_{\vec{\ell}}u_{\vec{\ell}}(\vec{\theta}) + B_{\vec{\ell}}v(\vec{\theta})\big)\Big)\\ &= R^{-m/2}\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)Z_{\vec{\ell}}e_{\vec{\ell}}(\vec{\theta}), \end{aligned} \tag{1.1}$$

where $Z_0 = A_0$ and

$$Z_{\vec{\ell}} = \begin{cases} \frac{1}{\sqrt{2}}\big(A_{\vec{\ell}} - \boldsymbol{i}B_{\vec{\ell}}\big), & \vec{\ell} \succ 0,\\ \bar{Z}_{-\vec{\ell}}, & \vec{\ell} \prec 0.\end{cases}$$

Let us observe that $F^R_{\mathfrak{a}}$ is a.s. smooth. To see this consider the Sobolev space $H^p(\mathbb{T}^m)$ of functions on $\mathbb{T}^m$ with L^2-weak derivatives up to order p; see [1, Sec. 10.2.4]. Denote by $C^k(\mathbb{T}^m)$ the Banach sapce of functions on $\mathbb{T}^m$ that are k-times continuously differentiable. Since $\partial_{\theta^j}e_{\vec{\ell}} = \boldsymbol{i}\ell_je_{\vec{\ell}}$ we deduce that for any positive integer p there exists $C = C_p > 0$ such that

$$\forall\vec{\ell} \in \mathbb{Z}^m,\quad \big\|e_{\vec{\ell}}\big\|_{H^p(\mathbb{T}^m)} \le C_p\big|\vec{\ell}\big|^p. \tag{1.2}$$

Since $\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)$ decays very fast as $|\vec{\ell}| \to \infty$ we deduce from Kolmogorov's two-series theorem that for any $p \in \mathbb{N}$, the scalar random series

$$\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)\big|Z_{\vec{\ell}}\big|\,\|\,e_{\vec{\ell}}\,\|_{H^p(\mathbb{T}^m)}$$

converges a.s. This implies that the functional random series

$$\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)Z_{\vec{\ell}}e_{\vec{\ell}}$$

converges a.s. in $H^p\big(\mathbb{T}^m\big)$, $\forall p \in \mathbb{N}$. Using the Sobolev embeddings $H^p(\mathbb{T}^m) \hookrightarrow C^k(\mathbb{T}^m)$, for any $0 \le k < p - \frac{m}{2}$, [1, Thm. 10.2.23], we deduce that the Gaussian function $F^R_{\mathfrak{a}}$ is a.s. smooth.

The covariance kernel of $F^R_{\mathfrak{a}}$ is the function $\mathcal{C}^R_{\mathfrak{a}} : \mathbb{T}^m \times \mathbb{T}^m \to \mathbb{R}$ given by

$$\mathcal{C}^R_{\mathfrak{a}}\big(\vec{\varphi},\vec{\theta}\big) = C^R_{\mathfrak{a}}\big(\vec{\varphi} - \vec{\theta}\big) = R^{-m}\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)^2e^{2\pi\boldsymbol{i}\langle\vec{\ell},\vec{\varphi}-\vec{\theta}\rangle}.$$

The covariance kernel $\mathcal{C}^R_{\mathfrak{a}}$ is the Schwarz kernel of the smoothing operator

$$S_R := R^{-m}\mathfrak{a}\big(R^{-1}\sqrt{\Delta}\big)^2 : L^2\big(\mathbb{T}^m\big) \to L^2\big(\mathbb{T}^m\big)$$

uniquely defined by the equalities

$$S_Re_{\vec{\ell}} = R^{-m}\mathfrak{a}\big(R^{-1}\lambda^{1/2}_{\vec{\ell}}\big)^2e_{\vec{\ell}},\quad \forall\vec{\ell} \in \mathbb{Z}^m.$$

By definition, its Schwartz kernel is

$$K_{S_R}(\vec{\varphi},\vec{\theta}) = \sum_{\vec{\ell}\in\mathbb{Z}^m}S_Re_{\vec{\ell}}(\vec{\varphi})\cdot\overline{e_{\vec{\ell}}(\vec{\theta})} = R^{-m}\sum_{\vec{\ell}\in\mathbb{Z}^m}\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big)^2e^{2\pi\boldsymbol{i}\langle\vec{\ell},\vec{\varphi}-\vec{\theta}\rangle}.$$

The estimates (1.2) and the Sobolev embeddings coupled with the fact that $\mathfrak{a}$ is a Schwartz function show that the above series converges in $C^k(\mathbb{T}^m \times \mathbb{T}^m)$ for any $k \in \mathbb{N}$. Thus the Schwartz kernel is smooth so S_R is a smoothing operator. For example, if $\mathfrak{a}(x) = e^{-x^2/2}$ and setting $t := R^{-2}$, then $\mathfrak{a}\big(R^{-1}\sqrt{\Delta}\big)^2 = e^{-t\Delta}$ is the heat operator.

The smoothing operator S_R is well defined on any compact Riemann manifold and for any amplitude $\mathfrak{a}$. The asymptotics of its kernel as $R \nearrow \infty$ are closely related to Hörmander's local Weyl asymptotics, [2, Sec.XIII.2]. These are based on short time asymptotic expansions for the wave kernel $\cos t\sqrt{\Delta}$. In the special case of the Laplacian one such explicit expansion is given by Hadamard's Ansatz, [3, Chap.2].

In the case of torus $\mathbb{T}^m$ we can avoid these general considerations and obtain sharper results by relying on Poisson's summation formula. Define

$$w_{\mathfrak{a}} : \mathbb{R}^m \to \mathbb{R}, \ \ w_{\mathfrak{a}}(\xi) = \mathfrak{a}\big(|\xi|\big)^2, \ \ |\xi|^2 := \sum_{j=1}^m \xi_j^2.$$

Its Fourier transform is

$$\hat{w}_{\mathfrak{a}}(\boldsymbol{x}) = \int_{\mathbb{R}^m} e^{i\langle \xi, \boldsymbol{x}\rangle} w_{\mathfrak{a}}(\xi) d\xi.$$

Set $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}) := \frac{1}{(2\pi)^m}\hat{w}_{\mathfrak{a}}(\boldsymbol{x})$. Using Poisson's summation formula [4, §7.2] we deduce

$$C_{\mathfrak{a}}^R(\vec{\tau}) = \sum_{\vec{k}\in\mathbb{Z}^m} \boldsymbol{K}_{\mathfrak{a}}\big((\vec{k}-\vec{\tau})R\big), \ \ \vec{\tau} = \vec{\varphi} - \vec{\theta}. \tag{1.3}$$

If we formally let $R \to \infty$ in the equality

$$R^{m/2} F_{\mathfrak{a}}^R(\vec{\theta}) = \sum_{\vec{\ell}\in\mathbb{Z}^m} \mathfrak{a}\big(|2\pi\vec{\ell}|/R\big) Z_{\vec{\ell}} e_{\vec{\ell}}(\vec{\theta})$$

we deduce

$$W_\infty(\vec{\theta})\}\} = \varepsilon \lim_{R\to\infty} R^{m/2} F_{\mathfrak{a}}^R(\vec{\theta}) = \sum_{\vec{\ell}\in\mathbb{Z}^m} Z_{\vec{\ell}} e_{\vec{\ell}}(\vec{\theta}).$$

The series on the right-hand-side is a.s. divergent but we can still assign a meaning to W_∞ as a random generalized function, see [5]. This is a random linear functional $C^\infty(\mathbb{T}^m) \to \mathbb{R}$,

$$W_\infty(f) = \sum_{\vec{\ell}\in\mathbb{Z}^m} Z_{\vec{\ell}}\big(f, e_{\vec{\ell}}(\vec{\theta})\big)_{L^2(\mathbb{T}^m)} = \lim_{R\to\infty} \sum_{\vec{\ell}\in\mathbb{Z}^m} \mathfrak{a}\big(|2\pi\vec{\ell}|/R\big) Z_{\vec{\ell}}\big(f, e_{\vec{\ell}}\big)_{L^2(\mathbb{T}^m)}.$$

The convergence of the above random scalar series is guaranteed by Kolmogorov's one-series theorem. A simple computation shows that for any functions $f_0, f_1 \in C^\infty(\mathbb{T}^m)$

$$Cov\big[W_\infty(f_0), W_\infty(f_1)\big] = \sum_{\vec{\ell}\in\mathbb{Z}^m} \big(f_0, e_{\vec{\ell}}\big)_{L^2(\mathbb{T}^m, g_1)} \big(f_1, e_{\vec{\ell}}\big)_{L^2(\mathbb{T}^m, g_1)} = \big(f_0, f_1\big)_{L^2(\mathbb{T}^m, g_1)}.$$

The last equality shows that W_∞ is the Gaussian white noise on $\mathbb{T}^m$ driven by the volume measure vol_{g_1}.

We could think of the family $\big(W_{\mathfrak{a}}^R = R^{m/2} F_{\mathfrak{a}}^R\big)_{R>0}$ as a white noise approximation because the covariance kernel of $W_{\mathfrak{a}}^R$ converges, in the sense of distributions, to the covariance kernel of W_∞. Note that $W_{\mathfrak{a}}^R = \mathfrak{a}\big(R^{-1}\sqrt{\Delta}\big)W_\infty$.

Remark 1.1. Here is a naive motivation for considering such limits. The distribution of critical points of a function does not change if we multiply the function by a nonzero constant. Thus, if we want to unbiasedly understand the distribution of critical points of a typical smooth function on $\mathbb{T}^m$ we ought to sample uniformly the unit sphere of $C^\infty(\mathbb{T}^m)$ determined by the L^2-inner product. Equivalently, we could sample $C^\infty(\mathbb{T}^m)$ with respect to a Gaussian measure whose variance is the L^2-inner product. There is no such measure on $C^\infty(\mathbb{T}^m)$, but there is one on the space $C^{-\infty}(\mathbb{T}^m)$ of generalized functions, namely the white noise.

For C fixed and $|\vec{\ell}| < C$, we have $\mathfrak{a}\big(|2\pi\vec{\ell}|/R\big) \to 1$ as $R \to \infty$. Thus, as $R \to \infty$, the contributions of the eigenfunctions $u_{\vec{\ell}}$ and $v_{\vec{\ell}}$ to the series (1.1) will be nearly equiprobable. Note that these eigenfunctions are highly oscillatory for $|\vec{\ell}|$ large, making $F_{\mathfrak{a}}^R$ highly oscillatory and thus gaining in critical points as $R \to \infty$. Roughly speaking, as $R \to \infty$, the random function $F_{\mathfrak{a}}^R$ is attempting to sample $C^\infty(\mathbb{T}^m)$ uniformly from the point of view of critical point distributions. □

For R sufficiently large, $F_{\mathfrak{a}}^R$ is a.s. Morse. The main goal of this paper is to investigate the distribution of the critical points of $F_{\mathfrak{a}}^R$ in the white noise limit, $R \to \infty$.

We can think of $F_{\mathfrak{a}}^R$ either as a function on $\mathbb{T}^m$, or as a $\mathbb{Z}^m$-periodic function on $\mathbb{R}^m$. Consider the rescaled $(R\mathbb{Z})^m$-periodic function $\Phi_{\mathfrak{a}}^R : \mathbb{R}^m \to \mathbb{R}$,

$$\Phi_{\mathfrak{a}}^R(\boldsymbol{x}) := F_{\mathfrak{a}}^R(\boldsymbol{x}/R) = R^{-m/2} \sum_{\vec{\ell}\in\mathbb{Z}^m} \mathfrak{a}\big(|2\pi\vec{\ell}|/R\big) Z_{\vec{\ell}} e^{2\pi i\langle \vec{\ell}, R^{-1}\boldsymbol{x}\rangle}.$$

We denote by $\mathcal{K}_{\mathfrak{a}}^R$ the covariance kernel of $\Phi_{\mathfrak{a}}^R$. Then $\mathcal{K}_{\mathfrak{a}}^R(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{K}_{\mathfrak{a}}^R(\boldsymbol{x} - \boldsymbol{y})$, where

$$\boldsymbol{K}_{\mathfrak{a}}^R(\boldsymbol{z}) \stackrel{(1.3)}{=} \sum_{\vec{k}\in\mathbb{Z}^m} \boldsymbol{K}_{\mathfrak{a}}(R\vec{k} - \boldsymbol{z}) = \sum_{t\in(R\mathbb{Z})^m} \boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{t} - \boldsymbol{z}). \tag{1.4}$$

Since $\boldsymbol{K}_{\mathfrak{a}}$ is a Schwartz function we deduce that

$$\lim_{R\nearrow\infty} \boldsymbol{K}^R_{\mathfrak{a}} = \boldsymbol{K}_{\mathfrak{a}} \quad \text{in } C^k\big(\mathbb{R}^m\big),\ \forall k\in\mathbb{N}. \tag{1.5}$$

The function $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})$ is the covariance kernel of an isotropic Gaussian function $\Phi_{\mathfrak{a}}$. The equality (1.5) suggests that $\Phi^R_{\mathfrak{a}}$ approximates $\Phi_{\mathfrak{a}}$ for $R >> 0$ since the covariance kernel of $\Phi^R_{\mathfrak{a}}$ converges in $C^\infty(\mathbb{R}^m)$ to that of $\Phi_{\mathfrak{a}}$.

Remark 1.2. This phenomenon is related to the one analyzed by L. Gass in [6]. The large band model investigated in [6] corresponds formally to a choice of singular amplitude $\mathfrak{a} = \boldsymbol{I}_{[-1,1]}$, the indicator of the interval $[-1,1]$.

The role of the large parameter R can be given a new geometric interpretation. On an arbitrary Riemann manifold (M,g), the rescaled metric $g_R = R^2 g$ becomes flatter and flatter. In the metric g_R a tiny compact region of M will look metrically more and more like a larger and larger region of a Euclidean space. The convergence $\Phi^R_{\mathfrak{a}} \to \Phi_{\mathfrak{a}}$ is one manifestation of this asymptotic flattening. □

Suppose that $G:\mathbb{R}^m\to\mathbb{R}$ is a Gaussian C^2-function such that $\nabla G(\boldsymbol{x})$ is a nondegenerate Gaussian vector for any $\boldsymbol{x}\in\mathbb{R}^m$. Then G is a.s. Morse (see Corollary 2.1). Define

$$\mathfrak{C}\big[-,G\big] := \sum_{\nabla G(\boldsymbol{x})=0} \delta_{\boldsymbol{x}}.$$

This is a locally finite random measure on $\mathbb{R}^m$ in the sense of [7] or [8]. For every Borel subset $S\subset\mathbb{R}^m$, $\mathfrak{C}[S,G]$ is the number of critical points of G in S.

Denote by $C^0_{\rm cpt}(\mathbb{R}^m)$ the space of continuous, compactly supported functions on $\mathbb{R}^m$. For any $f\in C^0_{\rm cpt}(\mathbb{R}^m)$ we set

$$\mathfrak{C}[f,G] := \int_{\mathbb{R}^m} f(\boldsymbol{x})\mathfrak{C}[d\boldsymbol{x},G] = \sum_{\nabla G(\boldsymbol{x})=0} f(\boldsymbol{x}) \in [0,\infty].$$

Clearly, $\nabla F^R_{\mathfrak{a}}(\boldsymbol{y})=0$ iff $\nabla\Phi^R_{\mathfrak{a}}(R\boldsymbol{y})=0$ so, for any box $B=[a,b]^m\subset\mathbb{R}^m$, and any $f\in C^0_{\rm cpt}(\mathbb{R}^m)$, we have

$$\mathfrak{C}\big[RB,\Phi^R_{\mathfrak{a}}\big] = \mathfrak{C}\big[B,F^R_{\mathfrak{a}}\big],\ \ \mathfrak{C}\big[f_R,\Phi^R_{\mathfrak{a}}\big] = \mathfrak{C}\big[f,F^R_{\mathfrak{a}}\big],$$

where $f_R(\boldsymbol{x}) = f\big(R^{-1}\boldsymbol{x}\big)$.

When the support of f is contained in the interior of a fundamental domain of the $\mathbb{Z}^m$-action on $\mathbb{R}^m$ we can identify f canonically with a function $\bar{f}$ on $\mathbb{T}^m$ and we have

$$\mathfrak{C}\big[f,F^R_{\mathfrak{a}}\big] = \sum_{\substack{\boldsymbol{x}\in\mathbb{T}^m,\\ \nabla F^R_{\mathfrak{a}}(\boldsymbol{x})=0}} \bar{f}(\boldsymbol{x}).$$

In [9,10] it is shown that the random function $\Phi_{\mathfrak{a}}$ is a.s. Morse and there exists a positive constant $C_m(\mathfrak{a})$ that depends only on $\mathfrak{a}$ and m, explicitly, such that for any box $B\subset\mathbb{R}^m$ and any $f\in C^0_{\rm cpt}(\mathbb{R}^m)$ we have

$$\mathbb{E}\Big[\mathfrak{C}[B,\Phi_{\mathfrak{a}}]\Big] = C_m(\mathfrak{a})vol\big[B\big], \tag{1.6a}$$

$$\mathbb{E}\Big[\mathfrak{C}[f,\Phi_{\mathfrak{a}}]\Big] = C_m(\mathfrak{a})\int_{\mathbb{R}^m} f(\boldsymbol{x})\lambda\big[d\boldsymbol{x}\big], \tag{1.6b}$$

where λ denotes the Lebesgue measure on $\mathbb{R}^m$.

Remark 1.3. The dependence of $C_m(\mathfrak{a})$ on m and $\mathfrak{a}$, while explicit, is rather complicated and it involves the Gaussian Orthogonal Ensemble of symmetric $(m+1)\times(m+1)$ matrices. The constant $C_m(\mathfrak{a})$ grows really fast with m; see [11, Sec.3]. For example, if $\mathfrak{a}(t)^2 = e^{-t^2}$ for $t\gg 1$, then

$$\log C_m(a) \sim \frac{m}{2}\log \quad \text{as } m\to\infty,$$

while if $\mathfrak{a}(t)^2 = e^{-c(\log)t)^\alpha}$, $\alpha>1$, $c>0$, $t\gg 1$ then, for some explicit constant $Z(\alpha,c)>0$,

$$\log C_m(a) \sim Z(\alpha,c)m^{\frac{\alpha}{\alpha-1}} \quad \text{as } m\to\infty.$$

□

Set $C_1 := [0,1]^m$. In [9] the second author proved that there exists a constant $C'_{\mathfrak{a}}(m)\geq 0$ such that

$$\lim_{R\to\infty} R^{-m} Var\big[\mathfrak{C}[C_1,F^R_{\mathfrak{a}}]\big] = C'_m(\mathfrak{a}). \tag{1.7}$$

The proof of (1.7) in [9] is very laborious and computationally intensive.

The first result of this paper is a functional version of (1.7). We achieve this using a less computational, more robust and more conceptual technique. One consequence of this asymptotic estimate is a functional strong law of large numbers concerning the random measures $\mathfrak{C}[-,F^N_{\mathfrak{a}}]$, $N\in\mathbb{N}$. Let us provide some more details.

First some notation. Denote by $|-|$ the Euclidean norm on $\mathbb{R}^m$ and by $|-|_\infty$ the sup-norm on $\mathbb{R}^m$. For $\boldsymbol{x}_0\in\mathbb{R}^m$ and $r>0$ we set

$$B_r(x_0) := \big\{\boldsymbol{x}\in\mathbb{R}^m;\ \ |\boldsymbol{x}-\boldsymbol{x}_0|\leq r\big\},\ \ B^\infty_r(x_0) := \big\{\boldsymbol{x}\in\mathbb{R}^m;\ \ |\boldsymbol{x}-\boldsymbol{x}_0|_\infty\leq r\big\}.$$

Clearly $B_r(x_0)\subset B^\infty_r(x_0)$.

The function $F^R_{\mathfrak{a}}$ is $\mathbb{Z}^m$-periodic. For $r\in(0,1/2)$, and $\boldsymbol{x},\boldsymbol{y}\in B_r(0)$, the difference $\boldsymbol{x}-\boldsymbol{y}$ is contained in the interior of a fundamental domain of the $\mathbb{Z}^m$-action since $|\boldsymbol{x}-\boldsymbol{y}|_\infty\leq 2r<1$ and $|\vec{\ell}|_\infty\geq 1$, $\forall\vec{\ell}\in\mathbb{Z}^m\setminus 0$. This reflects the fact that the injectivity radius of the flat torus $\mathbb{T}^m=\mathbb{R}^m/\mathbb{Z}^m$ is $\leq\frac{1}{2}$ so $B_r(0)$ can be viewed as a geodesic ball. We can now state the main technical result of this paper.

Theorem 1.1. *Fix an amplitude $\mathfrak{a}$, a positive integer $m \in \mathbb{N}$, a radius $r_0 \in (0, 1/2)$ and a nonnegative continuous function $f : \mathbb{R}^m \to \mathbb{R}$ with support contained in $B_{r_0}(0)$. Then the following hold.*

i.

$$\lim_{R\to\infty} R^{-m}\mathbb{E}\big[\mathfrak{C}[f, F_{\mathfrak{a}}^R]\big] = \mathbb{E}\big[\mathfrak{C}[f, \Phi_{\mathfrak{a}}]\big] = C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})\lambda\big[dx\big]. \tag{1.8}$$

ii. There exists a constant $V_m(\mathfrak{a}) \geq 0$ that depends only on m and $\mathfrak{a}$ such that

$$\lim_{R\to\infty} R^{-m} Var\big[\mathfrak{C}[f, F_{\mathfrak{a}}^R]\big] = V_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})^2 d\boldsymbol{x}. \tag{1.9}$$

If we consider the normalized random measures on $\mathbb{T}^m$.

$$\bar{\mathfrak{C}}_R := \frac{1}{R^m}\mathfrak{C}[-, F_{\mathfrak{a}}^R], \quad R > 0$$

then we deduce that for any nonnegative $f \in C^0_{\text{cpt}}(\mathbb{R}^m)$, $\operatorname{supp} f \in B_{r_0}(0)$, we have

$$\lim_{R\to\infty} \mathbb{E}\big[\bar{\mathfrak{C}}_R[f]\big] = C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x})\lambda\big[d\boldsymbol{x}\big], \tag{1.10}$$

and

$$Var\big[\bar{\mathfrak{C}}_R[f]\big] \sim V_m(\mathfrak{a})R^{-m} \;\text{ as } R \to \infty. \tag{1.11}$$

Clearly the above results hold for any continuous function supported in any geodesic ball of $(\mathbb{T}^m, g_1)$ of radius r_0. Using finite partitions of unity we deduce from (1.11) that for any nonnegative $f \in C^0(\mathbb{T}^m)$ we have

$$Var\big[\bar{\mathfrak{C}}_R[f]\big] = O\big(R^{-m}\big) \;\text{ as } R \to \infty.$$

If $m \geq 2$, then

$$\sum_{N\in\mathbb{N}} \frac{1}{N^m} < \infty$$

Borel-Cantelli and (1.11) imply that for any nonnegative $f \in C^0(\mathbb{T}^m)$ we have

$$\lim_{N\to\infty} \bar{\mathfrak{C}}_N[f] = C_m(\mathfrak{a}) \int_{\mathbb{T}^m} f(\boldsymbol{x})vol\big[dx\big] \;\text{ a.s. and in } L^2. \tag{1.12}$$

Thus, in the white noise limit ($R \to \infty$), the critical points of $F_{\mathfrak{a}}^R$ will equidistribute with probability 1. In the case $m = 1$, this law of large numbers is proved in the recent work of L. Gass [12, Thm. 1.6].

To put (1.12) in its proper context we need to recall a few facts about the convergence of random measures. For proofs and details we refer to [7, Chap. 11] and [8, Chap. 4].

Suppose that X is a Heine-Borel metric space, i.e., it is complete and the closed bounded sets are compact; see [13] for a topological characterization of such spaces. The torus and $\mathbb{R}^m$ are Heine-Borel metric spaces. We denote by $Prob(X)$ the space of Borel probability measures on X, by $Meas(X)$ the space of finite measures on X and by $Meas_{\text{loc}}(X)$ the space of locally finite Borel measures on $\mathbb{R}^m$, i.e., Borel measures μ such that $\mu\big[S\big] < \infty$ for any bounded Borel set $S \subset X$. Any such set S defines an additive map

$$L_S : Meas_{\text{loc}}(X) \to \mathbb{R}, \quad Meas_{\text{loc}}(X) \ni \mu \mapsto L_S(\mu) = \mu\big[S\big] \in \mathbb{R}.$$

The *vague* topology on the space $Meas_{\text{loc}}(X)$ is the smallest topology such that all the maps L_S, S bounded Borel, are continuous. The space $Meas_{\text{loc}}(X)$ equipped with the vague topology is a Polish space, i.e., it is separable and the topology is induced by a complete (Prokhorov-like) metric.

A random locally finite measure on X is a measurable map

$$\mathfrak{M} : \big(\Omega, \mathcal{S}, \mathbb{P}\big) \to Meas_{\text{loc}}\big(X\big),$$

where $\big(\Omega, \mathcal{S}, \mathbb{P}\big)$ is a probability space. Its distribution is a Borel probability measure $\mathbb{P}_{\mathfrak{M}}$ on $Meas_{\text{loc}}\big(X\big)$. Its mean intensity is the measure $\bar{\mathfrak{M}}$ on X

$$\bar{\mathfrak{M}}[S] = \mathbb{E}\big[\mathfrak{M}[S]\big]$$

for any Borel subset $S \subset X$. We say that $\mathfrak{M}$ is locally integrable if its mean intensity is locally finite.

A sequence of random measures $\mathfrak{M}_N : \big(\Omega, \mathcal{S}, \mathbb{P}\big) \to Meas_{\text{loc}}\big(X\big)$ is said to converge vaguely a.s. to the random measure $\mathfrak{M}$ if

$$\mathbb{P}\Big[\big\{\omega;\; \mathfrak{M}_N(\omega) \to \mathfrak{M}(\omega) \;\text{ vaguely in } Meas_{\text{loc}}(X)\big\}\Big] = 1.$$

One can show that the following statements are equivalent

- $\mathfrak{M}_N \to \mathfrak{M}$ vaguely a.s.
- $\mathfrak{M}_N[S] \to \mathfrak{M}[S]$, a.s., for any bounded Borel $S \subset X$.
- $\mathfrak{M}_N[f] \to \mathfrak{M}[f]$, a.s., $\forall f \in C^0_{\text{cpt}}(X)$.

Similarly we say that $\mathfrak{M}_N \to \mathfrak{M}$ vaguely L^p if the following equivalent conditions hold.

- $\mathfrak{M}_N[S] \to \mathfrak{M}[S]$, in L^p, for any bounded Borel $S \subset X$.
- $\mathfrak{M}_N[f] \to \mathfrak{M}[f]$, in L^p, $\forall f \in C^0_{\mathrm{cpt}}(X)$.

We can rephrase the equality (1.12) as a law of large numbers.

Corollary 1.1 (Strong Law of Large Numbers)**.** *Suppose that $m \geq 2$. In white noise limit ($N \to \infty$) the random measures $\frac{1}{N^m}\mathfrak{C}\left[-, F_{\mathfrak{a}}^N\right]$ on $\mathbb{T}^m$ converge vaguely a.s. and L^2 to the deterministic measure $C_m(\mathfrak{a})vol_1$. In particular, for any Borel subset $S \subset \mathbb{T}^m$ we have*

$$\lim_{N\to\infty} \frac{1}{N^m}\mathfrak{C}\left[S, F_{\mathfrak{a}}^N\right] = C_m(\mathfrak{a})vol_1\left[S\right]$$

a.s. and L^2.

In [14] the second author proved that in the white noise limit the random measures $\mathfrak{C}\left[-, F_{\mathfrak{a}}^N\right]$ also satisfy a Central Limit Theorem. More precisely, it states that $V_m(\mathfrak{a}) > 0$ and, for any $r \in (0, 1)$

$$\frac{1}{N^{m/2}}\Big(\mathfrak{C}[B^{\infty}_{r/2}, F_{\mathfrak{a}}^N] - \mathbb{E}\big[\mathfrak{C}[B^{\infty}_{r/2}, F_{\mathfrak{a}}^N]\big]\Big)$$

converges in distribution to a centered normal random variable with variance $V_m(\mathfrak{a})$.

The positivity of the limiting variance $V_m(\mathfrak{a})$ is a rather nontrivial fact. In dimension 1 it was first established in 1976 by J. Cuzick, [15]. In higher dimensions it was first proved in [14]. For a recent and more general result of this nature we refer to L. Gass' paper [16].

On the other hand, we can also view $F_{\mathfrak{a}}^R$ as a $\mathbb{Z}^m$-periodic function on $\mathbb{R}^m$ we obtain a $\mathbb{Z}^m$-periodic random measure $R^{-m}\mathfrak{C}\left[-, F_{\mathfrak{a}}^R\right]$ on $\mathbb{R}^m$. We obtain another law of large numbers.

Corollary 1.2. *Suppose that $m \geq 2$. In white noise limit ($N \to \infty$) the random measures $\frac{1}{N^m}\mathfrak{C}\left[-, F_{\mathfrak{a}}^N\right]$ on $\mathbb{R}^m$ converge vaguely a.s. and L^2 to the deterministic measure $C_m(\mathfrak{a})\lambda$, where λ denotes the Lebesgue measure. In particular, for any bounded Borel subset $S \subset \mathbb{R}^m$ we have*

$$\lim_{N\to\infty} \frac{1}{N^m}\mathfrak{C}\left[S, F_{\mathfrak{a}}^N\right] = C_m(\mathfrak{a})\lambda\left[S\right]$$

a.s. and L^2.

The random functions $F_{\mathfrak{a}}^N$ on $\mathbb{R}^m$ are stationary and so the random measures $\mathfrak{C}\left[-, F_{\mathfrak{a}}^N\right]$ are stationary as well, i.e., their distributions are invariant with respect to the natural action of $\mathbb{R}^m$ by translations on $Meas_{\mathrm{loc}}(\mathbb{R}^m)$.

As discussed in [7, Chap.12] or [8, Chap.5], every *stationary* random locally finite measure $\mathfrak{M}$ on $\mathbb{R}^m$ with locally finite mean has an *asymptotic intensity* $\widehat{\mathfrak{M}}$. This is an integrable random variable $\widehat{\mathfrak{M}} \in L^1\left(\Omega, \mathcal{S}, \mathbb{P}\right)$ with an ergodic meaning, [7, Sec. 12.2] or [8, Sec. 5.4]. More precisely, for any compact convex subset $C \subset \mathbb{R}^m$ containing the origin in its interior we have

$$\widehat{\mathfrak{M}} = \lim_{N\to\infty} \frac{1}{vol\left[NC\right]}\mathfrak{M}[NC] \;\text{ a.s. and } L^1.$$

The random measure $\mathfrak{M} = \mathfrak{C}[-, \Phi_{\mathfrak{a}}]$ is stationary and the results of [10] show that the asymptotic intensity of $\mathfrak{C}[-, \Phi_{\mathfrak{a}}]$ is the constant $\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}} = C_m(\mathfrak{a})$.

We set $C_R = [-R/2, R/2]^m$. For fixed $R \in \mathbb{N}$, the random function $\Phi_{\mathfrak{a}}^R$ is $\left(R\mathbb{Z}\right)^m$-periodic and we deduce that for any $N \in \mathbb{N}$ we have

$$N^m\mathfrak{C}[C_R, \Phi_{\mathfrak{a}}^R] = \mathfrak{C}[NC_R, \Phi_{\mathfrak{a}}^R].$$

Hence

$$\mathfrak{C}[C_R, \Phi_{\mathfrak{a}}^R] = \lim_{N\to\infty}\frac{1}{N^m}\mathfrak{C}[NC_R, \Phi_{\mathfrak{a}}^R] = \widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}^R} vol\left[C_R\right],$$

where $\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}^R}$ denotes the asymptotic intensity of the stationary random measure $\mathfrak{C}[-, \Phi_{\mathfrak{a}}^R]$. Hence

$$\widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}^R} = \frac{1}{R^m}\mathfrak{C}\left[C_R, \Phi_{\mathfrak{a}}^R\right].$$

Theorem 1.1 shows that

$$\lim_{N\to\infty} \widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}^N} = \widehat{\mathfrak{C}}_{\Phi_{\mathfrak{a}}} = C_m(\mathfrak{a}),$$

a.s. and L^2.

To prove Theorem 1.1 we relate the critical points of $F_{\mathfrak{a}}^R$ to the critical points of $\Phi_{\mathfrak{a}}^R$. The advantage is that $\Phi_{\mathfrak{a}}^R$ converges in distribution to the smooth isotropic function $\Phi_{\mathfrak{a}}$. We use the Kac-Rice formula to express $Var\left[\mathfrak{C}[f_R, \Phi_{\mathfrak{a}}^R]\right]$ as an integral

$$\int_{(\mathbb{R}^m\times\mathbb{R}^m)\setminus\Delta} \rho_R^{(2)}(\boldsymbol{x}, \boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y},$$

where Δ is the diagonal and $\rho_R^{(2)}$ is a certain integrand that blows-up along Δ.

To deal with integrability of $\rho_R^{(2)}$ far away from the diagonal we rely on the estimates in Lemma 3.1 that, roughly speaking, states that as $R \to \infty$ the covariance kernel $\boldsymbol{K}_{\mathfrak{a}}$ of $\Phi_{\mathfrak{a}}$ approximates well the covariance kernel $\boldsymbol{K}_{\mathfrak{a}}^R$ of $\Phi_{\mathfrak{a}}^R$ over a *large* ball, of radius $\approx R/2$. The local integrability of $\rho_R^{(2)}$ near the diagonal is an established fact, [17–19]. It follows from the blow-up estimate

$$\sup_{|\boldsymbol{x}-\boldsymbol{y}|<1} |\boldsymbol{x}-\boldsymbol{y}|^{m-2}\rho_R^{(2)}(\boldsymbol{x},\boldsymbol{y}) < \infty.$$

In Appendix B we refine the strategy in [18,19] and prove that the above estimate is *uniform* in R.

In a recent (January 2025) preprint [20] the authors prove very general strong laws of large numbers and central limit theorem similar in spirit with ours. However, our main result does not follow from [20] since the covariance kernels $\boldsymbol{K}_{\mathfrak{a}}^R$ are $R\mathbb{Z}^m$-periodic, so they do not satisfy the hypothesis H_2, [20, page 7].

Let us say a few things about the organization of the paper. In Section 2 we survey a few facts about Gaussian measures and Gaussian random fields and the Kac-Rice formula. Theorem 1.1 is proved in Section 3. We have subdivided this section into several subsections corresponding to the conceptually different steps in proof of Theorem 1.1.

The paper also includes two technical appendices. Our proof is based on the Kac-Rice formula that requires certain nodegeneracy or ampleness conditions on the random function. These can be quite tricky to verify in concrete situations. Appendix A describes several general conditions guaranteeing various forms of ampleness (nondegeneracy) of Gaussian fields. To the best of our knowledge these seem to be new and we believe they will find many other applications. In particular, these results imply that we can apply the various Kac-Rice formulæ to the function $F_{\mathfrak{a}}^R$ if R is sufficiently large.

Appendix B contains explicit upper bounds for the variance of the number of zeros of a C^2 Gaussian field F in a given region $\mathcal{R}$ in terms of C^k norms of its covariance kernel of F and the size of $\mathcal{R}$. The finiteness of the variance is ultimately due to Fernique's inequality [21] guaranteeing that $\mathbb{E}\big[\|F\|_{C^2}^p\big] < \infty$ for all $p \in [1,\infty)$. The fact that these L^p-norms can be controlled by the covariance kernel of F follows from a result of Nazarov and Sodin [22]; see Theorem 2.1.

2. Basic Gaussian concepts and the Kac-Rice formula

Let first recall a few things about Gaussian vectors. For details and proofs we refer to [23].

Suppose that $\boldsymbol{X}$ is a finite dimensional real Euclidean space with inner product $(-,-)$. An $\boldsymbol{X}$-valued *Gaussian vector* is a measurable map $X : (\Omega, \mathcal{S}, \mathbb{P}) \to \boldsymbol{X}$ such that, for any $\xi \in \boldsymbol{X}$, the random variable (ξ, X) is Gaussian. The Gaussian vector X is called *centered* if (ξ, X) has mean zero, $\forall \xi \in \boldsymbol{X}$.

If X is a centered Gaussian random vector valued in the finite dimensional Euclidean space $\boldsymbol{X}$, then its distribution is uniquely determined by its *variance*. This is the symmetric nonnegative operator

$$Var\big[X\big] : \boldsymbol{X} \to \boldsymbol{X}$$

uniquely determined by the equality

$$\mathbb{E}\big[e^{\boldsymbol{i}(\xi,X)}\big] = e^{-\frac{1}{2}(Var[X]\xi,\xi)}, \;\; \forall \xi \in \boldsymbol{X}.$$

The Gaussian vector X is called nondegenerate if $Var\big[X\big]$ is nonsingular.

Suppose that X_1, X_2 are centered Gaussian vectors valued in the Euclidean spaces $\boldsymbol{X}_1$ and respectively $\boldsymbol{X}_2$. If X_1, X_2 are *jointly Gaussian,* i.e., $X_1 \oplus X_2$ is also Gaussian, then the variance of $X_1 \oplus X_2$ has the block decomposition

$$Var\big[X_i \oplus X_2\big] = \begin{bmatrix} Var\big[X_1\big] & Cov\big[X_1, X_2\big] \\ Cov\big[X_2, X_1 & Var\big[X_2\big] \end{bmatrix}$$

where the *covariance* $Cov\big[X_1, X_2\big]$ is a linear map $\boldsymbol{X}_2 \to \boldsymbol{X}_1$ and

$$Cov\big[X_1, X_2\big] = Cov\big[X_1, X_2\big]^*.$$

If X_1, X_2 are jointly Gaussian and X_1 is nondegenerate, then $\mathbb{E}\big[X_2 \,\|\, X_1\big]$, the conditional expectation[1] of X_2 given X_1, is an explicit linear function of X_1. Moreover, for any continuous function $f : \boldsymbol{X}_2 \to \mathbb{R}$ with at most polynomial growth at ∞ we have the *regression formula* (see [24, Prop. 12] or [12, Sec. 2.3.2])

$$\mathbb{E}\big[f(X_2)\big| X_1 = 0\big] = \mathbb{E}\big[f(Z)\big],$$

where $Z = X_2 - \mathbb{E}\big[X_2 \,\|\, X_1\big]$ is a centered Gaussian vector independent of X_1 with variance

$$\Delta_{X_2,X_1} = Var\big[X_2\big] - Cov\big[X_2, X_1\big]Var[X_1]^{-1}Cov\big[X_1, X_2\big]. \tag{2.1}$$

Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are finite dimensional real Euclidean spaces and $\mathscr{V} \subset \boldsymbol{V}$ is an open set. For a map $F \in C^k(\mathscr{V}, \boldsymbol{U})$ we denote by $F^{(k)}(v)$ the k-order differential of F at v. It is an element of the vector space $\mathbf{Sym}_k\big(\boldsymbol{V}, \boldsymbol{U}\big)$ consisting of symmetric k-linear maps

$$\underbrace{\boldsymbol{V} \times \cdots \times \boldsymbol{V}}_{k} \to \boldsymbol{U}.$$

[1] **About notation**: $\mathbb{E}\big[X_2 \,\|\, X_1\big]$ is a random quantity, $f(x_1) = \mathbb{E}\big[X_2\big| X_1 = x_1\big]$ is a *deterministic* quantity and $\mathbb{E}\big[X_2 \,\|\, X_1\big] = f(X_1)$.

The *k-th jet* of F at $\boldsymbol{v} \in \mathscr{V}$ is the vector

$$J_k F(\boldsymbol{v}) := F(\boldsymbol{v}) \oplus F'(\boldsymbol{v}) \oplus \cdots \oplus F^{(k)}(\boldsymbol{v}).$$

The *Jacobian of* F *at* $v \in \mathscr{V}$ is

$$J_F(v) = \sqrt{\det\left(F'(v)F'(v)^*\right)}.$$

When $\boldsymbol{U} = \boldsymbol{V}$ we have

$$J_F(v) = \left|\det F'(v)\right|.$$

A $\boldsymbol{U}$-valued *random field* on $\mathscr{V}$ is a family of random variables

$$F(v) : \left(\Omega, \mathcal{S}, \mathbb{P}\right) \to \boldsymbol{U},\ \ \Omega \ni \omega \mapsto F_\omega(\boldsymbol{v}) \in \boldsymbol{U},\ \ \boldsymbol{v} \in \mathscr{V}.$$

We will work with measurable random fields , i.e., random fields F such that the map

$$\Omega \times \mathscr{V} \to \boldsymbol{U},\ \ (\omega, \boldsymbol{v}) \mapsto F_\omega(\boldsymbol{v})$$

is measurable with respect to the product sigma-algebra on $\Omega \times \mathscr{V}$. The maps $F_\omega : \mathscr{V} \to \boldsymbol{U}$ are called the *sample maps* of the random field F. The random field is called C^ℓ if all its sample maps belong to $C^\ell(\mathscr{V}, \boldsymbol{U})$.

The random field is called *Gaussian* if, for any $n \in \mathbb{N}$, and any $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathscr{V}$, the random vector

$$\left(F(\boldsymbol{v}_1), \ldots, F(\boldsymbol{v}_n)\right) \in \boldsymbol{U}^n$$

is Gaussian. In the sequel we will work exclusively with centered Gaussian fields, i.e., $F(\boldsymbol{v})$ is centered, $\forall \boldsymbol{v} \in \mathscr{V}$.

If $F : \Omega \times \mathscr{V} \to \boldsymbol{U}$, the covariance kernel of F is the map

$$\mathcal{K}_F : \mathscr{V} \times \mathscr{V} \to End\left(\boldsymbol{U}\right),\ \mathcal{K}_F(\boldsymbol{v}_1, \boldsymbol{v}_0) = Cov\left[F(\boldsymbol{v}_1), F(\boldsymbol{v}_0)\right]$$

Work going back to Kolmogorov shows that if the covariance kernel of F is sufficiently regular, then so is F. More precisely, we will need the following more precise result, [22, Appendices A.9-A.11].

Theorem 2.1. *Fix* $\ell \in \mathbb{N}_0$ *and* $\alpha \in (0,1)$*. Suppose that* $\mathcal{V}$ *is an open subset of* $\mathbb{R}^m$ *and* $X : \Omega \times \mathcal{V} \to \mathbb{R}$ *is a centered Gaussian function with covariance kernel* $\mathcal{K}_X$*. Assume that* $\mathcal{K} \in C^{2\ell+2}\left(\mathcal{V} \times \mathcal{V}\right)$*. Then* X *admits a* $C^{\ell,\alpha}$*-modification. Moreover, for every box* $B \subset \mathcal{V}$ *and every* $p \geq 1$ *there exists a constant* $C_p = C(B, \mathcal{V}, \ell, \alpha)$ *such that*

$$\mathbb{E}\left[\|X\|^p_{C^{\ell,\alpha}(B)}\right] \leq C\left\|\mathcal{K}_X\right\|^{p/2}_{C^{2\ell+2}(B\times B)}, \tag{2.2}$$

where $C^{k,\alpha}$ *denotes the spaces of functions that are* k *times differentiable and the* k*-th differential is Hölder continuous with exponent* α*.* □

Suppose that $F : \Omega \times \mathscr{V} \to \boldsymbol{U}$ is a C^ℓ Gaussian field. F is said to be *ample* if for any $\boldsymbol{v} \in \mathscr{V}$ the Gaussian vector $F(\boldsymbol{v})$ is nondegenerate. More generally, if n is a positive integer, then we say that F is n*-ample* if, for any pairwise distinct points $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathscr{V}$, the Gaussian vector $F(\boldsymbol{v}_1) \oplus \cdots \oplus F(\boldsymbol{v}_n)$ is nondegenerate. Let $0 \leq k \leq \ell$. We say that F is J_k*-ample* if, for any $\boldsymbol{v} \in \mathscr{V}$, the k-th jet $J_k F(\boldsymbol{v})$ is a nondegenerate Gaussian vector.

We have the following result. For a proof we refer to [25, Lemma 11.2.10] or [26, Sec. 4].

Lemma 2.1 (Bulinskaya). *Suppose now that* $\dim \boldsymbol{U} = \dim \boldsymbol{V} = m$ *and* $F : \Omega \times \mathcal{V} \to \boldsymbol{V}$ *is an ample* C^1*, Gaussian random field. If* $K \subset \mathscr{V}$ *is a compact set of Hausdorff dimension* $\leq m-1$*, then* $F^{-1}(0) \cap K = \emptyset$ a.s.. *Moreover*

$$\mathbb{P}\left[\{F(v) = 0,\ \ J_F(v) = 0\}\right] = 0$$

□

If $\boldsymbol{V}$ is an m-dimensional Euclidean space, then a box in $\boldsymbol{V}$ is a subset isometric to a parallelepiped $[a_1, b_1] \times \cdots \times [a_m, b_m] \subset \mathbb{R}^m$. The Kac-Rice formula plays a central role in this paper. We state below one version of this formula. For a proof we refer to [26] or [24, Thm.6.2].

Theorem 2.2 (Local Kac-Rice formula)**.** *Suppose that* $F : \Omega \times \mathcal{V} \to \boldsymbol{U}$ *is an ample* C^1*, Gaussian random field,* $m = \dim \boldsymbol{V} = \dim \boldsymbol{U}$*. Denote by* $p_{F(v)}$ *the probability density of the nondegenerate Gaussian vector* $F(v)$*. For any box* $B \subset \mathscr{V}$ *and any nonnegative continuous function* $w \in C(B)$ *we set*

$$\mathcal{Z}_B(w, F) = \sum_{\substack{F(v)=0,\\ v\in B}} w(v) \in [0, \infty].$$

In particular, $\mathcal{Z}(B, F) := \mathcal{Z}_B(1, F)$ *is the number of zeros of* F *in* B*. Then* $\mathcal{Z}_B(\varphi, F)$*, is measurable,* a.s. *finite and*

$$\mathbb{E}\left[\mathcal{Z}_B(w, F)\right] = \int_B w(v)\rho_F(v)dv \tag{2.3}$$

where ρ_{KR} *is the Kac-Rice density*

$$\rho_F(v) := \mathbb{E}\left[J_F(v)\middle|F(v) = 0\right]p_{F(v)}(0). \tag{2.4}$$

Corollary 2.1. *Let $\mathcal{V} \subset \mathbb{R}^m$ an open set. Suppose that $\Phi : \mathcal{V} \to \mathbb{R}$ a Gaussian random function that is* a.s. C^2 *and such that the Gaussian vector $\nabla\Phi(\boldsymbol{v})$ is nondegenerate for any $\boldsymbol{v} \in \mathcal{V}$. We denote by $p_{\nabla\Phi(v)}$ is probability density. The following hold.*

i. The random function Φ is a.s. *Morse*
ii. We set

$$\mathfrak{C}[-,\Phi] := \sum_{\nabla F(\boldsymbol{v})=0} \delta_v \tag{2.5}$$

Then $\mathfrak{C}[-,\Phi]$ is a random locally finite measure on $\mathscr{V}$ in the sense of [7] or [8]. For any nonnegative measurable function $\varphi : \mathscr{V} \to [0,\infty)$ we set

$$\mathfrak{C}[\varphi,\phi] = \int \varphi(v)\mathfrak{C}[dv,\Phi] = \sum_{\nabla\Phi(v)=0} \varphi(v).$$

iii. For any box $B \subset \mathscr{V}$, the function Φ a.s. *has no critical points on ∂B and*

$$\mathbb{E}\big[\mathfrak{C}^{\Phi}[\boldsymbol{I}_B\varphi]\big] = \int_B \rho_{\nabla\Phi}(v)\varphi(\boldsymbol{v})d\boldsymbol{v}, \tag{2.6}$$

where quantity

$$\rho_{\nabla\Phi} := \mathbb{E}\big[|\det Hess_{\Phi}(\boldsymbol{v})| \;\big|\; \nabla\Phi(\boldsymbol{v}) = 0\big] p_{\nabla F(v)}(0) \tag{2.7}$$

is called the Kac-Rice (or KR) density of Φ.

We conclude this section with a technical result that will be used several times in the proof of Theorem 1.1.

Suppose that $\boldsymbol{V}$ is an m-dimensional real Euclidean space with inner product $(-,-)$. Denote by $S_1(\boldsymbol{V})$ the unit sphere in $\boldsymbol{V}$, by $\mathbf{Sym}(\boldsymbol{V})$ the space of symmetric operators $\boldsymbol{V} \to \boldsymbol{V}$, and by $\mathbf{Sym}_{\geq 0}(\boldsymbol{V}) \subset \mathbf{Sym}(\boldsymbol{V})$ the cone of nonnegative ones. For $A \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we denote by Γ_A the centered Gaussian measure on $\boldsymbol{V}$ with variance A.

The space $\mathbf{Sym}(\boldsymbol{V})$ is equipped with an inner product

$$\big(A, B\big)_{\mathrm{op}} = \mathrm{tr}(AB), \quad \forall A, B \in \mathbf{Sym}(\boldsymbol{V}).$$

Denote by $\|-\|_{\mathrm{op}}$ the associated norm. We have the following result, [27, Prop.2.1].

Proposition 2.1. *For any $\mu > 0$ and $\forall A, B \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$, such that $A^{1/2} + B^{1/2} \geq \mu\mathbb{1}$ we have*

$$\mu\big\|A^{1/2} - B^{1/2}\big\|_{\mathrm{op}} \leq \big\|A - B\big\|_{\mathrm{op}}^{1/2}. \tag{2.8}$$

□

Lemma 2.2. *Fix $A_0 \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $A_0^{1/2} \geq \mu_0\mathbb{1}$, $\mu_0 > 0$. Suppose that $f : \mathbb{V} \to \mathbb{R}$ is a locally Lipschitz function that is homogeneous of degree $k \geq 1$. For $A \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we set*

$$\mathfrak{I}_A(f) := \int_{\boldsymbol{V}} f(\boldsymbol{v})\Gamma_A\big[d\boldsymbol{v}\big].$$

Then for any $R \geq \|A_0\|_{\mathrm{op}}$ there exists a constant $C{=}C(f, R, \mu_0) > 0$ with the following property: for any $A \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $\|A\|_{\mathrm{op}} \leq R$

$$\Big|\mathfrak{I}_{A_0}(f) - \mathfrak{I}_A(f)\Big| \leq C\|A - A_0\|^{1/2} \leq C(k,R)\|A - A_0\|_{\mathrm{op}}^{1/2}. \tag{2.9}$$

In other words, $A \mapsto \mathfrak{I}_A(f)$ is locally Hölder continuous with exponent $1/2$ in the open set $\mathbf{Sym}_{>0}\big(\boldsymbol{V}\big)$.

Proof. The function f is Lipschitz on the ball

$$B_R(\boldsymbol{V}) := \big\{\, \boldsymbol{v} \in \boldsymbol{V};\, \|\boldsymbol{v}\| \leq R \,\big\},$$

so there exists $L = L(R) > 0$ such that

$$\big[f(\boldsymbol{u}) - f(\boldsymbol{v})\big| \leq L\|\boldsymbol{u} - \boldsymbol{v}\|, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in B_R(\boldsymbol{V}). \tag{2.10}$$

Note that

$$\mathfrak{I}_A(f) = \int_{\boldsymbol{V}} f\big(A^{1/2}\boldsymbol{v}\big)\Gamma_{\mathbb{1}}\big[d\boldsymbol{v}\big],$$

so

$$\begin{aligned}
\Big|\mathfrak{I}_{A_0}(f) - \mathfrak{I}_A(f)\Big| &\leq \int_{\boldsymbol{V}} \Big|f\big(A^{1/2}\boldsymbol{v}\big) - f\big(A_0^{1/2}\boldsymbol{v}\big)\Big|\,\Gamma_{\mathbb{1}}\big[d\boldsymbol{v}\big] \\
&= \underbrace{\frac{1}{(2\pi)^{m/2}}\left(\int_0^\infty r^{n+k-1}e^{-r^2/2}dr\right)}_{C_{m,k}} \int_{S_1(\boldsymbol{V})} \Big|f\big(A^{1/2}\boldsymbol{v}\big) - f\big(A_0^{1/2}\boldsymbol{v}\big)\Big| vol_{S_1(\boldsymbol{V})}\big[d\boldsymbol{v}\big] \\
&\overset{(2.10)}{\leq} C_{m,k}L(R)\int_{S_1(\boldsymbol{V})} \|A^{1/2} - A_0^{1/2}\|_{\mathrm{op}} vol_{S^1(\boldsymbol{V})}\big[d\boldsymbol{v}\big] \overset{(2.8)}{\leq} C(k,R,\mu_0)\|A - A_0\|_{\mathrm{op}}^{1/2}.
\end{aligned}$$

□

Lemma 2.3. *Suppose that* $f : \boldsymbol{V} \to \mathbb{R}$ *is a continuous function that is homogeneous of degree* $k \geq 1$. *Set*

$$M(f) := \sup_{\|\boldsymbol{w}\|\leq 1} |f(\boldsymbol{v})|.$$

Then there exists $C{=}C(m,k) > 0$ *such that* $\forall A \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$

$$\left|\mathcal{I}_A(f)\right| \leq \mathcal{I}_A(|f|) \leq C(m,k)M(f)\|A\|_{\mathrm{op}}^{k/2}. \tag{2.11}$$

Proof. Note that

$$\sup_{\|\boldsymbol{u}\|\leq R} |f(\boldsymbol{u})| = M(f)R^k.$$

As in the proof of Lemma 2.2 we have

$$\mathcal{I}_A(|f|) = \int_{\boldsymbol{V}} f\big(A^{1/2}\boldsymbol{w}\big)\boldsymbol{\Gamma}_{\mathbb{1}}\big[d\boldsymbol{v}\big]$$

$$= \underbrace{\frac{1}{(2\pi)^{m/2}}\left(\int_0^\infty r^{m+k-1}e^{-r^2/2}dr\right)}_{=:C_{m,k}}\int_{S_1(\boldsymbol{V})}\left|f\big(A^{1/2}\boldsymbol{v}\big)\right| vol_{S_1(\boldsymbol{V})}\big[d\boldsymbol{v}\big]$$

$(\|A^{1/2}\boldsymbol{v}\| \leq \|A^{1/2}\|_{\mathrm{op}}\|\boldsymbol{v}\|)$

$$\leq C_{m,k}M(f)\|A^{1/2}\|_{\mathrm{op}}^k vol\big[S_1(\boldsymbol{V})\big] = C(m,k)M(f)\|A\|_{\mathrm{op}}^{k/2}.$$

□

Corollary 2.2. *Suppose that* $f : \boldsymbol{V} \to \mathbb{R}$ *is a continuous function that is homogeneous of degree* $k \geq 1$. *Suppose that* $A, B \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ *and* $B \leq A$. *Then*

$$\left|\mathcal{I}_B(f)\right| \leq \mathcal{I}_B(|f|) \leq C(m,k)M(f)\|A\|_{\mathrm{op}}^{k/2} \tag{2.12}$$

Proof. Indeed, $0 \leq B \leq A \implies \|B\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}}$. □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 relies on various Kac-Rice formulas. In turn, their applicability depends heavily on a good understanding of the asymptotic behavior of the covariance kernel $\mathcal{C}^R_{\mathfrak{a}}\big(\vec{\varphi},\vec{\theta}\big)$ as $R\to\infty$.

The covariance kernel $\mathcal{C}^R_{\mathfrak{a}}$ is the Schwarz kernel of the smoothing operator $R^{-m}\mathfrak{a}\big(R^{-1}\sqrt{\Delta}\big)^2$. The asymptotic behavior of this kernel is well understood along the diagonal $\vec{\varphi}=\vec{\theta}$; see e.g.[2, Chap.XII]. However, we need to understand what happens away from the diagonal. Off-diagonal estimates are a lot harder to come by on arbitrary Riemann manifolds. However, on a flat torus, Poisson's summation formula will help us produce very sharp off-diagonal estimates avoiding more sophisticated considerations. This is the goal of the next subsection.

3.1. The key estimate

Lemma 3.1. *Fix* $r_0 \in (0,1)$ *and a box* $B = B^\infty_{r_0/2}(0) = [-r_0/2, r_0/2]^m$. *Then the following hold.*

i. *For any* $\ell \in \mathbb{N}_0$ *and any* $p > m$ *there exists* $C{=}C(p,m,\ell,\mathfrak{a}) > 0$, *independent of* R, *such that,* $\forall R > 2$

$$\left\|\boldsymbol{K}^R_{\mathfrak{a}} - \boldsymbol{K}_{\mathfrak{a}}\right\|_{C^\ell(RB)} \leq CR^{-p}$$

ii. *For any* $\ell \in \mathbb{N}_0$ *and any* $p > m$ *there exists* $C{=}C(p,m,\ell,\mathfrak{a}) > 0$, *independent of* R, *such that,* $\forall R > 2$, $\forall \boldsymbol{x}, \boldsymbol{y} \in RB$

$$\left|D^\ell \boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})\right| \leq \frac{C}{\big(1+|\boldsymbol{x}-\boldsymbol{y}|_\infty\big)^p}.$$

Proof.

(i) Denote by $\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}$ the translate

$$\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}) := \boldsymbol{K}\big(\boldsymbol{x}-R\vec{k}\big).$$

We have

$$\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{x}) - \boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}) = \sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\mathcal{T}_{R\vec{k}}\boldsymbol{K}_{\mathfrak{a}}\big(\boldsymbol{x}\big).$$

Now observe that $\forall R > 0$, $\forall \boldsymbol{x} \in RB$, and any $\vec{k} \in \mathbb{Z}^m \setminus 0$ we have

$$\left|\boldsymbol{x} - R\vec{k}\right|_\infty \geq R\left|\vec{k}\right|_\infty - \left|\boldsymbol{x}\right|_\infty \geq R\big(\left|\vec{k}\right|_\infty - r_0/2\big).$$

Since $\boldsymbol{K}_\mathfrak{a}$ and all its derivatives are Schwartz functions we deduce that for any $p > m$, and any $\vec{k} \in \mathbb{Z}^m \setminus 0$

$$\left\|\mathcal{T}_{R\vec{k}}\boldsymbol{K}_\mathfrak{a}\right\|_{C^\ell(NB)} \leq C(p, m, \ell, \mathfrak{a})R^{-p}\big(\left|\vec{k}\right|_\infty - r_0/2\big)^{-p}.$$

The last expression is well defined since $r_0 < 1 \leq \left|\vec{k}\right|_\infty$ for any $\vec{k} \in \mathbb{Z}^m \setminus 0$. Hence

$$\left\|\boldsymbol{K}^R_\mathfrak{a} - \boldsymbol{K}_\mathfrak{a}\right\|_{C^\ell(NB)} \leq C(p, m, \ell, \mathfrak{a})R^{-p}\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0}\big(\left|\vec{k}\right|_\infty - r_0/2\big)^{-p}$$

The above series is convergent since $p > m$.

(ii) Note that $\forall \boldsymbol{x}, \boldsymbol{y} \in RB$ we have $\left|\boldsymbol{x} - \boldsymbol{y}\right|_\infty \leq Rr_0$. Set $\boldsymbol{z} := \boldsymbol{x} - \boldsymbol{y}$. We discuss only the case $\ell = 0$. The general case can be reduced to this case by taking partial derivatives.

Using (i) we deduce that

$$C = \sup_R \sup_{\left|z\right|_\infty < r_0} \left|\boldsymbol{K}^R_\mathfrak{a}(\boldsymbol{z})\right| < \infty$$

and thus, $\forall R \geq 2$, $\forall \left|\boldsymbol{z}\right|_\infty < r_0$,

$$\left|\boldsymbol{K}^R_\mathfrak{a}(\boldsymbol{z})\right| < \frac{C\left(1 + r_0\right)^p}{\left(1 + \left|\boldsymbol{z}\right|_\infty\right)^p}.$$

Assume now that $\left|\boldsymbol{z}\right|_\infty \geq r_0$. We have

$$\boldsymbol{K}^R_\mathfrak{a}(\boldsymbol{z}) = \boldsymbol{K}_\mathfrak{a}(\boldsymbol{z}) + \sum_{\vec{k}\in\mathbb{Z}^m\setminus 0} \mathcal{T}_{R\vec{k}}\boldsymbol{K}_\mathfrak{a}\big(\boldsymbol{z}\big),$$

and thus,

$$\left|\boldsymbol{K}^R_\mathfrak{a}(\boldsymbol{z})\right| \leq \left|\boldsymbol{K}_\mathfrak{a}(\boldsymbol{z})\right| + \sum_{\vec{k}\in\mathbb{Z}^m\setminus 0} \left|\mathcal{T}_{R\vec{k}}\boldsymbol{K}_\mathfrak{a}\big(\boldsymbol{z}\big)\right|.$$

Since $\boldsymbol{K}_\mathfrak{a}(\boldsymbol{x})$ is Schwartz we deduce that there exists a constant $C{=}C(p, \mathfrak{a}) > 0$ such that

$$\left|\boldsymbol{K}^R_\mathfrak{a}(\boldsymbol{z})\right| \leq \frac{C_p}{\left(1 + \left|\boldsymbol{z}\right|_\infty\right)^p} + C_p \sum_{\vec{k}\in\mathbb{Z}^m\setminus 0} \frac{1}{\left(1 + \left|\boldsymbol{z} - R\vec{k}\right|_\infty\right)^p}.$$

We have $\left|\boldsymbol{z}\right|_\infty \leq Rr_0$ and

$$\left|\boldsymbol{z} - Z\vec{k}\right|_\infty \geq \left|\boldsymbol{z}\right|_\infty\left(\frac{R\left|\vec{k}\right|_\infty}{\left|\boldsymbol{z}\right|_\infty} - 1\right) \geq \left|\boldsymbol{z}\right|_\infty\left(\frac{1}{r_0}\left|\vec{k}\right|_\infty - 1\right).$$

Thus

$$\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0} \frac{1}{\left(1 + \left|\boldsymbol{z} - R\vec{k}\right|_\infty\right)^p} \leq \left|\boldsymbol{z}\right|_\infty^{-p} \underbrace{\sum_{\vec{k}\in\mathbb{Z}^m\setminus 0} \left(\frac{1}{r_0}\left|\vec{k}\right|_\infty - 1\right)^{-p}}_{<\infty}.$$

□

3.2. An integral formula

We will use the Kac-Rice formula to describe the variance of $\bar{\mathfrak{C}}_R[f]$ as an integral of $\mathbb{R}^m \times \mathbb{R}^m \setminus \Delta$. Set $B := B^\infty_{r_0/2}(0)$, $f_R(\boldsymbol{x}) := f(\boldsymbol{x}/R)$,

$$Z^R[f] := \mathfrak{C}[f, F^R_\mathfrak{a}] = \mathfrak{C}[f_R, \Phi^R_\mathfrak{a}], \quad Z[f] := \mathfrak{C}[f, \Phi_\mathfrak{a}].$$

Denote by $\rho^R_\mathfrak{a}$ the Kac-Rice density of $\Phi^R_\mathfrak{a}$ and by $\rho_\mathfrak{a}$ the Kac-Rice density of $\Phi_\mathfrak{a}$; see (2.7). Since both $\Phi^R_\mathfrak{a}$ and $\Phi_\mathfrak{a}$ are stationary random functions we deduce that both $\rho^R_\mathfrak{a}$ and $\rho_\mathfrak{a}$ are constant functions. We set and $C_m(\mathfrak{a}) := \rho_\mathfrak{a}(0)$. For an explicit description of $C_m(\mathfrak{a})$ we refer to [10,11] .

The covariance functions $\boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})$ and $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{z})$ are even so the odd order derivatives of these functions vanish at 0. This implies that the Gaussian vectors $Hess_{\Phi^R_{\mathfrak{a}}}(0)$ and $\nabla\Phi^R_{\mathfrak{a}}(0)$ are independent. A similar phenomenon is true for $\Phi_{\mathfrak{a}}$. Thus, the conditional expectations in (2.7) are usual expectations. Using Lemma 2.2 and Lemma 3.1(i) we deduce that for any $\boldsymbol{x}\in\mathbb{R}^m$

$$\sup_{\boldsymbol{x}\in RB}\Big|\rho^R_{\mathfrak{a}}(\boldsymbol{x})-\rho_{\mathfrak{a}}(\boldsymbol{x})\Big| = \Big|\rho^R_{\mathfrak{a}}(0)-\rho_{\mathfrak{a}}(0)\Big| = O\big(R^{-\infty}\big), \tag{3.1}$$

where $O(R^{-\infty})$ is short-hand for $O(R^{-p})$, $\forall p>0$. We deduce that

$$R^{-m}\big(\mathbb{E}\big[Z^R[f]\big]-\mathbb{E}\big[Z[f]\big]\big) = R^{-m}\int_{RB} f_R(\boldsymbol{x})\big(\rho^R_{\mathfrak{a}}(0)-\rho_{\mathfrak{a}}(0)\big)d\boldsymbol{x}$$

$$= \int_B f(\boldsymbol{y})\big(\rho^R_{\mathfrak{a}}(0)-\rho_{\mathfrak{a}}(0)\big)d\boldsymbol{y} = O\big(R^{-\infty}\big).$$

On the other hand

$$\mathbb{E}\big[Z[f]\big] = C_m(\mathfrak{a})\int_{\mathbb{R}^m} f(\boldsymbol{x})d\boldsymbol{x}.$$

We need to introduce some notation. Set

- $\Phi^\infty_{\mathfrak{a}} = \Phi_{\mathfrak{a}}$.
- For any $R\in(0,\infty]$ we define

$$\widehat{\Phi}^R,\ \widehat{\Phi}:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R},$$

$$\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}) = \Phi^R_{\mathfrak{a}}(\boldsymbol{x})+\Phi^R_{\mathfrak{a}}(\boldsymbol{y}),\ \ \widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}) = \Phi_{\mathfrak{a}}(\boldsymbol{x})+\Phi_{\mathfrak{a}}(\boldsymbol{y}),$$

$$\hat{\mathfrak{C}}^R := \mathfrak{C}[-,\widehat{\Phi}^R_{\mathfrak{a}}],\ \ \widehat{H}_R(\boldsymbol{x},\boldsymbol{y}) := Hess_{\widehat{\Phi}^R}(\boldsymbol{x},\boldsymbol{y}),\ \ H_R(\boldsymbol{x}) := Hess_{\Phi^R_{\mathfrak{a}}}(\boldsymbol{x}).$$

- Choose an independent copy $\Psi^R_{\mathfrak{a}}$ of $\Phi^R_{\mathfrak{a}}$ and for $R\in(0,\infty]$ set

$$\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y}) := \Phi^R_{\mathfrak{a}}(\boldsymbol{x})+\Psi^R_{\mathfrak{a}}(\boldsymbol{y}),\ \ \widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}) := Hess_{\widetilde{\Phi}^R}(\boldsymbol{x},\boldsymbol{y}),$$

$$\tilde{\mathfrak{C}}^R := \mathfrak{C}[-,\widetilde{\Phi}^N].$$

- For $R\in(0,\infty)$ define

$$f^{\boxtimes 2}_R:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R},\ \ f^{\boxtimes}_R(\boldsymbol{x},\boldsymbol{y}) = f_R(\boldsymbol{x})f_R(\boldsymbol{y})$$

  and set $\|f\| := \|f\|_{C^0(\mathbb{R}^m)}$.
- Set

$$\mathfrak{X} = \mathbb{R}^m\times\mathbb{R}^m\setminus\Delta = \big\{(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^m\times\mathbb{R}^m;\ \ \boldsymbol{x}\neq\boldsymbol{y}\big\}.$$

Observe that the random function on $\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})$ is *stationary* with respect to the action of $\mathbb{R}^{2m}$ on itself by translations.

We have

$$\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}}f^{\boxtimes 2}_R] = \sum_{\substack{\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})=\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})=0,\\ \boldsymbol{x}\neq\boldsymbol{y}}} f_R(\boldsymbol{x})f_R(\boldsymbol{y}) = Z^R[f]^2 - Z^R[f^2].$$

Bulinskaya's lemma implies that

$$\mathbb{P}\big[\exists\boldsymbol{x}:\ \ \nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}) = \nabla\Psi_{\mathfrak{a}}(\boldsymbol{x}) = 0\big] = 0$$

and we deduce

$$\tilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}}f^{\boxtimes 2}_R] = \sum_{\substack{\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})=\nabla\Psi^R_{\mathfrak{a}}(\boldsymbol{y})=0,\\ \boldsymbol{x}\neq\boldsymbol{y}}} f_R(\boldsymbol{x})f_R(\boldsymbol{y})$$

$$= \sum_{\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})=\nabla\Psi^R_{\mathfrak{a}}(\boldsymbol{y})=0} f_R(\boldsymbol{x})f_R(\boldsymbol{y}) = \mathfrak{C}[f,\Phi^R_{\mathfrak{a}}]\mathfrak{C}[f,\Psi^R_{\mathfrak{a}}],\ \ \text{a.s.}.$$

Hence

$$\mathbb{E}\big[\mathfrak{C}[f,\Phi^R_{\mathfrak{a}}]\mathfrak{C}[f,\Psi^R_{\mathfrak{a}}]\big] = \mathbb{E}\big[\mathfrak{C}[f,\Phi^R_{\mathfrak{a}}]\big]\cdot\mathbb{E}\big[\mathfrak{C}[f,\Psi^R_{\mathfrak{a}}]\big] = \mathbb{E}\big[\mathfrak{C}[f,\Phi^R_{\mathfrak{a}}]\big]^2$$

so that

$$\mathbb{E}\big[\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}}f^{\boxtimes 2}_R]\big]-\mathbb{E}\big[\tilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}}f^{\boxtimes 2}_R]\big] = \underbrace{\mathbb{E}\big[Z^R[f]^2\big]-\mathbb{E}\big[Z^R[f]\big]^2}_{=Var\left[Z^R[f]\right]} - \mathbb{E}\big[Z^R[f^2]\big]. \tag{3.2}$$

We have seen that

$$\lim_{R\to\infty} R^{-m}\mathbb{E}\big[Z^R[f^2]\big] = C_m(\mathfrak{a})\int_{\mathbb{R}^m} f^2(\boldsymbol{x})d\boldsymbol{x}$$

so we have to show that

$$I(R) := \mathbb{E}\big[\widehat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\big] - \mathbb{E}\big[\widetilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\big] \sim Z_m(\mathfrak{a})R^m\int_{\mathbb{R}^m} f^2(\boldsymbol{x})d\boldsymbol{x} \ \text{ as } R\to\infty \tag{3.3}$$

for some constant $Z_m(\mathfrak{a})\in\mathbb{R}$ that depends only on m and $\mathfrak{a}$.

According to Corollary A.5, there exists $R_0 > 0$ such that for $R \geq R_0$, the gradient $\nabla\Phi^R_{\mathfrak{a}}$ is 2-ample and $\Phi^R_{\mathfrak{a}}$ is J_1-ample so, for $R\geq R_0$ the gradient $\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate for any $\boldsymbol{x}\neq\boldsymbol{y}$ and the random vector $\big(\Phi^R_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}\big)$ is nondegenerate for any $\boldsymbol{x}\in\mathbb{R}^n$. As shown in [10] this is true also for $R=\infty$, where we recall that $\Phi^\infty_{\mathfrak{a}} = \Phi_{\mathfrak{a}}$.

We can apply the Kac-Rice formula and we deduce that for any $R > R_0$ we have

$$\begin{aligned}&\mathbb{E}\big[\widehat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\big]\\ &= \int_{\mathbb{R}^m\times\mathbb{R}^m\setminus\Delta} \underbrace{\mathbb{E}\Big[|\det \widehat{H}_R(\boldsymbol{x},\boldsymbol{y})|\,\Big|\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})=0\Big]p_{\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y})}\, f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})\lambda\big[d\boldsymbol{x}d\boldsymbol{y}\big].\end{aligned} \tag{3.4}$$

The gradient $\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate for any $\boldsymbol{x},\boldsymbol{y}$ and invoking Kac-Rice again we obtain

$$\begin{aligned}&\mathbb{E}\big[\widetilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\big]\\ &= \int_{\mathbb{R}^m\times\mathbb{R}^m\setminus\Delta} \underbrace{\mathbb{E}\Big[|\det \widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})|\,\Big|\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})=0\Big]p_{\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})}\, f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})\lambda\big[d\boldsymbol{x}d\boldsymbol{y}\big].\end{aligned} \tag{3.5}$$

The function $\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})$ is independent of $\boldsymbol{x},\boldsymbol{y}$ since the random function $\widetilde{\Phi}^R$ is stationary. Thus

$$\begin{aligned}I(R) &= \int_{\mathfrak{X}} \big(\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})\big) f_R(\boldsymbol{x})f_R(\boldsymbol{y})\lambda\big[d\boldsymbol{x}d\boldsymbol{y}\big]\\ &= \int_{\substack{|\boldsymbol{x}|,\,|\boldsymbol{y}|\leq Rr_0/2,\\ \boldsymbol{x}\neq\boldsymbol{y}}} \big(\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})\big) f_R(\boldsymbol{x})f_R(\boldsymbol{y})\lambda\big[d\boldsymbol{x}d\boldsymbol{y}\big].\end{aligned} \tag{3.6}$$

Let us observe that for any $\boldsymbol{x}\neq\boldsymbol{y}$ we have

$$\lim_{R\to\infty}\big(\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})\big) = \big(\widehat{\rho}_\infty(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_\infty(\boldsymbol{x},\boldsymbol{y})\big).$$

Moreover

$$\lim_{R\to\infty} f_R(\boldsymbol{x}) = f(0)$$

uniformly on compacts.

We divide the proof of (3.3) into several conceptually distinct parts.

3.3. Off-diagonal behavior

Note that

$$Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\big] = \begin{bmatrix} Var\big[H_R(\boldsymbol{x})\big] & 0\\ 0 & Var\big[H_R(\boldsymbol{y})\big]\end{bmatrix}.$$

For every $\boldsymbol{z}\in\mathbb{R}^m$ we set

$$T_R(\boldsymbol{z}) := \sum_{|\alpha|\leq 4}\Big|\partial^\alpha \boldsymbol{K}^R_{\mathfrak{a}}(\boldsymbol{z})\Big|.$$

Lemma 3.1(ii) shows that for every $p>0$ there exists $C_p = C_p(\mathfrak{a},m,r) > 0$ such that, $\forall R$, $\forall\big|\boldsymbol{z}\big|_\infty < Nr$

$$\forall N,\ \ \forall\big|\boldsymbol{z}\big|_\infty < Rr_0,\ \ T_R(\boldsymbol{z}) \leq C_p\big(1+\big|\boldsymbol{z}\big|_\infty\big)^{-p}. \tag{3.7}$$

We want to emphasize that C_p *is independent of* R.

Observe next that

$$Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] = \begin{bmatrix} Var\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & 0\\ 0 & Var\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big]\end{bmatrix},$$

is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] = \begin{bmatrix} Var\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & Cov\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big]\\ Cov\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & Var\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big]\end{bmatrix}$$

$$= Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] + \underbrace{\begin{bmatrix} 0 & Cov\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big] \\ Cov\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & 0\end{bmatrix}}_{=:\mathcal{E}^R_\nabla(\boldsymbol{x},\boldsymbol{y})}.$$

Hence

$$\Big\|Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] - Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]\Big\|_{\rm op} = \|\mathcal{E}^R_\nabla(\boldsymbol{x},\boldsymbol{y})\|_{\rm op} = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big), \tag{3.8}$$

where $\|-\|_{\rm op}$ denotes the operator norm. Above and in the sequel, the *constant implied by the Landau symbol O is independent of R as long as $\boldsymbol{x},\boldsymbol{y}\in RB$*. In particular

$$\begin{gathered} Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1} = \Big(Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] + \mathcal{E}^R_\nabla(\boldsymbol{x},\boldsymbol{y})\Big)^{-1} \\ = \Big(\mathbb{1} + Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1}\mathcal{E}^R_\nabla(\boldsymbol{x},\boldsymbol{y})\Big)^{-1} Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1}. \end{gathered} \tag{3.9}$$

As shown in [10], there exists an explicit positive constant d_m such that

$$Var\big[\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\big] = d_m\mathbb{1}_m,\ \ \forall\boldsymbol{x}.$$

Then $Var\big[\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] = Var\big[\nabla\Phi^R_{\mathfrak{a}}(0)\big]$, $\forall\boldsymbol{x}\in\mathbb{R}^m$ and

$$Var\big[\nabla\Phi^R_{\mathfrak{a}}(0)\big] = d_m\mathbb{1}_m + O\big(R^{-\infty}\big).$$

The variance $Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]$ is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$ and

$$Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] = Var\big[\nabla\Phi^R_{\mathfrak{a}}(0)\big]\oplus Var\big[\nabla\Phi^R_{\mathfrak{a}}(0)\big] = d_m\mathbb{1}_{2m} + O\big(R^{-\infty}\big). \tag{3.10}$$

From (3.9) and (3.10) we conclude that there exists $C_0>0$, independent of $R>R_0$, such that

$$\|Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1}\mathcal{E}^R_\nabla(\boldsymbol{x},\boldsymbol{y})\|_{\rm op} < \frac{1}{2},\ \ \forall\boldsymbol{x},\boldsymbol{y}\in RB,\ \ |\boldsymbol{x}-\boldsymbol{y}|_\infty > C_0,$$

and thus

$$\begin{gathered} \Big\|Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1} - Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1}\Big\|_{\rm op} \\ = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big),\ \ \forall\boldsymbol{x},\boldsymbol{y}\in RB,\ \ |\boldsymbol{x}-\boldsymbol{y}|_\infty > C_0. \end{gathered} \tag{3.11}$$

Note that since $\Phi^R_{\mathfrak{a}}$ is stationary, $Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\big]$ is *independent* of $\boldsymbol{x}$ and $\boldsymbol{y}$.

$$Var\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\big] = \begin{bmatrix} Var\big[H_R(\boldsymbol{x})\big] & Cov\big[H_R(\boldsymbol{x}),H_R(\boldsymbol{y})\big] \\ Cov\big[H_R(\boldsymbol{y}),H_R(\boldsymbol{x})\big] & Var\big[H_R(\boldsymbol{y})\big]\end{bmatrix}$$

$$= Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\big] + \underbrace{\begin{bmatrix} 0 & Cov\big[H_R(\boldsymbol{x}),H_R(\boldsymbol{y})\big] \\ Cov\big[H_R(\boldsymbol{y}),H_R(\boldsymbol{x})\big] & 0\end{bmatrix}}_{=:\mathcal{E}^R_H(\boldsymbol{x},\boldsymbol{y})}.$$

We deduce

$$\Big\|Var\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\big] - Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\big]\Big\|_{\rm op} = \|\mathcal{E}^R_H(\boldsymbol{x},\boldsymbol{y})\|_{\rm op} = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big). \tag{3.12}$$

We denote by $\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ the Gaussian random matrix

$$\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat = \widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}) - \mathbb{E}\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\,\|\,\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big],$$

where we recall that $\mathbb{E}\big[X\,\|\,Y\big]$ denotes the conditional expectation of X given Y. We define $\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ similarly

$$\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat = \widehat{H}_R(\boldsymbol{x},\boldsymbol{y}) - \mathbb{E}\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\,\|\,\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big].$$

The distributions of $\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ and $\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat$ are determined by the Gaussian regression formula (2.1). We have

$$Cov\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] = \begin{bmatrix} Cov\big[H_R(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & Cov\big[H_R(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big] \\ Cov\big[H_R(\boldsymbol{y}),\nabla\Phi^N_{\mathfrak{a}}(\boldsymbol{x})\big] & Cov\big[H_R(\boldsymbol{y}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big]\end{bmatrix}$$

$$= \begin{bmatrix} Cov\big[H_R(0),\nabla\Phi^R_{\mathfrak{a}}(0)\big] & Cov\big[H_R(\boldsymbol{x}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big] \\ Cov\big[H_R(\boldsymbol{y}),\nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & Cov\big[H_R(0),\nabla\Phi^R_{\mathfrak{a}}(0)\big]\end{bmatrix}.$$

The covariance $Cov\big[H_R(0), \nabla\Phi^R_{\mathfrak{a}}(0)\big]$ involves only third order partial derivatives of $\boldsymbol{K}^N_{\mathfrak{a}}$ at 0, and these are all trivial since $\boldsymbol{K}^R_{\mathfrak{a}}$ is an even function. Hence

$$Cov\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] = \begin{bmatrix} 0 & Cov\big[H_R(\boldsymbol{x}), \nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big] \\ Cov\big[H_R(\boldsymbol{y}), \nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & 0 \end{bmatrix}.$$

Similarly

$$Cov\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] = \begin{bmatrix} Cov\big[H_R(\boldsymbol{x}), \nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{x})\big] & 0 \\ 0 & Cov\big[H_R(\boldsymbol{y}), \nabla\Phi^R_{\mathfrak{a}}(\boldsymbol{y})\big] \end{bmatrix} = 0.$$

Lemma 3.1(ii) implies that

$$\Big\| Cov\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]\Big\|_{\mathrm{op}} = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big),$$
$$\Big\| Cov\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]\Big\|_{\mathrm{op}} = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big).$$

The regression formula (2.1) shows that

$$\begin{aligned} Var\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\big] &= Var\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\big] \\ &- Cov\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1} Cov\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}), \widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\big]. \\ &= Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\big] + O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big) \\ &- Cov\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1} Cov\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}), \widehat{H}^R(\boldsymbol{x},\boldsymbol{y})\big]. \end{aligned}$$

Since $Cov\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big)$ we deduce from (3.10) and (3.11) that there exists $C_1 > 0$, independent of $R > R_0$, such that

$$\begin{aligned} &Cov\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y}), \nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big] Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1} Cov\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y}), \widehat{H}_R(\boldsymbol{x},\boldsymbol{y})\big] \\ &\qquad = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big), \;\; \forall \boldsymbol{x},\boldsymbol{y} \in RB, \;\; |\boldsymbol{x}-\boldsymbol{y}|_\infty > C_1, \end{aligned}$$

and thus

$$\begin{aligned} &\Big\| Var\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\big] - Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\big]\Big\|_{\mathrm{op}} \\ &= O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big), \;\; \forall \boldsymbol{x},\boldsymbol{y} \in RB, \;\; |\boldsymbol{x}-\boldsymbol{y}|_\infty > C_2 = \max(C_0, C_1). \end{aligned}$$

Since $Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})\big] = Var\big[H_R(0)\big] \oplus Var\big[H_R(0)\big]$ we deduce from Lemma 3.1(i) that there exists $\mu_0 > 0$ such that

$$Var\big[\widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\big] \geq \mu_0 \mathbb{1}, \;\; \forall R \geq R_0.$$

Note also that (3.8) implies that there exists $C_3 > 0$, independent of $R > R_0$, such that

$$\sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in RB \\ |\boldsymbol{x}-\boldsymbol{y}|_\infty > C_3}} \|Var\big[\widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat\big]\|_{\mathrm{op}} = O(1).$$

Lemma 2.2 implies that

$$\Big|\mathbb{E}\big[|\det \widehat{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat|\big] - \mathbb{E}\big[|\det \widetilde{H}_R(\boldsymbol{x},\boldsymbol{y})^\flat|\big]\Big| = O\big(T_R(\boldsymbol{x}-\boldsymbol{y})^{1/2}\big). \tag{3.13}$$

Using (3.11) we deduce that there exists $C_4 > 0$, independent of $R > R_0$, such that

$$\begin{aligned} &\Big| p_{\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0) - p_{\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})}(0)\Big| \\ &= \frac{1}{(2\pi)^{m/2}}\Big| \det Var\big[\nabla\widehat{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1} - \det Var\big[\nabla\widetilde{\Phi}^R(\boldsymbol{x},\boldsymbol{y})\big]^{-1}\Big| \\ &= O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big), \;\; \forall \boldsymbol{x},\boldsymbol{y} \in RB, \;\; |\boldsymbol{x}-\boldsymbol{y}|_\infty > C_4. \end{aligned} \tag{3.14}$$

We can now estimate the right-hand-side of (3.6). For any $\boldsymbol{x},\boldsymbol{y} \in RB$

$$O\big(T_R(\boldsymbol{x}-\boldsymbol{y})\big) \overset{(3.7)}{=} O\big(\big|\boldsymbol{x}-\boldsymbol{y}\big|_\infty^{-p/2}\big), \;\; \forall p > 0.$$

Using (3.11), (3.12), (3.13) and (3.14) we conclude that there exists $C_5 > 1$, independent of $R > R_0$ such that, for any $p > m$,

$$\forall \boldsymbol{x},\boldsymbol{y} \in RB, \;\; |\boldsymbol{x}-\boldsymbol{y}|_\infty > C_5, \;\; \Big|\underbrace{\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}) - \widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})}_{=\Delta_R(\boldsymbol{x},\boldsymbol{y})}\Big| = O\big(\big|\boldsymbol{x}-\boldsymbol{y}\big|_\infty^{-p/2}\big). \tag{3.15}$$

3.4. Conclusion

Since the random function $\Phi_{\mathfrak{a}}^R$ is stationary, we deduce that for any $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^m$ such that $\boldsymbol{x}\neq\boldsymbol{y}$ we have

$$\Delta_R(\boldsymbol{x}+\boldsymbol{z},\boldsymbol{y}+\boldsymbol{z})=\Delta_R(\boldsymbol{x},\boldsymbol{y})$$

so $\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y})$, $\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})$ and $\Delta_R(\boldsymbol{x},\boldsymbol{y})$ depend only on $\boldsymbol{y}-\boldsymbol{x}$. Assume now that $\boldsymbol{x},\boldsymbol{y}\in RB$ and $|\boldsymbol{x}-\boldsymbol{y}|_\infty\leq C_5$.

Denote by $\widehat{\mathfrak{X}}$ the radial-blowup of $\mathbb{R}^m\times\mathbb{R}^m$ along the diagonal. It is diffeomorphic to the product $\mathbb{R}^m\times S^{m-1}\times[0,\infty)$.

Choose new orthogonal coordinates (ξ,η) given by

$$\xi=\boldsymbol{x}+\boldsymbol{y},\;\;\eta=\boldsymbol{x}-\boldsymbol{y}\Longleftrightarrow \boldsymbol{x}=\frac{1}{2}(\xi+\eta),\;\;\boldsymbol{y}=\frac{1}{2}(\xi-\eta)$$

then

$$|x-y|=|\eta|,\;\;d\boldsymbol{x}d\boldsymbol{y}=2^{-2m}d\xi d\eta.$$

Note that if $\boldsymbol{x},\boldsymbol{y}\in\operatorname{supp}f_R$, then $|\boldsymbol{x}|,|\boldsymbol{y}|\leq Rr_0/2$ and thus

$$\boldsymbol{x},\boldsymbol{y}\in\operatorname{supp}f_R\;\Rightarrow\;|\xi|,\;|\eta|<\frac{1}{2}\big(|\xi+\eta|+|\xi-\eta|\big)=|\boldsymbol{x}|+|\boldsymbol{y}|\leq Rr_0. \tag{3.16}$$

The natural projection $\pi:\widehat{\mathfrak{X}}\to\mathbb{R}^m\times\mathbb{R}^m$ can given the explicit description

$$\mathbb{R}^m\times S^{m-1}\times[0,\infty)\ni(\xi,\boldsymbol{v},r)\mapsto(\xi,\eta)=(\xi,r\boldsymbol{v})\in\mathbb{R}^m\times\mathbb{R}^m.$$

For $R\in(R_0,\infty]$ we set

$$w_R(\boldsymbol{x},\boldsymbol{y})=|\boldsymbol{x}-\boldsymbol{y}|^{m-2}\widehat{\rho}_R(\boldsymbol{x},\boldsymbol{y}).$$

Lemma 3.1(i) implies that

$$\lim_{R\to\infty}\|K_{\mathfrak{a}}^R-K_{\mathfrak{a}}\|_{C^6(RB)}=0.$$

Since $w_R(\boldsymbol{x},\boldsymbol{y})$ depends only on $\boldsymbol{x}-\boldsymbol{y}$ we deduce from Proposition B.1 that

$$\limsup_{R\to\infty}\sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in RB\\ 0<|\boldsymbol{x}-\boldsymbol{y}|\leq C_5}}\Big|w_R(\boldsymbol{x},\boldsymbol{y})\Big|<\infty. \tag{3.17}$$

Using (3.15) and (3.17) we deduce that for any $p>0$ there exists a constant $K_p>0$, independent of R, such that

$$|x-y|^{m-1}\Big|\Delta_R(\boldsymbol{x},\boldsymbol{y})\Big|\leq K_p\big(1+|x-y|\big)^{-p+m-1},\;\;\forall\boldsymbol{x},\boldsymbol{y}\in RB \tag{3.18}$$

Set

$$\delta_R(\xi,\eta):=\Delta_R\big(\pi(\xi,\eta)\big).$$

Since $\Delta_R(\boldsymbol{x},\boldsymbol{y})$ depends only on $\boldsymbol{y}-\boldsymbol{x}$ we deduce that $\delta_R(\xi,\eta)$ is independent of ξ. We have

$$I(R)=\int_{\mathfrak{X}}\Delta_R(\boldsymbol{x},\boldsymbol{y})f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}=\int_{\substack{|\boldsymbol{x}|,|\boldsymbol{y}|\leq Rr_0/2\\ \boldsymbol{x}\neq\boldsymbol{y}}}\Delta_R(\boldsymbol{x},\boldsymbol{y})f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

$$\stackrel{(3.16)}{=}\frac{1}{2^{2m}}\int_{\substack{|\xi|<Rr_0,\\ |\boldsymbol{v}|=1,\,r\in(0,Rr_0)}}r^{m-1}\delta_R\big(\xi,r\boldsymbol{v}\big)f_R\Big(\frac{\xi+r\boldsymbol{v}}{2}\Big)f_R\Big(\frac{\xi-r\boldsymbol{v}}{2}\Big)dr\,vol_{S^{m-1}}\big[d\boldsymbol{v}\big]d\xi$$

$(\xi=2R\zeta,\;\delta_R(\xi,r\boldsymbol{v})=\delta_R(0,r\boldsymbol{v}))$

$$=R^m\int_{|\zeta|\leq}\underbrace{\left(2^{-m}\int_{\substack{|\boldsymbol{v}|=1\\ r\in(0,Rr_0)}}r^{m-1}\delta_R\big(0,r\boldsymbol{v}\big)f\Big(\zeta+\frac{r\boldsymbol{v}}{2R}\Big)f(\zeta-\frac{r\boldsymbol{v}}{2R}\Big)dr\,vol_{S^{m-1}}\big[d\boldsymbol{v}\big]\right)}_{=:J(R)}d\zeta\,.$$

Note that

$$\delta_R\big(0,r\boldsymbol{v}\big)=\widehat{\rho}_R(r\boldsymbol{v}/2,-r\boldsymbol{v}/2)-\widetilde{\rho}_R(r\boldsymbol{v}/2,-r\boldsymbol{v}/2)$$

and for $r>0$, $|\boldsymbol{v}|=1$ fixed

$$\lim_{R\to\infty}\delta_R\big(0,r\boldsymbol{v}\big)=\delta_\infty(0,r\boldsymbol{v})=\widehat{\rho}_\infty(r\boldsymbol{v}/2,-r\boldsymbol{v}/2)-\widetilde{\rho}_\infty(r\boldsymbol{v}/2,-r\boldsymbol{v}/2).$$

We deduce from (3.16) and (3.17) that, for any $p>0$, there exists $K_p>0$ such that, for any $R>R_0$, $|\zeta|<r_0/2$, $|\boldsymbol{v}|=1$ and $r\leq Rr_0$, we have

$$\Big|r^{m-1}\delta_R\big(0,r\boldsymbol{v}\big)f\Big(\zeta+\frac{r\boldsymbol{v}}{2R}\Big)f\Big(\zeta-\frac{r\boldsymbol{v}}{2R}\Big)\Big|\leq K_p\|f\|^2\big(1+r\big)^{-p+m-1}.$$

For $p > m$ we have

$$\int_{|\zeta|\leq r_0/2}\left(\int_{(0,\infty)\times S^{m-1}}(1+r)^{-p+m-1}drvol_{S^{m-1}}\big[dv\big]\right)d\zeta<\infty.$$

The dominated convergence theorem implies that $J(R)$ has a finite limit as $R\to\infty$. More precisely

$$\lim_{R\to\infty}J(R)=\int_{|\zeta|\leq r_0/2}\underbrace{2^{-m}\left(\int_{\substack{|v|=1\\ r>0}}r^{m-1}\delta_\infty(0,rv)drvol_{S^{m-1}}\big[dv\big]\right)}_{=:Z_m(\mathfrak{a})}f(\zeta)^2d\zeta.$$

This concludes the proof of Theorem 1.1.

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Appendix A. Some abstract ampleness criteria

We begin we an abstract result that will be our main tool for detecting ample Gaussian fields.

Proposition A.1. *Let $\boldsymbol{X}$ be a separable Banach space with norm $\|-\|$. Let $(x_n)_{n\geq 0}$ be a sequence in $\boldsymbol{X}$ and $(c_n)_{n\geq 0}$ a sequence of positive real numbers such that*

$$\sum_{n\geq 1}c_n\|x_n\|<\infty.$$

Denote by $\boldsymbol{Y}$ the closure of the span of $(x_n)_{n\geq 1}$. Let $(A_n)_{n\geq 1}$ be a sequence of independent standard normal random variables defined on the probability space $(\Omega,\mathcal{S},\mathbb{P})$. Then the following hold.

i. *There exists a negligible subset $\mathcal{N}\in\mathcal{S}$ such that the series*

$$\sum_{n\geq 1}A_n(\omega)c_nx_n$$

converges in $\boldsymbol{X}$ to an element in $\boldsymbol{Y}$ for any $\omega\in\Omega\setminus\mathcal{N}$.

ii. *The map $S:\Omega\to\boldsymbol{Y}$ defined by*

$$S(\omega)=\begin{cases}\sum_{n\geq 1}A_n(\omega)c_nx_n, & \omega\in\Omega\setminus\mathcal{N},\\ 0, & \omega\in\mathcal{N}\end{cases}$$

is Borel measurable and the push-forward $\Gamma_S:=S_\#\mathbb{P}$ is a nondegenerate Gaussian measure on $\boldsymbol{Y}$.

iii. *For any nonempty open subset $\mathcal{O}\subset\boldsymbol{Y}$, $\mathbb{P}\big[S\in\mathcal{O}\big]>0$.*

Proof.

(i) We will show that the random scalar series

$$\sum_n|A_n|c_n\|x_n\|$$

is a.s. convergent. According to Kolmogorov's two-series theorem this happens if the positive random variables $X_n=|A_n|\cdot c_n\|x_n\|$ satisfy

$$\sum_{n\geq 1}\mathbb{E}\big[X_n\big]<\infty\ \text{ and }\ \sum_{n\geq 1}\mathbb{E}\big[X_n^2\big]<\infty.$$

Now observe that

$$\mathbb{E}\big[|A_n|\big]=2\frac{1}{\sqrt{2\pi}}\int_0^\infty xe^{-x^2/2}dx=\sqrt{\frac{2}{\pi}},$$

$$\sum_{n\geq 1}\mathbb{E}\big[X_n\big]=\sqrt{\frac{2}{\pi}}\sum_{n\geq 1}c_n\|x_n\|<\infty$$

and

$$\sum_{n\geq 1}\mathbb{E}\big[X_n^2\big]=\sum_{n\geq 1}c_n^2\|x_n\|^2<\infty.$$

(ii) Define $S_n : \Omega \to \boldsymbol{Y}$

$$S_n(\omega) = \begin{cases} \sum_{k=1}^n A_k(\omega) c_k x_k, & \omega \in \Omega \setminus \mathcal{N}, \\ 0, & \omega \in \mathcal{N}. \end{cases}$$

The maps S_n are measurable since the addition operation on a separable Banach space is a measurable map. The map S is measurable since for any $\xi \in \boldsymbol{Y}^*$ the function $\langle \xi, S \rangle$ is measurable as limit of the measurable functions $\langle \xi, S_n \rangle$.

To see that Γ_S is a Gaussian measure let $\xi \in \boldsymbol{Y}^*$. Then

$$\langle \xi, S(\omega) \rangle = \lim_{n \to \infty} \langle \xi, S_n(\omega) \rangle.$$

The random variables

$$\langle \xi, S_n \rangle = \sum_{k=1}^n A_n c_n \langle \xi, x_n \rangle$$

are Gaussian as sum of independent Gaussians. Since the limit of Gaussian random variables is also Gaussian we deduce that $\langle \xi, S \rangle$ is Gaussian with variance

$$v[\xi] = \sum_{n \geq 1} c_n^2 \left| \langle \xi, x_n \rangle \right|^2.$$

Since (x_n) spans a dense subspace of $\boldsymbol{Y}$, we deduce that for any $\xi \in \boldsymbol{Y}^* \setminus 0$ there exists n such that $\langle \xi, x_n \rangle \neq 0$. This proves that Γ_S is nondegenerate. Part (iii) now follows from the Support Theorem [28, Thm. 3.6.1] or [23, Cor. 4.2.1].

□

Proposition A.2. *Suppose that $\boldsymbol{U}$ is a Banach space with norm $\| - \|$, $\boldsymbol{T}$ is a compact metric space, $N \in \mathbb{N}$ and*

$$G : \boldsymbol{U}^N \times \boldsymbol{T} \to [0, \infty), \;\; (u_1, \dots, u_N, t) \mapsto G(u_1, \dots, u_N, t) \in [0, \infty)$$

is a continuous function. We define

$$G_* : \boldsymbol{U}^N \to [0, \infty), \;\; G_*(u_1, \dots, u_N) := \min_{t \in \boldsymbol{T}} G(u_1, \dots, u_N, t).$$

Suppose that there exist $v_1, \dots, v_N \in \boldsymbol{U}$ such that $G_*(v_1, \dots, v_N) = r_0 > 0$. Then, for any $r \in (0, r_0)$, there exists $\varepsilon = \varepsilon(r) > 0$ such that

$$\forall u_1, \dots, u_N \in \boldsymbol{U}, \;\; \forall i = 1, \dots, N, \;\; \| u_i - v_i \| < \varepsilon \Rightarrow G_*(u_1, \dots, u_N) > r.$$

In particular, if $U_1 \subset U_2 \subset \cdots$ is an increasing sequence of finite dimensional subspaces of $\boldsymbol{U}$ whose union is a dense subspace of $\boldsymbol{U}$, then there exists $\nu \in \mathbb{N}$ and

$$u_{1,\nu}, \dots, u_{N,\nu} \in U_\nu$$

such that $G_*\big(u_{1,\nu}, \dots, u_{N,\nu}\big) > 0$.

Proof. We argue by contradiction. Suppose that there exist $r_1 \in (0, r_0)$ and sequences in $\boldsymbol{U}$

$$\big(u_{i,\nu}\big)_{\nu \in \mathbb{N}}, \;\; i = 1, \dots, N,$$

such that

$$\lim_{\nu \to \infty} \| u_{i,\nu} - v_i \| = 0, \;\; \forall i = 1, \dots, N,$$

and

$$G_*(u_{1,\nu}, \dots, u_{N,\nu}) \leq r_1, \;\; \forall \nu.$$

Next choose $t_\nu \in \boldsymbol{T}$ such that

$$G\big(u_{1,\nu}, \dots, u_{N,\nu}, t_\nu\big) = G_*\big(u_{1,\nu}, \dots, u_{N,\nu}\big).$$

Upon extracting a subsequence we can assume that t_ν converges in $\boldsymbol{T}$ to some point t_∞. Then

$$r_1 \geq \liminf_{\nu \to \infty} G_*\big(u_{1,\nu}, \dots, u_{N,\nu}\big) = \liminf_{\nu \to \infty} G\big(u_{1,\nu}, \dots, u_{N,\nu}, t_\nu\big)$$

$$= G\big(v_1, \dots, v_N, t_\infty\big) \geq r_0 > r_1.$$

□

With $\boldsymbol{T}$ a compact metric space as above, let $E \to \boldsymbol{T}$ be a rank r topological real vector bundle over $\boldsymbol{T}$ equipped with a continuous metric h. We will refer to the pair (E, h) as a metric vector bundle. For $t \in \boldsymbol{T}$ we denote by $| - |_t$ the norm on the fiber E_t induced by h. The space $C^0(E)$ of continuous sections E is a Banach space with respect to the norm

$$\|u\| := \sup_{t \in \boldsymbol{T}} \left| u(t) \right|_t, \;\; u \in C(E).$$

Definition A.1. An *ample Banach space of sections* of E is a Banach space $\boldsymbol{U} \subset C^0(E)$ continuously embedded in $C^0(E)$ such that

$$\forall t \in \boldsymbol{T},\ \ span\{\, u(t),\ \ u \in \boldsymbol{U}\,\} = E_t.$$

Let $k \in \mathbb{N}$. We say that the Banach space $\boldsymbol{U}$ is k-*ample* if for any *distinct* points $t_1, \dots, t_k \in \boldsymbol{T}$ the map

$$\boldsymbol{U} \ni u \mapsto u(t_1) \oplus \cdots \oplus u(t_k) \in E_{t_1} \oplus \cdots \oplus E_{t_k}$$

is onto. □

Example A.1. The space $C^0(E)$ is a k-ample Banach space of continuous sections of $E \to \boldsymbol{T}$ for any $k \in \mathbb{N}$. If $\boldsymbol{T}$ is a compact smooth manifold and $E \to \boldsymbol{T}$ is a smooth vector bundle, then each of the spaces $C^\ell(E)$, $\ell \in \mathbb{N}$, is a k-ample Banach space of sections of E for any $k \in \mathbb{N}$. □

Corollary A.1. *Let $E \to \boldsymbol{T}$ be a real metric vector bundle over the compact metric space $\boldsymbol{T}$. Suppose that $\boldsymbol{U} \subset C^0(E)$ is an ample Banach space of sections of E and $U_1 \subset U_2 \cdots$ is an increasing sequence of finite dimensional subspaces of $\boldsymbol{U}$ such that*

$$U_\infty = \bigcup_{\nu \in \mathbb{N}} U_\nu$$

is dense in $\boldsymbol{U}$. Then there exists $\nu \in \mathbb{N}$, for any $t \in \boldsymbol{T}$, the evaluation map

$$\mathbf{Ev}_t : U_\nu \to E_t \ \text{ is onto.}$$

Proof. Using the compactness of $\boldsymbol{T}$ and the openess of the surjectivity condition we can find $v_1, \dots, v_N \in \boldsymbol{U}$ such that

$$\forall t \in \boldsymbol{T},\ \ span\{\, v_1(t), \dots, v_N(t)\,\} = E_t.$$

For every $u_1, \dots, u_N \in U$ and $t \in \boldsymbol{T}$ define

$$S_{u_1,\dots,u_N,t} : \mathbb{R}^N \to E_t,\ \ S_{u_1,\dots,u_N,t}(\boldsymbol{x}) = \sum_{k=1}^N x_k u_k(t)$$

and

$$G(u_1, \dots, u_N, t) = \det\left(S_{u_1,\dots,u_N,t} S^*_{u_1,\dots,u_N,t}\right) \geq 0.$$

Note that

$$span\{\, u_1(t), \dots, u_N(t)\,\} = E_t \Longleftrightarrow G(u_1, \dots, u_N, t) > 0.$$

The resulting map $G : \boldsymbol{U}^N \times \boldsymbol{T} \to [0, \infty)$ is continuous and

$$G(v_1, \dots, v_N, t) > 0,\ \ \forall t \in \boldsymbol{T}$$

Hence

$$G_*(v_1, \dots, v_N) = \inf_{t \in \boldsymbol{T}} G(v_1, \dots, v_N, t) > 0.$$

Using Proposition A.2, we deduce that there exists $\nu \in \mathbb{N}$ and $u_{1,\nu}, \dots, u_{N,\nu} \in U_\nu$ such that

$$G_*(u_{1,\nu}, \dots, u_{N,\nu}) > 0.$$

Hence

$$\mathbf{Ev}_t : span\{\, u_{1,\nu}, \dots, u_{N,\nu}\,\} \subset \boldsymbol{U} \to E_t \text{ is onto},\ \ \forall t \in \boldsymbol{T}.$$

A fortiori, this implies that $\mathbf{Ev}_t : U_\nu \to E_t$ is onto, $\forall t \in \boldsymbol{T}$. □

Corollary A.2. *Let $E \to \boldsymbol{T}$ be a real metric vector bundle over the compact metric space $\boldsymbol{T}$. Suppose that $\boldsymbol{U} \subset C^0(E)$ is a 2-ample Banach space of sections and $U_1 \subset U_2 \subset \cdots$ is an increasing sequence of finite dimensional subspaces of $\boldsymbol{U}$ such that*

$$U_\infty = \bigcup_{\nu \in \mathbb{N}} U_\nu$$

is dense in $\boldsymbol{U}$. Then, for any open neighborhood $\mathcal{O}$ of the diagonal $\Delta \subset \mathbb{T} \times \mathbb{T}$, there exists $\nu \in \mathbb{N}$ such that for any $(t_1, t_2) \in \boldsymbol{T}^2 \setminus \mathcal{O}$, the evaluation map

$$\mathbf{Ev}_{t_1,t_2} : U_\nu \to E_{t_1} \oplus E_{t_2} \ \text{ is onto.}$$

Proof. For $\underline{t} \in \boldsymbol{T}^2$ and $u \in \boldsymbol{U}$ we set

$$u(\underline{t}) := u(t_1) \oplus u(t_2),\ \ E_{\underline{t}} = E_{t_1} \oplus E_{t_2},\ \ \mathbf{Ev}_{\underline{t}}(u) = u(\underline{t}).$$

Using the compactness of $\boldsymbol{T}^2 \setminus \mathcal{O}$ and the openness of the surjectivity condition we can find $v_1, \dots, v_N \in \boldsymbol{U}$ such that

$$\forall \underline{t} \in \boldsymbol{T}^2 \setminus \mathcal{O},\ \ span\{\, v_1(\underline{t}), \dots, v_N(\underline{t})\,\} = E_{\underline{t}}.$$

For every $u_1, \dots, u_N \in \boldsymbol{U}$ and $\underline{t} \in \boldsymbol{T}^2$ define

$$S_{u_1,\dots,u_N,\underline{t}} : \mathbb{R}^N \to E_{\underline{t}}, \quad S_{u_1,\dots,u_N,\underline{t}}(\boldsymbol{x}) = \sum_{k=1}^{N} x_k u_k(\underline{t})$$

and

$$G(u_1, \dots, u_N, \underline{t}) = \det\left(S_{u_1,\dots,u_N,\underline{t}} S^*_{u_1,\dots,u_N,\underline{t}}\right) \geq 0.$$

Note that

$$span\left\{ u_1(\underline{t}), \dots, u_N(\underline{t}) \right\} = E_{\underline{t}} \Longleftrightarrow G(u_1, \dots, u_N, \underline{t}) > 0.$$

Thus

$$G(u_1, \dots, u_N, \underline{t}) > 0 \iff \mathbf{Ev}_t : span\left\{ u_1, \dots, u_N \right\} \subset \boldsymbol{U} \to E_{\underline{t}} \text{ is onto.}$$

The resulting map $G : \boldsymbol{U}^N \times \left(\boldsymbol{T}^2 \setminus \mathcal{O}\right) \to [0, \infty)$ is continuous and since $\boldsymbol{T}^2 \setminus \mathcal{O}$ is compact we have

$$G_*(v_1, \dots, v_N) = \inf_{\underline{t} \in \boldsymbol{T}^2 \setminus \mathcal{O}} G(v_1, \dots, v_N, \underline{t}) > 0.$$

Using Proposition A.2, we deduce that there exists $\nu \in \mathbb{N}$ and $u_{1,\nu}, \dots, u_{N,\nu} \in U_\nu$ such that

$$G_*(u_{1,\nu}, \dots, u_{N,\nu}) > 0.$$

Hence

$$\mathbf{Ev}_{\underline{t}} : span\left\{ u_1, \dots, u_N \right\} \subset \boldsymbol{U} \to E_{\underline{t}} \text{ is onto,} \quad \forall \underline{t} \in \boldsymbol{T}^2 \setminus \mathcal{O},$$

and thus $\mathbf{Ev}_{\underline{t}} : U_\nu \to E_{\underline{t}}$ is onto, $\forall \underline{t} \in \boldsymbol{T}^2 \setminus \mathcal{O}$. □

Proposition A.3. *Suppose that $E \to \boldsymbol{T}$ is a topological metric vector bundle over the compact metric space $\boldsymbol{T}$. Let $\boldsymbol{U} \subset C^0(E)$ be an ample Banach space of sections of E embedded continuously in $C^0(T)$.*

Suppose that $(u_n)_{n \in \mathbb{N}}$ is a sequence of sections in $\boldsymbol{X}$ such that $span\left\{u_n, \ n \in \mathbb{N}\right\}$ is dense in $C^0(\boldsymbol{E})$ and there exists $\alpha > 0$ such that

$$\|u_n\|_{\boldsymbol{X}} = O(n^\alpha) \text{ as } n \to \infty. \tag{A.1}$$

Fix a sequence of positive real numbers $(\lambda_n)_{n \geq 0}$ such that

$$\liminf_{n \to \infty} \frac{\lambda_n}{n^\beta} > 0. \tag{A.2}$$

for some $\beta > 0$. Let $\mathfrak{a} \in \mathcal{S}(\mathbb{R})$ be an even Schwartz function such that $\mathfrak{a}(0) = 1$. Fix a sequence of i.i.d. *standard normal random variables $(X_n)_{n \geq 0}$. Then the following hold.*

i. *For any $R > 0$ the random series*

$$\sum_{n \in \mathbb{N}} \mathfrak{a}\left(\lambda_n / R\right) X_n u_n \tag{A.3}$$

converges a.s. *in $\boldsymbol{U} \subset C^\infty(E)$. Denote by Φ^R the resulting continuous Gaussian section of E.*

ii. *There exists R_0 such that $\forall R > R_0$ the Gaussian section Φ^R is ample.*

Proof.

(i) Since $\mathfrak{a}$ is a Schwartz function we deduce from (A.1) and (A.2) that

$$\sum_{n \to \infty} \left|\mathfrak{a}\left(\lambda_n / R\right)\right| \|u_n\|_U < \infty, \quad \forall \hbar > 0$$

The convergence of the random series (A.3) in $\boldsymbol{U}$ follows from Proposition A.1.

(ii) For $\hbar > 0$ we set

$$\mathcal{N}_R := \left\{ n \in \mathbb{N}; \ \mathfrak{a}(\lambda_n / R) \neq 0 \right\}$$

and denote by $\boldsymbol{U}^R$ the closure in $\boldsymbol{U}$ of

$$span\left\{ u_n; \ n \in \mathcal{N}_R \right\}.$$

According to Proposition A.1 the above random series defines a *nondegenerate* Gaussian $\boldsymbol{\Gamma}^R$ measure on the Banach space $\boldsymbol{U}^R$.

Set $U_\nu := span\left\{ u_1, \dots, u_\nu \right\}$. Since $\mathfrak{a}(0) = 1$, we deduce that

$$\exists r_0 > 0, \quad \forall |t| \leq r_0, \quad \left|\mathfrak{a}(t)\right| \geq 1/2.$$

Hence, for any $\nu \in \mathbb{N}$ there exists $R = \hbar(\nu) > 0$ so that

$$\forall \hbar \leq \hbar(\nu), \quad \max_{1 \leq k \leq \nu} \lambda_k / R < r_0,$$

i.e., $U_\nu \subset \boldsymbol{U}^R$, $\forall R \geq R(\nu)$. Corollary A.1 implies that there exists $\nu_0 \in \mathbb{N}$ such that

$$\forall t \in \boldsymbol{T},\ \ \mathbf{Ev}_t : U_{\nu_0} \to E_t \ \text{ is onto.}$$

Set $R_0 = R(\nu_0)$ such that $U_{\nu_0} \subset \boldsymbol{U}^R$, $\forall R \geq R_0$.

We will show that for any $t \in \boldsymbol{T}$ and any $R \geq R_0$, the Gaussian vector $\Phi^R(t)$ is nondegenerate, i.e., for any open set $\mathcal{O} \subset E_t$, $\mathbb{P}\big[\Phi^R(t) \in \mathcal{O}\big] > 0$. Equivalently, this means

$$\boldsymbol{\Gamma}^R\big[\mathbf{Ev}_t^{-1}(\mathcal{O})\big] > 0.$$

Since $\boldsymbol{\Gamma}^R$ is a nondegenerate Gaussian measure on $\boldsymbol{U}^\hbar$, it suffice to show that the open subset $\mathbf{Ev}_t^{-1}(\mathcal{O}) \subset \boldsymbol{U}^R$ is nonempty. This is indeed the case since $\mathbf{Ev}_t^{-1}(\mathcal{O}) \cap U_{\nu_0} \neq \emptyset$.

□

Suppose that M is a smooth compact manifold. Denote by $C^k(E)$ the vector space of sections of E that are k-times continuously differentiable. We need to define on $C^k(E)$ a structure of separable Banach space and to do so we need to make some choices.

- Fix a smooth Riemannian metric g on M.
- Fix a smooth h metric on E. We denote by $(-,-)_{E_x}$ the induced inner product on E_x.
- Fix a connection (covariant derivative) ∇^h on E that is compatible with the metric h.

We will refer to such choices as *standard choices*. There are several geometric objects canonically induced by these choices; see [1, Sec. 3.3].

First, the metric g determines a Borel measure vol_g on M, classically referred to as the *volume element* or the *volume density*. Next, the metric determines the *Levi-Civita connection* ∇^g on TM. The metric g also determines metrics on all the tensor bundles $TM^{\otimes p} \otimes (T^*M)^{\otimes q}$ and the connection ∇^g determines connections on these bundles compatible with the metrics induced by g. To ease the notational burden we will denote by ∇^g each of these connections.

Similarly, the metric h induces metrics in all the bundles $E^{\otimes p} \otimes (E^*)^{\otimes q}$ and the connection ∇^h determines connections on these bundles compatible with the induced metrics. We will denote by $\big|-\big|_x$ the Euclidean norms in any of the spaces $(T_x^*M)^{\otimes q} \otimes E^{\otimes p}$. We define the *jet bundle*

$$J_k(E) := \bigoplus_{j-0}^{k} T^*M^{\otimes j} \otimes E. \tag{A.4}$$

The connections ∇^g and ∇^h induce a connection $\nabla = \nabla^{g,h}$ on the bundle $(T^*M)^{\otimes k} \otimes E$

$$\nabla : C^1\big((T^*M)^{\otimes k} \otimes E\big) \to C^0\big((T^*M)^{\otimes k+1} \otimes E\big).$$

We denote by ∇^q the composition

$$C^q(E) \stackrel{\nabla}{\to} C^{m-1}\big(T^*M \otimes E\big) \stackrel{\nabla}{\to} \cdots \stackrel{\nabla}{\to} C^1\big((T^*M)^{\otimes q-1} \otimes E\big) \stackrel{\nabla}{\to} C^0\big((T^*M)^{\otimes q} \otimes E\big).$$

For every section $\psi \in C^k(E)$ we set

$$J_k(\psi) = J_k(\psi, \nabla) = \bigoplus_{k=0}^{k} \nabla^k \psi,\ \ \|u\|_{C^k} = \sum_{j=0}^{q} \|\nabla^j \psi\|,$$

where

$$\|\nabla^j u\| = \sup_{x \in M} \Big|\nabla^j u(x)\Big|_x.$$

Note that $J_k(\psi)$ is a continuous section of $J_k(E)$.The resulting normed spaces is a separable Banach space. The norm $\|-\|_{C^k}$ depends on the standard choices, but different standard choices yield equivalent norms. Fix one such norm and denote by $C^k(E)$ the resulting separable Banach space.

Corollary A.3. *Suppose that $E \to M$ is a smooth real vector bundle over the compact smooth manifold M. Fix a smooth Riemann metric g on M, a smooth metric h on E and a smooth connection on E compatible with h. Let $k \in \mathbb{N}$ and suppose that $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of C^k sections of E that span a dense subset of $C^k(E)$. Suppose that*

$$\|\phi_n\|_{C^k(E)} = O(n^\alpha) \text{as } n \to \infty, \tag{A.5}$$

for some $\alpha > 0$. Fix a sequence of positive numbers $(\lambda_n)_{n\in\mathbb{N}}$ satisfying (A.2). Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. *standard normal random variables and suppose that $\mathfrak{a} \in \mathcal{S}(\mathbb{R})$ is an even Schwartz function such that $\mathfrak{a}(0) = 1$. Then the following hold.*

i. For any $R > 0$ the random series

$$\sum_{n\in\mathbb{N}} \mathfrak{a}\big(\lambda_n/R\big) X_n \phi_n \tag{A.6}$$

converges a.s. *in $C^k(E)$. Denote by Φ^R the resulting C^k Gaussian section of E.*

ii. *There exists $R_0 >$ such that $\forall R > R_0$ the Gaussian section Φ^R is J_k-ample, i.e., for any $\boldsymbol{x} \in M$ the Gaussian vector*

$$J_k\Phi^R(\boldsymbol{x}) = \bigoplus_{j=0}^{k} \nabla^j \Phi^R(\boldsymbol{x})$$

is nondegenerate.

Proof.

(i) This follows from Proposition A.3.
(ii) Consider the jet bundle $J_k(E) \to M$; see (A.4) We have a continuous linear map

$$C^k(E) \to C^0\big(J^k(E)\big), \;\; \phi \mapsto J_k(\phi).$$

Denote by $\boldsymbol{U}$ the image of this map. It is a closed[2] subspace of $C^0\big(J^k(E)\big)$. Then the random series

$$\sum_{n\in\mathbb{N}} \mathfrak{a}\big(\lambda_n/R\big) X_n J_k(\phi_n)$$

converges a.s. uniformly to $J_k(\Phi^R)$. Now observe that $\boldsymbol{U}$ is an ample Banach space of sections of $J^k(E)$. Indeed, using smooth partitions of unity we can find $\psi_1, \dots, \psi_N \in C^k(E)$ such that, for any $x \in M$,

$$span\big\{ J_k(\psi_1(x), \dots, J_k(\psi_N)(x) \big\} = J_k(E)_x.$$

Proposition A.3 now implies that $J_k(\Phi^R)$ is an ample Gaussian section of $J_k(E)$.

□

Remark A.1. In applications (ϕ_n) are eigenfunctions of the Laplacian Δ on a Riemann manifold (M, g) and $\Delta\phi_n = \lambda_n^2\phi_n$. The covariance kernel of Φ^R is then the Schwartz kernel of the smoothing operator $\mathfrak{a}\big(\hbar\sqrt{\Delta}\big)^2$, $\hbar = R^{-1}$. If $\mathfrak{a}(x) = e^{-x^2/2}$ $\hbar = t^{1/2}$, then $\mathfrak{a}\big(\hbar\sqrt{\Delta}\big)^2 = e^{-t\Delta}$, the heat operator on M. □

Corollary A.4. *Fix an even Schwartz function $\mathfrak{a} \in \mathcal{S}(\mathbb{R})$ and consider the random Fourier series $F_{\mathfrak{a}}^R$ defined in (1.1). We regard it as a random smooth function on the torus $\mathbb{T}^m$. Then for any $k \in \mathbb{R}$ there exists $R = R_k > 0$ such that, for any $R > R_k$ the function $F_{\mathfrak{a}}^R$ is J_k-ample. In particular it is* a.s. *Morse for $R > R_1$.* □

Lemma A.1. *Suppose that $E \to M$ is a smooth real vector bundle over the compact smooth manifold M. Fix a smooth Riemann metric g on M, a smooth metric h on E and a smooth connection on E compatible with h. Let $k \in \mathbb{N}$ and suppose that $(\phi_n)_{n\in\mathbb{N}}$ is a sequence of C^k sections of E that span a dense subset of $C^k(E)$. Set*

$$U_\nu := span\{\phi_1, \dots, \phi_\nu\}$$

Then there exists $\nu_0 > 0$ such that $\forall \nu \geq \nu_0$ the following hold.

i. *For any $t \in M$ and any $\nu \geq \nu_0$ the map*

$$U_\nu \ni u \mapsto J_1(u)_t \in J_1(E_t)$$

is onto. Above, $J_1(u)_t$ is the 1-jet of u at t, $J_1(u)_t = u(t) \oplus \nabla u(t) \in E_t \oplus T_t^*M \otimes E_t$.
ii. *For any $\underline{t} \in M^2 \setminus \Delta$ the map $U_\nu \ni u \mapsto u(\underline{t}) \in E_{\underline{t}}$ is onto.*

Proof. The space $C^k(E)$ is J_1-ample and arguing as in the proof of Corollary A.1 we deduce that there exists $\nu_1 \in \mathbb{N}$ such that for any $\nu \geq \nu_1$ and $t \in M$ the map

$$U_\nu \ni u \mapsto J_1(u)_t \in J_1(E)_t$$

is onto.

The argument at the beginning of [29, Sec. 3.3] based on Kergin interpolation shows that there exists an open neighborhood $\mathcal{O}$ of the diagonal $\Delta \in M^2$ such that $\forall \nu \geq \nu_1$ and any $\underline{t} \in \mathcal{O} \setminus \Delta$ the map

$$U_\nu \ni u \mapsto u(\underline{t}) \in E_{\underline{t}}$$

is onto.

Corollary A.2 implies that there exists $\nu_0 > 0$ such that $\forall \nu \geq \nu_2$ and any $\underline{t} \in M^2 \setminus \mathcal{O}$ the map

$$U_\nu \ni u \mapsto u(\underline{t}) \in E_{\underline{t}}$$

is onto. Then $\nu_0 = \max(\nu_1, \nu_2)$ has all the claimed properties. □

Corollary A.5. *Fix an amplitude $\mathfrak{a} \in \mathcal{S}(\mathbb{R})$ and consider the random Fourier series $F_{\mathfrak{a}}^R$ defined in (1.1). We regard it as a random smooth function on the torus $\mathbb{T}^m$. Then there exists $R = R_{2,2} > 0$ such that, for any $R > R_{2,2}$ the function $F_{\mathfrak{a}}^R$ is J_2-ample and $\nabla F_{\mathfrak{a}}^R$ is 2-ample.* □

[2] Here we are using the classical fact that if a sequence of C^1-function (u_n) has the property that both (u_n) and their differentials (du_n) converge uniformly to u and respectively v, then u is C^1 and $du = v$.

Appendix B. Variance estimates

It has been known for some time that under certain conditions the number of zeros in a box of a Gaussian field F has finite variance, [17–19,29,30]. In this appendix we use the ideas in the above references to obtain estimates for the variance in terms of the covariance kernel.

Suppose that $\boldsymbol{U}$ and $\boldsymbol{V}$ are finite dimensional real Euclidean spaces of the same dimension m and $\mathscr{V} \subset \boldsymbol{V}$ is an open set. If $f : \mathscr{V} \to \boldsymbol{U}$ is a C^k-map, we denote by $f^{(k)}(v)$ its k-th differential at $v \in \mathscr{V}$. We view $f^{(k)}(v)$ as an element of $\mathbf{Sym}^k(\boldsymbol{V}, \boldsymbol{U})$ the space of symmetric k-linear maps $\boldsymbol{V}^k \to \boldsymbol{U}$.

Let $F : \mathscr{V} \to \boldsymbol{U}$ be a centered Gaussian random field whose covariance kernel $\mathcal{K}_F$ is C^6. In particular, this implies that F is a.s. C^2.

For any box $B \subset \mathscr{V}$ we denote by Z_B the number of zeros of F in B, i.e., $Z_B = Z[B, F]$. Note that the correspondence $F \mapsto Z[B, F]$ is homogeneous of degree 0. Any upper estimate of momentums of $Z[B, F]$ will have to be homogeneous of degree 0 in F.

Let $\mathscr{V}_*^2 := \mathscr{V}^2 \setminus \Delta$, where Δ is the diagonal

$$\Delta := \big\{ (v_0, v_1) \in \mathscr{V}^2;\ \ v_0 = v_1 \big\}.$$

Define B_*^2 in a similar fashion. Consider the random field

$$\hat{F} =: \mathscr{V}_*^2 \to \boldsymbol{U} \oplus \boldsymbol{U},\ \ \hat{F}(v_0, v_1) = F(v_0) \oplus F(v_1).$$

Note that

$$Z[\hat{F}, B_*^2] = Z_B\big(Z_B - 1\big).$$

Suppose that $F|_B$ is 2-ample, i.e., for any $\underline{v} = (v_0, v_1) \in B_*^2$ the Gaussian vector $F(v_0) \oplus F(v_1)$ is nondegenerate. We deduce from the local Kac-Rice formula (2.3) that $E\big[Z_B\big] < \infty$, and

$$\mathbb{E}\big[Z_B\big(Z_B - 1\big)\big] = \int_{B_*^2} \rho_G^{(2)}(v_0, v_1)dv_0 dv_1,$$

where $\rho_F^{(2)}$ is the Kac-Rice density

$$\rho_F^{(2)}(v_0, v_1) := \mathbb{E}\big[|\det F'(v_0)\det F'(v_1)|\ \big| F(v_0) = F(v_1) = 0\big] p_{\hat{F}(v_0,v_1)}(0). \tag{B.1}$$

Note that

$$p_{\hat{F}(v_0,v_1)}(0) = \frac{1}{\sqrt{\det\big(2\pi Var[F(v_0) \oplus F(v_1)]\big)}},$$

so $p_{\hat{F}(v_0,v_1)}(0)$ explodes as (v_0, v_1) approaches the diagonal since $F(v) \oplus F(v)$ is degenerate for any $v \in \mathscr{V}$. Thus the function $\rho_F^{(2)}(v_0, v_1)$ might have a non integrable singularity along the diagonal so $E\big[Z_B^2\big]$ could be infinite. We want to show that this is not the case and a bit more.

Proposition B.1. *Fix a box $B \subset \mathscr{V}$ and $\delta < dist(B, \partial\mathscr{V})$. Denote by $S = S(\delta, B)$ the compact set set*

$$S = \big\{ v \in \mathscr{V};\ \ dist(v, B) \le \delta \big\}.$$

Suppose that $F\Big|_B$ is C^2, 2-ample and J_1-ample, i.e., for any $\boldsymbol{v} \in B$ the Gaussian vector $\big(F(v), F'(v)\big)$ is nondegenerate. Define

$$w_F : B_*^2 \to \mathbb{R},\ \ w_F(\boldsymbol{x}, \boldsymbol{y}) = |\boldsymbol{x} - \boldsymbol{y}|^{m-2}\rho_F^{(2)}(\boldsymbol{x}, \boldsymbol{y}).$$

There exists a constant $C_m(\mathcal{K}_F) > 0$, that depends only on m and $\mathcal{K}_F$ such that the correspondence $\mathcal{K}_F \to C(\mathcal{K}_F)$ is homogeneous of degree 0, continuous with respect to the $C^6\big(S \times S\big)$-topology of covariance kernels and

$$\sup_{p \in B_*^2} |w_F(\boldsymbol{p})\Big| \le C_m(\mathcal{K}_F). \tag{B.2}$$

In particular

$$Var\big[Z_B\big] \le C_m(\mathcal{K}_F) \int_{B^2} \|\boldsymbol{u} - \boldsymbol{v}\|^{2-m} d\boldsymbol{u} d\boldsymbol{v} < \infty.$$

For a more explicit description of $C_m(\mathcal{K}_F)$ we refer to (B.8) and (B.5).

Proof. We will use the gauge-change trick in [18, Sec. 4.2]. This is a higher-dimensional version of the divided-difference trick used in the one-dimensional case by J. Cuzik, [30].

For any $\underline{v} = (\boldsymbol{v}_0, \boldsymbol{v}_1) \in B_*^2$, the Gaussian vector $\hat{F}(\boldsymbol{v}_0, \boldsymbol{v}_1) = F(\boldsymbol{v}_0) \oplus F(\boldsymbol{v}_1)$ is nondegenerate. We denote by $p_{F(\boldsymbol{v}_0),F(\boldsymbol{v}_1)}$ its probability density. We set

$$r(\underline{v}) := \|\boldsymbol{v}_1 - \boldsymbol{v}_0\|,\ \ \Xi(\underline{v}) := \frac{1}{r(\underline{v})}\big(F(\boldsymbol{v}_1) - F(\boldsymbol{v}_0)\big).$$

Note that

$$\hat{F}(\underline{v}) = 0 \Longleftrightarrow F(v_0) = \Xi(\underline{v}) = 0.$$

Denote by $A(\underline{v})$ the linear map $\boldsymbol{U}^2 \to \boldsymbol{U}^2$ given by

$$A(\underline{v})\begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_0 + r(\underline{v})\boldsymbol{u}_1 \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{\boldsymbol{U}} & 0 \\ \mathbb{1}_{\boldsymbol{U}} & r(\underline{v})\mathbb{1}_{\boldsymbol{U}} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{u}_0 \\ \boldsymbol{u}_1 \end{bmatrix}. \tag{B.3}$$

Thus

$$\begin{bmatrix} F(\boldsymbol{v}_0) \\ F(\boldsymbol{v}_1) \end{bmatrix} = A(\underline{v})\begin{bmatrix} F(\boldsymbol{v}_0) \\ \Xi(\underline{v}) \end{bmatrix}.$$

The gauge transformation $A(\underline{v})$ desingularizes $\hat{F}$. Denote by $Z(\underline{v})$ the Gaussian vector $(F(\boldsymbol{v}_0), \Xi(\underline{v}))$.

The Gaussian regression formula implies that

$$\begin{aligned} &\mathbb{E}\big[|\det F'(\boldsymbol{v}_0)\det F'(\boldsymbol{v}_1)| \,\big| F(v_0) = F(\boldsymbol{v}_1) = 0\big] \\ &\qquad = \mathbb{E}\big[|\det F'(\boldsymbol{v}_0)\det F'(\boldsymbol{v}_1)| \,\big| Z(\underline{v}) = 0\big]. \end{aligned}$$

Note that

$$\begin{aligned} p_{F(\boldsymbol{v}_0),F(\boldsymbol{v}_1)} &= \frac{1}{\sqrt{\det\big(2\pi Var[F(\boldsymbol{v}_0)\oplus F(\boldsymbol{v}_1)]\big)}} \\ &= \frac{1}{|\det A|\sqrt{\det\big(2\pi Var[F(\boldsymbol{v}_0)\oplus \Xi(\underline{v})]\big)}} = r(\underline{v})^{-m} p_{F(\boldsymbol{v}_0)\oplus\Xi(\underline{v})}(0). \end{aligned}$$

We deduce that for any $\underline{v} \in B_*^2$ we have

$$\rho_F^{(2)}(\underline{v}) := r(\underline{v})^{-m}\mathbb{E}\big[|\det F'(\boldsymbol{v}_0)\det F'(v_1)| \,\big| Z(\underline{v}) = 0\big] p_{F(v_0)\oplus\Xi(\underline{v})}(0). \tag{B.4}$$

Since F is 2-ample and J_1-ample the strictly positive function

$$B_*^2 \ni (v_0, v_1) \mapsto \det Var\big[F(v_0) \oplus \Xi(\underline{v})\big|$$

admits an extension to the radial blow-up of B^2 along the diagonal as a strictly positive continuous function. Indeed, if $r(\underline{v}) \to 0$ and

$$\frac{1}{r(\underline{v})}(\boldsymbol{v}_1 - \boldsymbol{v}_0) \to \boldsymbol{v},$$

then

$$Var\big[F(\boldsymbol{v}_0) \oplus \Xi(\underline{v})\big| \to Var\big[F(\boldsymbol{v}_0) \oplus \partial_{\boldsymbol{v}} F(\boldsymbol{v}_0)\big] > 0.$$

Set

$$\mathfrak{q}\big(\mathcal{K}_F\big) := \inf_{(v_0,\boldsymbol{v}_1)\in B_*^2} \det Var\big[F(\boldsymbol{v}_0) \oplus \Xi(\underline{v})\big|. \tag{B.5}$$

Note that this is homogeneous of degree $2m$ in $\mathcal{K}_F$.

Lemma B.1. *There exists a constant $C{=}C(m) > 0$ depending only on m such that, for $i = 0, 1$, and any $\underline{v} \in B_*^2$*

$$\Big|\mathbb{E}\big[|\det F(\boldsymbol{v}_i)|^2 \big| Z(\underline{v}) = 0\big]\Big| \le C(m)\|\mathcal{K}_F\|^m_{C^6(S\times S)} r(\underline{v})^2.$$

Proof. It suffices to consider only the case $i = 0$. Set

$$\boldsymbol{v} = \boldsymbol{v}(\underline{v}) := \frac{1}{r(\underline{v})}\big(\boldsymbol{v}_1 - \boldsymbol{v}_0\big), \quad Z = Z(\underline{v}).$$

Let $f(t) = F(v_0 + t\boldsymbol{v})$. Since $F(v)$ is a.s. C^2 we deduce from the first order Taylor approximation with integral remainder that

$$F(\boldsymbol{v}_1) - F(\boldsymbol{v}_0) = f(r) - f(0) = rf'(0) + \int_0^r f''(t)(r-t)dt = \partial_{\boldsymbol{v}} F(v_0) + \underbrace{\int_0^r f''(t)(r-t)dt}_{=:W}.$$

Hence

$$r\partial_{\boldsymbol{v}} F(\boldsymbol{v}_0) = F(\boldsymbol{v}_0) - F(v_1) - W$$

For any $p \ge 1$ we have

$$\mathbb{E}\big[\|r\partial_{\boldsymbol{v}} F(\boldsymbol{v}_0)\|^p \,\big| Z = 0\big] = \mathbb{E}\big[\|F(\boldsymbol{v}_0) - F(\boldsymbol{v}_1) - W\|^p \big| Z = 0\big] = \mathbb{E}\big[\|W\|^p \,\big| Z = 0\big].$$

The random variable W is a centered $\boldsymbol{U}$-valued Gaussian vector. We deduce that for any $p \geq 1$ we have

$$\Big|\mathbb{E}\big[\|\partial_v F(v_0)\|^p \big| Z=0\big]\Big| = \frac{1}{r^p}\mathbb{E}\big[\|W\|^p \,\big| Z=0\big]^p.$$

Note that

$$\|W\| \leq \int_0^r \|f''(t)\|_{\boldsymbol{U}}(r-t)dt \leq \frac{r^2}{2}\|F\|_{C^2(B)}.$$

We deduce that

$$\Big\|Var\big[W\big]\Big\|_{\mathrm{op}} \leq C(m)\frac{r^4}{4}\mathbb{E}\big[\|F\|^2_{C^2(B)}\big].$$

Using the regression formula and Corollary 2.2 we deduce that

$$\mathbb{E}\big[\|W\|^p \,\big| Z=0\big] \leq C(m,p) r(\underline{v})^{2p}\mathbb{E}\big[\|F\|^2_{C^2(B)}\big]^{p/2},$$

where $C(m,p)$ is a universal constant that depends only on the dimension m and on p. We will continue to denote by the same symbol $C(m,p)$ various positive constants that depend only on m and p. We deduce

$$\mathbb{E}\big[\|\partial_v F(\boldsymbol{v}_0)\|^p \big| Z=0\big] \leq C(m,p) r(\underline{v})^p \mathbb{E}\big[\|F\|^2_{C^2(B)}\big]^{p/2}. \tag{B.6}$$

Extend v to an orthonormal basis $\{v=\boldsymbol{e}_1, \boldsymbol{e}_2, \dots \boldsymbol{e}_m\}$ of $\boldsymbol{V}$. Using Hadamard's inequality [31, Cor. 7.8.2] we deduce

$$\Big|\det F'(\boldsymbol{v}_0)\Big| = \Big|\det\big(\partial_{e_1}F(\boldsymbol{v}_0), \partial_{e_2}F(\boldsymbol{v}_0), \dots, \partial_{e_m}F(\boldsymbol{v}_0)\big)\Big|$$

$$\leq \Big\|\partial_{e_1}F(\boldsymbol{v}_0)\Big\| \prod_{k=2}^m \Big\|\partial_{e_k}F(\boldsymbol{v}_0)\Big\|.$$

Using Hölder's inequality we deduce

$$\mathbb{E}\Big[\Big|\det F'(\boldsymbol{v}_0)\Big|^2 \Big| Z=0\,\Big] \leq \prod_{k=1}^m \mathbb{E}\Big[\|\partial_{e_k}F(\boldsymbol{v}_0)\Big|^{2m}\| \,\Big| Z=0\Big]\Big]^{\frac{1}{m}}.$$

For $k=2,\dots,m$ we have

$$Var\big[\partial_{e_k}F(\boldsymbol{v}_0)\big| Z=0\big] \leq Var\big[\partial_{e_k}F(\boldsymbol{v}_0)\big]$$

and

$$\Big\|Var\big[\partial_{e_k}F(\boldsymbol{v}_0)\big]\Big\|_{\mathrm{op}} \leq C(m)\|\mathcal{K}_F\|_{C^2(B\times B)}.$$

Using again Corollary 2.2 we deduce that for $k=2,\dots,m$ we have

$$\mathbb{E}\big[\|\partial_{e_k}F(\boldsymbol{v}_0)\|^{2m}\big| Z=0\big]^{\frac{1}{m}} \leq C(m)\|\mathcal{K}_F\|_{C^2(B\times B)}.$$

Using (B.6), we deduce that

$$\mathbb{E}\Big[\Big|\det F'(\boldsymbol{v}_0)\Big|^2 \Big| Z=0\Big] \leq C(m) r(\underline{v})^2 \mathbb{E}\big[\|F\|^2_{C^2(B)}\big]\|\mathcal{K}_F\|^{m-1}_{C^2(B\times B)}.$$

Invoking (2.2) we conclude that

$$\mathbb{E}\big[\|F\|^2_{C^2(B)}\big] \leq C(m,B)\|\mathcal{K}_F\|_{C^6(S\times S)}.$$

This completes the proof of Lemma B.1. □

Lemma B.1 implies

$$\mathbb{E}\big[|\det F'(v_0)\det F'(v_1)| \,\big| Z(\underline{v})=0\big]$$

$$\leq \mathbb{E}\Big[\Big|\det F'(v_0)\Big|^2 \Big| Z(\underline{v})=0\Big]\Big|^{1/2}\mathbb{E}\Big[\Big|\det F'(v_1)\Big|^2\Big| Z(\underline{v})=0\Big]\Big|^{1/2}$$

$$\leq C(m)\|\mathcal{K}_F\|^m_{C^6(S\times S)} r(\underline{v})^{-2}.$$

Hence

$$\rho^{(2)}_F(\underline{v}) \leq C(m)\|\mathcal{K}_F\|^m_{C^6(S\times S)} r(\underline{v})^{2-m}\sup_{\underline{v}} p_{F(v_0)\oplus\Xi(\underline{v})}(0) = C(m)\frac{\|\mathcal{K}_F\|^m_{C^6(S\times S)}}{\sqrt{\mathfrak{q}(\mathcal{K}_F)}} r(\underline{v})^{2-m}. \tag{B.7}$$

This completes the proof of Proposition B.1 with

$$C_m(\mathcal{K}_F) = \frac{\|\mathcal{K}_F\|^m_{C^6(S\times S)}}{\sqrt{\mathfrak{q}(\mathcal{K}_F)}}. \tag{B.8}$$

□

Remark B.1.

(a) One can show that if F is a.s. C^3, then the function w_F in Proposition B.1 admits an extension to a continuous function on the radial blow-up of B^2 along the diagonal.
(b) For any box B in a Euclidean space $\boldsymbol{V}$ we set

$$I(B) := \int_{B_*^2} r(\underline{v})^{2-m} dv_0 dv_1.$$

Note that $\mathfrak{q}(B)$ is a translation invariant and for any $t > 0$, $I(tB) = t^{m+2} I(B)$. In particular, if B is the cube $B_c = [0, c]^m$, then

$$I(B_c) = \mathfrak{q}(B_1) c^{m+2} = C(m) I(B_1) vol\left[B_c\right]^{\frac{m+2}{m}}.$$

References

[1] L.I. Nicolaescu, Lectures on the Geometry of Manifolds, World Scientific, 3rd edition, 2021.
[2] M.E. Taylor, Pseuddifferential Operators, Princeton University Press, 1981.
[3] C.D. Sogge, Hangzhou Lectures on Eigenfunctions of the Laplacian, Ann. Math. Stud., 188, Princeton University Press, 2014.
[4] L. Hörmander, The Analysis of Linear Partial Differential Operators I Springer Verlag, 2003.
[5] I.M. Gelfand, N, Ya Vilenkin: Generalized Functions, 4, Academic Press, 1964, Applications of Harmonic Analysis.
[6] L. Gass, Almost sure asymptotics for Riemannian random waves, Bernoulli 29 (2023) 625–651. arXiv:2005.06389.
[7] D.J. Daley, D. Vere-Jones, An Introduction to the Theory of Point Processes, II, Springer Verlag, general theory and structure, 2nd edition edition, 2008.
[8] O. Kallenberg, Random Measures, Theory and Applications, Springer Verlag, 2017.
[9] L.I. Nicolaescu, Critical points of multidimensional random Fourier series: variance estimates, J. Math. Phys. 57 (2016) 83304. arXiv:1310.5571.
[10] L.I. Nicolaescu, A Law of Large Numbers concerning the number of critical points of isotropic gaussian functions, arXiv:2408.14383.
[11] L.I. Nicolaescu, Random Morse functions and spectral geometry, arXiv:1209.0639.
[12] L. Gass, Cumulants asymptotics of the zeros counting measure of real Gaussian processes, Electron. J. Probab. 28 (2023) 1–45. arXiv:2112.08247.
[13] R. Williamson, L. Janos, Constructing metrics with the Heine-Borel property, Proc. A. M. S. 100 (1987) 567–573.
[14] L.I. Nicolaescu, Critical points of multidimensional random Fourier series: central limits, Bernoulli 24 (2018) 1128–1170. arXiv:1511.04965v2.
[15] J. Cuzick, A central limit theorem for the number of zeros of a stationary Gaussian process, Ann. Probab. 4 (1976) 547–556.
[16] L. Gass, Spectral criteria for the asymptotics of local functionals of Gaussian fields and their applications to nodal volumes and critical points, arXiv:2501.07356.
[17] M. Ancona, T. Letendre, Multijet bundles and applications to the finiteness of moments for zeros of Gaussian fields, Anal. PDE. 18 (2025) 1433–1476. arXiv:2307.10659
[18] D. Beliaev, M. Mcauley, S. Muirhead, A central limit theorem for the number of excursion set components of Gaussian fields, Ann. Probab. 52, (2024) 882–922.
[19] A. Estrade, R. J, León: A central limit theorem for the Euler characteristic of a Gaussian excursion set, Ann. Probab. 44 (2016) 3849–3878. https://hal.archives-ouvertes.fr/hal-00943054v3.
[20] M. Ancona, L. Gass, T. Letendre, M. Stecconi, Zeros and critical points of Gaussian fields: cumulants asymptotics and limit theorems, arXiv:2501.10226
[21] X, Fernique: Integrabilité des vecteurs gaussiens, C. R. Acad. 270 (1970) 1698–1699.
[22] F. Nazarov, M. Sodin, Asymptotic laws for the spatial distribution and the number of components of zero sets of Gaussian random functions, J. Math. Phys. Anal. Geom. 12 (2016) 205–278. https://core.ac.uk/download/pdf/159626247.pdf.
[23] D.W. Stroock, Gaussian Measures in Finite and Infinite Dimensions, Springer Verlag, 2023.
[24] J.-M. Azaïs, M. Wschebor, Level Sets and Extrema of Random Processes, John Wiley & Sons, 2009.
[25] R. Adler, R.J.E. Taylor, Random Fields and Geometry, Springer Verlag, 2007.
[26] D. Armentano, J.-M. Azaïs, J.-R. León, On the general Kac-Rice formula for the measure of a level set, Ann. Probab. 35 (2025) 1828–1951. arXiv:2304.0742v1.
[27] J.L.V. Hemmen, J. Ando, An inequality for trace ideals, Comm. Math. Phys. 76 (1980) 143–148.
[28] V.I. Bogachev, Gaussian Measures, Amer. Math. Soc, 1998.
[29] L. Gass, M. Stecconi, The number of critical points of a Gaussian field: finiteness of moments, Prob. Th. Rel. Fields. 190 (2024) 1167–1197, arXiv:2305.17586v2.
[30] J. Cuzick, Conditions for finite moments of the zero crossings for Gaussian processes, Ann. Probab. 3 (1975) 848–858.
[31] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, 1990. Reprint.