# A LAW OF LARGE NUMBERS CONCERNING THE DISTRIBUTION OF CRITICAL POINTS OF ISOTROPIC GAUSSIAN FUNCTIONS

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Abstract. We investigate the distribution of critical points of certain isotropic random functions $\Phi$ on $\mathbb{R}^m$. We show that the distribution of critical points of $\Phi(Rx)$, suitably normalized, converges a.s. and $L^2$ to the Lebesgue measure as $R \to \infty$. We achieve this by producing precise asymptotics of the second moments of these distributions as $R \to \infty$.

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## 1. Introduction

Denote by $\mathrm{Meas}(\mathbb{R}^m)$ the space of finite Borel measures on $\mathbb{R}^m$. Suppose that $\mathfrak{a} : \mathbb{R} \to \mathbb{R}$ is an even Schwartz function such that $\mathfrak{a}(0) = 1$. We will refer to such functions as *amplitudes*. Consider the measure $\mu_{\mathfrak{a}} \in \mathrm{Meas}(\mathbb{R}^m)$

$$\mu_{\mathfrak{a}}\big[\, d\xi\,\big] = \frac{1}{(2\pi)^m} w_{\mathfrak{a},m}\big(\,\xi\,\big)\boldsymbol{\lambda}\big[\,d\xi\,\big],\;\; w_{\mathfrak{a},m}\big(\,\xi\,\big) = \mathfrak{a}\big(\,|\xi|\,\big)^2,$$

where $\boldsymbol{\lambda}$ denotes the Lebesgue measure on $\mathbb{R}^m$.

The characteristic function of $\mu_{\mathfrak{a}}$ is the nonnegative definite function

$$\boldsymbol{K}_{\mathfrak{a}}\big(\,\boldsymbol{x}\,\big) = \int_{\mathbb{R}^m} e^{\boldsymbol{i}\langle \xi, \boldsymbol{x}\rangle}\mu_{\mathfrak{a}}\big[\,d\xi\,\big] = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{\boldsymbol{i}\langle \xi, \boldsymbol{x}\rangle}\mathfrak{a}\big(\,|\xi|\,\big)^2\boldsymbol{\lambda}\big[\,d\xi\,\big]. \tag{1.1}$$

Clearly $\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x})$ is an $O(n)$-invariant, real valued Schwartz function. Then $\boldsymbol{K}_{\mathfrak{a}}\big(\,\boldsymbol{x}-\boldsymbol{y}\,\big)$ is the covariance kernel of a real valued, smooth isotropic Gaussian function $\Phi = \Phi_{\mathfrak{a}}$ on $\mathbb{R}^m$ with spectral measure $\mu_{\mathfrak{a}}$.

For $R > 0$ we set

$$\mathfrak{a}_R(t) := \mathfrak{a}\big(t/R\big),\;\; \forall t \in \mathbb{R}.$$

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Consider the finite Borel measure $\mu_{\mathfrak{a}}^R \in \operatorname{Meas}(\mathbb{R}^m)$

$$\mu_{\mathfrak{a}}^R\big[\,d\xi\,\big] = \frac{1}{(2\pi)^m} w_{\boldsymbol{a}_R,m}\big(\,\xi\,\big)\boldsymbol{\lambda}\big[\,d\xi\,\big] = \frac{1}{(2\pi)^m}\mathfrak{a}\big(\,|\xi|/R\,\big)^2\boldsymbol{\lambda}\big[\,d\xi\,\big].$$

Its characteristic function is the nonnegative definite function

$$\boldsymbol{K}_{\mathfrak{a}}^R\big(\,\boldsymbol{x}\,\big) = \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} e^{\boldsymbol{i}\langle\xi,\boldsymbol{x}\rangle}\mathfrak{a}\big(\,|\xi|/R\,\big)^2 d\xi = R^m\boldsymbol{K}_{\mathfrak{a}}\big(\,R\boldsymbol{x}\,\big). \tag{1.2}$$

We deduce that $\boldsymbol{K}_{\mathfrak{a}}^R\big(\,\boldsymbol{x}-\boldsymbol{y}\,\big)$ is the covariance kernel of the Gaussian function

$$\Phi_{\mathfrak{a}}^R(\boldsymbol{x}) := R^{m/2}\Phi_{\mathfrak{a}}\big(\,R\boldsymbol{x}\,\big).$$

The Fourier inversion formula shows that

$$\int_{\mathbb{R}^m}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x})d\boldsymbol{x} = \mathfrak{a}(0)^2 = 1.$$

Since $K_{\mathfrak{a}}(\boldsymbol{x})$ is $O(m)$-invariant and smooth, it has the form $\Psi\big(\,|\boldsymbol{x}|^2\,\big)$ for some function $\Psi : [0,\infty) \to \mathbb{R}$. According to Schoenberg's characterization theorem [17, Thm.7.13], the function $\Psi$ must be completely monotone. In particular, $\Psi$ is non-increasing, nonnegative and convex, [17, Lemma.7.3]. This implies that the probability measures $\boldsymbol{K}_{\mathfrak{a}}^R\big(\,\boldsymbol{x}\,\big)dx$ converge weakly to the Dirac measure $\delta_0$. For example, if $\mathfrak{a}(t) = e^{-t^2/4}$, then

$$\boldsymbol{K}_{\mathfrak{a}}^R(\boldsymbol{x}) = \frac{1}{(2\pi\hbar^2)^{m/2}}e^{-\frac{|\boldsymbol{x}|^2}{2\hbar^2}},\;\; \hbar = R^{-1}$$

is the probability density the Gaussian measure on $\mathbb{R}^m$ with variance $\hbar^2\mathbb{1}_m$.

To use a terminology favored by physicists,

$$\boldsymbol{K}_{\mathfrak{a}}^R\big(\,\boldsymbol{x}-\boldsymbol{y}\,\big) \to \delta(\boldsymbol{x}-\boldsymbol{y}).$$

In other words, as $R\nearrow\infty$, the Gaussian random function $\Phi_{\mathfrak{a}}^R$ converges in distribution to a Gaussian random "function" $\Phi^\infty$ whose covariance kernel is $\boldsymbol{K}^\infty(\boldsymbol{x}-\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})$. This is the Gaussian white noise determined by the Lebesgue measure, [12].

For every $R>0$ the random function $\Phi_{\mathfrak{a}}^R$ is a.s. Morse and there is an associated critical random measure

$$\mathfrak{C}_{\mathfrak{a}}^R := \sum_{\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x})=0}\delta_{\boldsymbol{x}}.$$

Thus, for any Borel subset $S\subset\mathbb{R}^m$, $\mathfrak{C}_{\mathfrak{a}}^R[S]$ is the number of critical points of $\Phi_{\mathfrak{a}}^R$ in $S$. More generally, if $f:\mathbb{R}^m\to\mathbb{R}$ is a continuous compactly supported function we set

$$\mathfrak{C}_{\mathfrak{a}}^R[f] = \int_{\mathbb{R}^m} f(\boldsymbol{x})\mathfrak{C}_{\mathfrak{a}}^R\big[\,d\boldsymbol{x}\,\big] = \sum_{\nabla\Phi_{\mathfrak{a}}^R(\boldsymbol{x})=0} f(\boldsymbol{x}).$$

Let $\mathfrak{C}_{\mathfrak{a}} := \mathfrak{C}_{\mathfrak{a}}^R\big|_{R=1}$. The main goal of this paper is to investigate the behavior of $\mathfrak{C}_{\mathfrak{a}}^R$ in the white noise limit, $R\nearrow\infty$. The main result is the following.

**Theorem 1.1.** *Fix an amplitude $\mathfrak{a}\in\mathcal{S}(\mathbb{R})$. Then the following hold.*

(i) *There exists a universal explicit constant $C_m(\mathfrak{a})>0$ such that for any $f\in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$ we have*

$$\mathbb{E}\big[\,\mathfrak{C}_{\mathfrak{a}}^R[f]\,\big] = C_m(\mathfrak{a})R^m\int_{\mathbb{R}^m} f(\boldsymbol{x})d\boldsymbol{x}..$$

(ii) *There exists a constant* $C_m^{(2)}(\mathfrak{a}) \geq 0$*, that depends only on* $m$ *and* $\mathfrak{a}$ *such that for any* $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$

$$\operatorname{Var}\big[\,\mathfrak{C}^R_{\mathfrak{a}}[B]\,\big] \sim C_m^{(2)}(\mathfrak{a}) R^m \int_{\mathbb{R}^m} f(\boldsymbol{x})^2 d\boldsymbol{x}, \;\; as\ R \to \infty$$

□

The case $m = 1$ of this theorem was proved by M. Ancona and T. Letendre [2, Thm. 1.16] and L. Gass [10, Thm.1.6].

One immediate application of this result is a law of large numbers. For any positive integer $N$ we denote by $\mathfrak{L}_N$ the random measure

$$\mathfrak{L}_N := \frac{1}{N^m}\mathfrak{C}^N_{\mathfrak{a}}.$$

Theorem 1.1 shows for any $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$ we have

$$\operatorname{Var}\big[\,\mathfrak{L}_N[f]\,\big] \sim \mathit{const} \times N^{-m} \;\text{ as } N \to \infty$$

Since $\sum_{n\geq 1} n^{-m} < \infty$ for $m \geq 2$, we deduce from Borel-Cantelli the following law of large numbers.

**Corollary 1.2.** *Let* $m \geq 2$*. Then for any* $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$*,*

$$\lim_{N\to\infty} \mathfrak{L}_N[f] = C_m(\mathfrak{a})\boldsymbol{\lambda}\big[\,f\,\big] = C_m(\mathfrak{a}) \int_{\mathbb{R}^m} f(\boldsymbol{x}), \;\; \text{a.s. } \textit{and } L^2, \tag{1.3}$$

*where* $\boldsymbol{\lambda}$ *denotes the Lebesgue measure.* □

As detailed in Appendix A the above result can be rephrased as saying that the random measures $\mathfrak{L}_N$ converge a.s. and $L^2$ to the deterministic measure $C_m(\mathfrak{a})\boldsymbol{\lambda}$. In particular, for any bounded Borel set $S$, the random variables $\mathfrak{L}_N[S]$ converge a.s. to $C_m(\mathfrak{a})\boldsymbol{\lambda}[S]$ a.s.. Thus, in the white noise limit $N \to \infty$, the critical points of $\Phi^N_{\mathfrak{a}}$ tend to equidistribute with high confidence.

For any bounded Borel set $S \subset \mathbb{R}^m$ we have $\mathfrak{C}^N_{\mathfrak{a}}[S] = \mathfrak{C}_{\mathfrak{a}}[N \cdot S]$ and thus Corollary 1.2 shows that

$$\lim_{N\to\infty} \frac{1}{N^m}\mathfrak{C}_{\mathfrak{a}}[N \cdot S] \to C_m(\mathfrak{a})\boldsymbol{\lambda}[S] \;\text{ a.s. and } L^2.$$

In [15] we proved a Central Limit Theorem stating that when $S = [0,\ell]^m$, then the random variables

$$N^{m/2}\Big(\frac{1}{N^m}\mathfrak{C}_{\mathfrak{a}}[N \cdot S] - C_m(\mathfrak{a})\boldsymbol{\lambda}[S]\Big)$$

converge in distribution to a centered normal random variable with positive variance.

The new contribution of Corollary 1.2 the explicit description of the a.s. limit in (1.3). The fact that this limit does indeed exist a.s., in any dimension $m$ follows from the general theory of stationary random measures on $\mathbb{R}^m$. Thus Corollary 1.2 also holds in the case $m = 1$. Here are the details.

For any $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$ we have

$$\mathfrak{C}^N_{\mathfrak{a}}[f] = \mathfrak{C}_{\mathfrak{a}}[f_N], \;\; f_N(\boldsymbol{x}) := f\big(\,N^{-1}\boldsymbol{x}\,\big).$$

Suppose that $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^n)$ is nonnegative and

$$\boldsymbol{\lambda}[f] = \int_{\mathbb{R}^m} f(\boldsymbol{x})d\boldsymbol{x}) > 0.$$

Then the sequence

$$\varphi_N(\boldsymbol{x}) = \frac{1}{N^m} f_N(\boldsymbol{x})$$

is asymptotically stationary in the precise sense defined in Appendix A. The random measure $\mathfrak{C}_{\mathfrak{a}}$ is stationary and Theorem A.2 implies that

$$\lim_{N\to\infty} \frac{1}{\lambda\big[\, f\,\big]} \mathfrak{C}^N_{\mathfrak{a}}[f]$$

exists a.s. and $L^1$ and it is a random variable $\widehat{\mathfrak{C}}_{\mathfrak{a}}$ independent of $f$. Moreover, we can identify $\widehat{\mathfrak{C}}_{\mathfrak{a}}$ with a measurable function on the space $\mathcal{M}$ of locally finite Borel measures on $\mathbb{R}^m$.

The distribution of the random measure $\mathfrak{C}_{\mathfrak{a}}$ is a Borel probability measure $\mathbb{P}_{\mathfrak{C}_{\mathfrak{a}}}$ on $\mathcal{M}$. The additive group $\mathbb{R}^m$ acts on $\mathcal{M}$ by translations and, since $\Phi_{\mathfrak{a}}$ is stationary, we deduce that $\mathbb{P}_{\mathfrak{C}_{\mathfrak{a}}}$ is invariant under the above action of $\mathbb{R}^m$.

In Appendix A we give an alternate ergodic description of $\widehat{\mathfrak{C}}_{\mathfrak{a}}$. The fact that $\widehat{\mathfrak{C}}_{\mathfrak{a}}$ is constant suggests that the action of $\mathbb{R}^m$ on $\big(\mathcal{M}, \mathbb{P}_{\mathfrak{C}_{\mathfrak{a}}}\big)$ might be ergodic. There is some circumstantial evidence.

The Gaussian function $\Phi_{\mathfrak{a}}$ defines a Gaussian measure $\boldsymbol{\Gamma}$ on $C^2(\mathbb{R}^m)$. The additive group $\mathbb{R}^m$ acts on $C^2(\mathbb{R}^m)$ by translations. Since the Gaussian function $\Phi_{\mathfrak{a}}$ is stationary, we deduce that the above action is $\boldsymbol{\Gamma}$-preserving. Since the spectral measure of $\Phi_{\mathfrak{a}}$ is absolutely continuous with respect to the Lebesgue measure, the above action of $\mathbb{R}^m$ on $\big(C^2(\mathbb{R}^m), \boldsymbol{\Gamma}\big)$ is ergodic; see [6]. This is the fact used in 1960 by V. Volkovski [16] to prove Corollary 1.2 in the case $m = 1$. We refer to [7, Sec. 11.5] for details.

The paper is organized as follows. In Section 2 we collect several probabilistic facts needed in the proof of Theorem 1.1: the Kac-Rice formula, the regression formula and Lemma 2.3, a Hölder continuity result concerning certain functions on the space of Gaussian measures on a fixed finite dimensional Euclidean space. Section 3 contains the proof of Theorem 1.1. We have subdivided this section into several parts corresponding to the conceptually distinct steps in the proof of Theorem 1.1. Ultimately, two facts lie behind this result: the stationarity of the random function $\Phi_{\mathfrak{a}}$ and the wery weak correlations its values at far apart points.

## 2. Some probabilistic preliminaries

We collect in this section several probabilistic facts used throughout the paper. We begin with the Kac-Rice formula, a key player in our proof.

**Theorem 2.1.** *Let $\mathcal{V} \subset \mathbb{R}^m$ an open set. Suppose that $F : \mathcal{V} \to \mathbb{R}$ a Gaussian random function that is* a.s. *$C^2$ and such that the Gaussian vector $\nabla F(\boldsymbol{v})$ is nondegenerate for any $\boldsymbol{v} \in \mathcal{V}$. We denote by $p_{\nabla F(v)}$ is probability* density*. The following hold.*

(i) *The random function $F$ is* a.s. *Morse*

(ii) *We set*

$$\mathfrak{C}_F := \sum_{\nabla F(\boldsymbol{v})=0} \delta_v \tag{2.1}$$

*Then $\mathfrak{C}_F$ is a random locally finite measure on $\mathscr{V}$ in the sense of [8] or [14]. For any nonnegative measurable function $\varphi : \mathscr{V} \to [0,\infty)$ we set*

$$\mathfrak{C}_F[\varphi] := \int \varphi(v)\mathfrak{C}^F[dv] = \sum_{\nabla F(v)=0} \varphi(v).$$

(iii) *For any box $B \subset \mathscr{V}$, the function $F$* a.s. *has no critical points on $\partial B$ and*

$$\mathbb{E}\big[\,\mathfrak{C}_F[\boldsymbol{I}_B\varphi]\,\big] = \int_B \mathbb{E}\big[\,|\det \operatorname{Hess} F(\boldsymbol{v})|\,\big|\,\nabla F(\boldsymbol{v}) = 0\,\big] p_{\nabla F(v)}(0)\varphi(\boldsymbol{v})d\boldsymbol{v}. \tag{2.2}$$

*We will refer to the function*

$$\rho_F(v) := \mathbb{E}\big[\,|\det \operatorname{Hess} F(\boldsymbol{v})|\,\big|\,\nabla F(\boldsymbol{v}) = 0\,\big] p_{\nabla F(v)}(0)$$

*as the Kac-Rice density of $F$.*

□

For proof and more details we refer to [1, 4]. In applications, the conditional expectation appearing in the Kac-Rice density is computed using a classical trick.

Suppose that $X, Y$ are jointly Gaussian centered random vectors valued in the finite dimensional Euclidean spaces $\boldsymbol{X}$ and $\boldsymbol{Y}$. If $X$ is nondegenerate, then for any measurable $f : \boldsymbol{Y} \to \mathbb{R}$ with polynomial growth at $\infty$ the conditional expectation $\mathbb{E}\big[\,f(Y)\,\big|\,X = 0\,\big]$ is computed using the *regression formula*

$$\mathbb{E}\big[\,f(Y)\,\big|\,X = 0\,\big] = \mathbb{E}\big[\,f(Z)\,\big]$$

where $Z$ is the centered Gaussian vector $Y - \mathbb{E}\big[\,Y \,\|\, X\,\big]$[1] whose variance is

$$\operatorname{Var}\big[\,Z\,\big] = \operatorname{Var}\big[\,Y\,\big] - \operatorname{Cov}\big[\,Y, X\,\big] \operatorname{Var}\big[\,X\,\big]^{-1} \operatorname{Cov}\big[\,X, Y\,\big]. \tag{2.3}$$

In the proof we will need to compare expectations with respect to different Gaussian measures. In last part of this section we describe a general method for doing this.

Suppose that $\boldsymbol{V}$ is an $m$-dimensional real Euclidean space with inner product $(-,-)$. Denote by $S_1(\boldsymbol{V})$ the unit sphere in $\boldsymbol{V}$ and by $\mathbf{Sym}(\boldsymbol{V})$ the space of symmetric operators $\boldsymbol{V} \to \boldsymbol{V}$, by $\mathbf{Sym}_{\geq 0}(\boldsymbol{V}) \subset \mathbf{Sym}(\boldsymbol{V})$ the cone of nonnegative ones. For $A \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we denote by $\Gamma_A$ the centered Gaussian measure on $\boldsymbol{V}$ with variance $A$.

The space $\mathbf{Sym}(\boldsymbol{V})$ is equipped with an inner product

$$\big(\,A, B\,\big)_{\mathrm{op}} = \operatorname{tr}(AB), \ \ \forall A, B \in \mathbf{Sym}(\boldsymbol{V}).$$

We denote by $\|-\|_{\mathrm{op}}$ the associated norm.

We have a natural map $\mathbf{Sym}_{\geq 0}(\boldsymbol{V}) \to \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$, $A \mapsto A^{1/2}$. We will need the following result, [13, Prop.2.1].

**Proposition 2.2.** *For any $\mu > 0$ and $\forall A, B \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$, such that $A^{1/2} + B^{1/2} \geq \mu \mathbb{1}$ we have*

$$\mu \|\,A^{1/2} - B^{1/2}\,\|_{\mathrm{op}} \leq \|\,A - B\,\|_{\mathrm{op}}^{1/2}. \tag{2.4}$$

□

**Lemma 2.3.** *Fix $A_0 \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $A_0^{1/2} \geq \mu_0 \mathbb{1}$, $\mu_0 > 0$. Suppose that $f : \mathbb{V} \to \mathbb{R}$ is a locally Lipschitz function that is homogeneous of degree $k \geq 1$. For $A \in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ we set*

$$\mathcal{I}_A(f) := \int_{\boldsymbol{V}} f(\boldsymbol{v}) \boldsymbol{\Gamma}_{\mathbf{A}}\big[\,\mathbf{d}\boldsymbol{v}\,\big].$$

[1]The conditional expectation $\mathbb{E}\big[\,Y \,\|\, X\,\big]$ is a random variable whereas $\mathbb{E}\big[\,f(Y)\,\big|\,X = 0\,\big]$ is a deterministic quantity, whence the difference in notation, "$\|$" vs "$|$".

*Then for and $R\geq \|A_0\|_{\mathrm{op}}$ there exists a constant $C=C(f,R,\mu_0)>0$ with the following property: for any $A\in \mathbf{Sym}_{\geq 0}(\boldsymbol{V})$ such that $\|,A\|_{\mathrm{op}}\leq R$*

$$\big|\,\mathcal{I}_{A_0}(f)-\mathcal{I}_A(f)\,\big|\leq C\|A-A_0\|^{1/2}\leq C(k,R)\|A-A_0\|^{1/2}_{\mathrm{op}}. \tag{2.5}$$

*In other words, $A\mapsto \mathcal{I}_A(f)$ is locally Hölder continuous with exponent $1/2$ in the open set $\mathbf{Sym}_{>0}\big(\boldsymbol{V}\big)$.*

*Proof.* The function $f$ is Lipschitz on the ball

$$B_R(\boldsymbol{V}):=\big\{\,\boldsymbol{v}\in\boldsymbol{V};\|\boldsymbol{v}\|\leq R\,\big\},$$

so there exists $L=L(R)>0$ such that

$$\big[\,f(\boldsymbol{u})-f(\boldsymbol{v})\,\big|\leq L\|\boldsymbol{u}-\boldsymbol{v}\|,\ \ \forall \boldsymbol{u},\boldsymbol{v}\in B_R(\boldsymbol{V}). \tag{2.6}$$

Note that

$$\mathcal{I}_A(f)=\int_{\boldsymbol{V}} f\big(\,A^{1/2}\boldsymbol{v}\,\big)\Gamma_{\mathbb{1}}\big[\,d\boldsymbol{v}\,\big],$$

so

$$\big|\,\mathcal{I}_{A_0}(f)-\mathcal{I}_A(f)\,\big|\leq \int_{\boldsymbol{V}}\big|\,f\big(\,A^{1/2}\boldsymbol{v}\,\big)-f\big(\,A_0^{1/2}\boldsymbol{v}\,\big)\,\big|\,\Gamma_{\mathbb{1}}\big[\,d\boldsymbol{v}\,\big]$$

$$=\underbrace{\frac{1}{(2\pi)^{m/2}}\left(\int_0^\infty r^{n+k-1}e^{-r^2/2}dr\right)}_{C_{m,k}}\int_{S_1(\boldsymbol{V})}\big|\,f\big(\,A^{1/2}\boldsymbol{v}\,\big)-f\big(\,A_0^{1/2}\boldsymbol{v}\,\big)\,\big|\,\mathrm{vol}_{S_1(\boldsymbol{V})}\big[\,d\boldsymbol{v}\,\big]$$

$$\stackrel{(2.6)}{\leq} C_{m,k}L(R)\int_{S_1(\boldsymbol{V})}\|A^{1/2}-A_0^{1/2}\|_{\mathrm{op}}\,\mathrm{vol}_{S_1(\boldsymbol{V})}\big[\,d\boldsymbol{v}\,\big]\stackrel{(2.4)}{\leq} C(k,R,\mu_0)\|A-A_0\|^{1/2}_{\mathrm{op}}.$$

□

## 3. Proof of Theorem 1.1

3.1. **Proof of Theorem 1.1(i).** To compute the expectation of $\mathfrak{C}^R_{\mathfrak{a}}[f]$ we rely on the Kac-Rice formula. Note that $\boldsymbol{x}$ is a critical point of $\Phi^R_{\mathfrak{a}}$ iff $R^{-1}\boldsymbol{x}$ is a critical point of $\Phi_{\mathfrak{a}}$ so that, for any $f\in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$, we have

$$\mathfrak{C}^R_{\mathfrak{a}}[f]=\mathfrak{C}_{\mathfrak{a}}[f_R],\ \ f_R(\boldsymbol{x})=f(\boldsymbol{x}/R).$$

We want to apply the Kac-Rice formula to $\Phi_{\mathfrak{a}}$.

For $k\in\mathbb{N}$ we denote by $D^k\Phi_{\mathfrak{a}}$ the $k$-th order differential of $\Phi_{\mathfrak{a}}$

**Proposition 3.1.** *Let $N\in\mathbb{N}$. Then the following hold.*

(i) *The function $\Phi_{\mathfrak{a}}$ is $N$-ample, i.e., for any distinct points $\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\in\mathbb{R}^m$, the Gaussian vector*

$$\big(\,\Phi_{\mathfrak{a}}(\boldsymbol{x}_1),\dots,\Phi_{\mathfrak{a}}(\boldsymbol{x}_N)\,\big).$$

*is nondegenerate.*

(ii) *The function $\Phi_{\mathfrak{a}}$ is $J_N$-ample*[2]*, i.e., the Gaussian vector*

$$\Phi_{\mathfrak{a}}(\boldsymbol{x})\oplus D\Phi_{\mathfrak{a}}(\boldsymbol{x})\oplus\cdots\oplus D^N\Phi_{\mathfrak{a}}(\boldsymbol{x})$$

*is nondegenerate for any $\boldsymbol{x}\in\mathbb{R}^m$.*

[2] $N$-th jet ample

*Proof.* (i) Since $w_{\mathfrak{a},m}\big(\,|\,\xi\,|\,\big)=\mathfrak{a}\big(\,|\xi|\,\big)^2$ is positive on an open neighborhood of $0\in\mathbb{R}^m$ we deduce from [17, Thm. 6.8] that if $\boldsymbol{x}_1,\dots,\boldsymbol{x}_N\in\mathbb{R}^m$ are distinct points, then the symmetric $N\times N$ matrix

$$\big(\,K_{\mathfrak{a}}(\boldsymbol{x}_i-\boldsymbol{x}_j)\,\big)_{1\leq i,j\leq N}$$

is *positive* definite. This matrix is the variance matrix of the Gaussian vector

$$\big(\,\Phi_{\mathfrak{a}}(\boldsymbol{x}_1),\dots,\Phi_{\mathfrak{a}}(\boldsymbol{x}_N)\,\big).$$

(ii) Observe that for any multi-indices $\alpha\in\big(\mathbb{Z}_{\geq 0}\big)^m$, we have

$$\mathbb{E}\big(\,\partial^\alpha\Phi_{\mathfrak{a}}(\boldsymbol{x})\partial^\beta\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big)=\partial_x^\alpha\partial_y^\beta\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y}\,)\big|_{\boldsymbol{x}=\boldsymbol{y}}$$

$$=\int_{\mathbb{R}^m}\xi^\alpha\xi^\beta\mu_{\mathfrak{a}}\big[\,d\xi\,\big],\;\;\xi^\alpha:=\xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}$$

This shows that for any $N\in\mathbb{N}$ and any $\boldsymbol{x}\in\mathbb{R}^n$ the variance the Gaussian vector $\big(\,\partial^\alpha\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big)_{|\alpha|\leq N}$ is the Gramian matrix of the functions $\big(\,\xi^\alpha\,\big)_{|\alpha|\leq N}$ with respect to the inner product in $L^2\big(\,\mathbb{R}^m,\mu_{\mathfrak{a}}\,\big)$. Since $\mathfrak{a}(0)=1$ we deduce that the functions $\xi^\alpha$ are linearly independent in $L^2\big(\,\mathbb{R}^m,\mu_{\mathfrak{a}}\,\big)$ so the determinant of their Gramian matrix is nonzero. Hence the Gaussian vector

$$\Phi_{\mathfrak{a}}(\boldsymbol{x})\oplus D\Phi_{\mathfrak{a}}(\boldsymbol{x})\oplus\cdots\oplus D^N\Phi_{\mathfrak{a}}(\boldsymbol{x})$$

is nondegenerate, for any $k\in\mathbb{N}$ and any $\boldsymbol{x}\in\mathbb{R}^m$. □

Results of Ancona-Letendre [3] or Gass-Stecconi [11] show that the $J_N$-ampleness of $\Phi_{\mathfrak{a}}$ implies that for any function $f\in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$, the random variable $\mathfrak{C}_{\mathfrak{a}}[f]$ has finite moments of any order.

Proposition 3.1 shows that the Gaussian vector $\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})$ is nondegenerate for any $\boldsymbol{x}\in\mathbb{R}^m$. For any multi-indices $\alpha,\beta\in\big(\,\mathbb{Z}_{\geq 0}\,\big)^m$ we have

$$\mathbb{E}\big[\,\partial^\alpha\Phi_{\mathfrak{a}}(\boldsymbol{x})\partial^\beta\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]_{\boldsymbol{x}=\boldsymbol{y}}=\partial^\alpha_{\boldsymbol{x}}\partial^\beta_{\boldsymbol{y}}\mathcal{K}^{\mathfrak{a}}(\boldsymbol{x},\boldsymbol{y})\big|_{\,\boldsymbol{x}=\boldsymbol{y}}$$

$$=\frac{(-1)^{|\beta|}\boldsymbol{i}^{|\alpha|+|\beta|}}{(2\pi)^m}\int_{\mathbb{R}^m}\xi^{\alpha+\beta}\mathfrak{a}(|\xi|)^2d\xi,\;\;\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_m^{\alpha_m}. \tag{3.1}$$

For any multi-index $\alpha\in\big(\,\mathbb{Z}_{\geq 0}\,\big)^m$ we set

$$M^{\mathfrak{a}}_\alpha:=\int_{\mathbb{R}^m}\xi^\alpha\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\xi^\alpha\mathfrak{a}(|\xi|)^2d\xi.$$

We say that the multi-index $\alpha=(\alpha_1,\dots,\alpha_m)$ is *even* if $\alpha_j$ is even for any $j=1,\dots,m$. The multi-index $\alpha$ is called *odd* if it is not even. The radial symmetry of $\mathfrak{a}\big(\,|\xi|\,\big)$ implies that

$$M^{\mathfrak{a}}_\alpha=0\;\text{ if }\alpha\text{ is odd}. \tag{3.2}$$

Using spherical coordinates on $\mathbb{R}^m$ we deduce that for any $\alpha$ we have

$$M^{\mathfrak{a}}_\alpha=\frac{1}{(2\pi)^m}\left(\int_0^\infty r^{m-1+|\alpha|}\mathfrak{a}(r)^2dr\right)\times\underbrace{\int_{S^{m-1}}\xi^\alpha\operatorname{vol}_{S^{m-1}}\big[\,d\xi\,\big]}_{=:\boldsymbol{m}_\alpha}, \tag{3.3}$$

$S^{m-1}=S_1(\mathbb{R}^m)$. Note that $\boldsymbol{m}_\alpha$ is independent of $\mathfrak{a}$. If we let $\mathfrak{a}_0:=(2\pi)^{m/2}e^{-\frac{t^2}{4}}$, then

$$M^{\mathfrak{a}_0}_\alpha=\int_{\mathbb{R}^m}\xi^\gamma e^{-|\xi|^2/2}d\xi=(2\pi)^{m/2}\prod_{j=1}^m\int_{\mathbb{R}}\xi^{\alpha_j}\boldsymbol{\gamma}_1\big[\,d\xi\,\big]$$

where $\boldsymbol{\gamma}_1$ denotes the Gaussian measure on $\mathbb{R}$ with mean zero and variance 1. If $\alpha$ is even, $\alpha=2\kappa$, then

$$M_{2\kappa}^{\mathfrak{a}_0}=(2\pi)^{m/2}\prod_{j=1}^m(2\kappa_j-1)!!.$$

On the other hand, using (3.3) we deduce

$$M_{2\kappa}^{\mathfrak{a}_0}=\boldsymbol{m}_{2\kappa}\int_0^\infty r^{m+2|\kappa|-1}e^{-r^2/2}dr=\sqrt{\frac{\pi}{2}}\boldsymbol{m}_{2\kappa}\int_{\mathbb{R}}|x|^{m+2|\kappa|-1}\boldsymbol{\gamma}_1\big[\,dx\,\big]$$

$$=2^{|\kappa|+m/2-1}\boldsymbol{m}_{2\kappa}\Gamma\big(\,|\kappa|+m/2\,\big).$$

Hence

$$\boldsymbol{m}_{2\kappa}=\frac{(2\pi)^{\frac{m}{2}}\prod_{j=1}^m(2\kappa_j-1)!!}{2^{|\kappa|+m/2-1}\Gamma\big(\,|\kappa|+m/2\,\big)}=\frac{2\prod_{j=1}^m\Gamma(\kappa_j+1/2)}{\Gamma\big(\,|\kappa|+m/2\,\big)}. \tag{3.4}$$

Above we used the classical identities

$$\Gamma(1/2)=\pi^{1/2},\;\;\Gamma\big(\,x+1\,\big)=x\Gamma(x).$$

For every $k\in\mathbb{Z}_{\geq 0}$ we set

$$I_k(\mathfrak{a}):=\int_0^\infty r^k\mathfrak{a}(r)^2dr.$$

We deduce

$$(2\pi)^mM_{2\kappa}^{\mathfrak{a}}=I_{m-1+2|\kappa|}(\mathfrak{a})\frac{2\prod_{j=1}^m\Gamma(\kappa_j+1/2)}{\Gamma\big(\,|\kappa|+m/2\,\big)}. \tag{3.5}$$

We set

$$s_m:=\int_{\mathbb{R}^m}\mu_{\mathfrak{a}}\big[\,d\xi\,\big],\;\;d_m=\int_{\mathbb{R}^m}\xi_1^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big],\;\;h_m:=\int_{\mathbb{R}^m}\xi_1^2\xi_2^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]. \tag{3.6}$$

Then

$$\int_{\mathbb{R}^m}\mathfrak{a}(|\xi|)^2d\xi=\frac{2\pi^{m/2}}{\Gamma(m/2)}I_{m-1}(\mathfrak{a})=(2\pi)^ms_m, \tag{3.7}$$

$$\int_{\mathbb{R}^m}\xi_j^2\mathfrak{a}(|\xi|)^2d\xi=\frac{2\pi^{m/2}}{\Gamma(m/2+1)}I_{m+1}(\mathfrak{a})=(2\pi)^md_m,\;\;\forall j, \tag{3.8}$$

$$\int_{\mathbb{R}^m}\xi_j^2\xi_k^2\mathfrak{a}(|\xi|)^2d\xi=\frac{(2\pi)^{m/2}}{\Gamma(m/2+2)}I_{m+3}(\mathfrak{a})=(2\pi)^mh_m,\;\;\forall j\neq k, \tag{3.9}$$

$$\int_{\mathbb{R}^m}\xi_j^4\mathfrak{a}(|\xi|)^2d\xi=\frac{6\pi^{m/2}}{\Gamma(m/2+1)}I_{m+3}(\mathfrak{a})=3(2\pi)^mh_m,\;\;\forall j. \tag{3.10}$$

Using (3.1) and (3.2) we deduce that for any $\boldsymbol{x}\in\mathbb{R}^m$ *the Gaussian vectors* $\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})$ *and* $\mathrm{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x})$ *are independent.* Hence

$$\mathbb{E}\big[\,\det\mathrm{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x})|\,\big|\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})=0\,\big]=\mathbb{E}\big[\,\det\mathrm{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x})|\,\big].$$

Using (3.1)and (3.8) we deduce that the variance matrix of $\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})$ is

$$\mathrm{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big]=d_m\mathbb{1}_m,\;\;\forall\boldsymbol{x}\in\mathbb{R}^m, \tag{3.11}$$

where $\mathbb{1}_m$ denotes the identity $m\times m$ matrix. Hence

$$p_{\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})}(0)=(2\pi d_m)^{-m/2}.$$

The space $\mathbf{Sym}(\mathbb{R}^m)$ of real symmetric $m\times m$ matrices is equipped with an inner product $(A,B)=\operatorname{tr}(A)$. Moreover, the linear functions $\ell_{ij},\omega_{ij}:\mathbf{Sym}(\mathbb{R}^m)\to\mathbb{R}$, $1\leq i\leq j\leq m$,

$$\ell_{ij}(A)=a_{ij},\;\;\omega_{ij}(A)=\begin{cases}a_{ii}, & i=j,\\ \sqrt{2}a_{ij}, & i<j\end{cases} \tag{3.12}$$

define coordinates on $\mathbf{Sym}(\mathbb{R}^m)$ that are *orthonormal* with respect to the above inner product. We set

$$L_{ij}(\boldsymbol{x}):=\ell_{ij}\big(\operatorname{Hess}_{\Phi}(\boldsymbol{x})\big),\;\;\Omega_{ij}(\boldsymbol{x}):=\omega_{ij}\big(\operatorname{Hess}_{\Phi}(\boldsymbol{x})\big). \tag{3.13}$$

Then

$$\mathbb{E}\big[\,\partial^2_{x_ix_j}\Phi_{\mathfrak{a}}(x)\partial^2_{x_kx_\ell}\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big]=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\xi_i\xi_j\xi_k\xi_\ell a\big(\,|\xi|^2\,\big)d\xi,\;\;i\leq j,\;\;k\leq\ell.$$

Note that if $i<j$, then the above integral is nonzero iff $(i,j)=(k,\ell)$ in which case

$$\mathbb{E}\big[\,L_{ij}(\boldsymbol{x})L_{ij}(\boldsymbol{x})\,\big]=\mathbb{E}\Big[\,\big(\,\partial^2_{x_ix_j}\Phi_{\mathfrak{a}}(x)\,\big)^2\,\Big]$$

$$=\frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\xi_i^2\xi_j^2\mathfrak{a}\big(\,|\xi|^2\,\big)d\xi\overset{(3.9)}{=}h_m.$$

If $i=j$, then the above integral is nonzero iff $k=\ell$, in which case we deduce from (3.9) and (3.10)

$$\mathbb{E}\big[\,\partial^2_{x_i}\Phi_{\mathfrak{a}}(x)\partial^2_{x_k}\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big]=\begin{cases}h_m & i\neq k,\\ 3h_m, & i=k.\end{cases}$$

The above equalities can be rewritten in the more compact form

$$\mathbb{E}\big[\,L_{ij}(\boldsymbol{x})L_{k\ell}(\boldsymbol{x})\,\big]=h_m\big(\,\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\,\big),\;\;\forall i\leq j,\;k\leq\ell. \tag{3.14}$$

These equalities show that the off-diagonal entries of $\operatorname{Hess}_{\Phi}$ are i.i.d., and also independent of the diagonal entries. The diagonal entries have identical distributions but are dependent. The parameter $h_m$ describes the various variances and covariances.

The Gaussian measure on $\mathbf{Sym}(\mathbb{R}^m)$ determined by these covariance equalities is invariant with respect to the action of $O(m)$ by conjugation on $\mathbf{Sym}(\mathbb{R}^m)$.

For $v>0$ denote by $\mathcal{S}^v_m$ the space $\mathbf{Sym}(\mathbb{R}^m)$ equipped with the $O(m)$-invariant Gaussian measure on $\mathbf{Sym}(\mathbb{R}^m)$ determined by the covariances

$$\mathbb{E}\big[\,L_{ij}(\boldsymbol{x})L_{k\ell}(\boldsymbol{x})\,\big]=v\big(\,\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}\,\big),\;\;\forall i\leq j,\;k\leq\ell.$$

Hence

$$\mathbb{E}\big[\,\det\operatorname{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x})|\,\big]=\mathbb{E}_{\mathcal{S}^{h_m}_m}\big[\,|\det H|\,\big].$$

We deduce from the Kac-Rice formula (2.2) that

$$\mathbb{E}\big[\,\mathfrak{C}_{\mathfrak{a}}[f_R]\,\big]=\int_{\mathbb{R}^m}\mathbb{E}_{\mathcal{S}^{h_m}_m}\big[\,|\det H|\,\big]p_{\nabla\Phi(\boldsymbol{x})}(0)f_R(\boldsymbol{x})\boldsymbol{\lambda}\big[\,d\boldsymbol{x}\,\big]$$

$$\overset{(3.11)}{=}(2\pi d_m)^{-m/2}\mathbb{E}_{\mathcal{S}^{h_m}_m}\big[\,|\det H|\,\big]\int_{\mathbb{R}^m}f_R(\boldsymbol{x})dx.$$

Note that

$$\int_{\mathbb{R}^m}f_R(\boldsymbol{x})dx.=R^m\int_{\mathbb{R}^m}f(\boldsymbol{y})d\boldsymbol{y}.$$

Using the linear change in variables $X=(2h_m)^{-1/2}H$ we deduce

$$(2\pi d_m)^{-m/2}\mathbb{E}_{\mathcal{S}_m^{h_m}}\big[\,|\det H|\,\big]=\underbrace{\left(\frac{h_m}{\pi d_m}\right)^{m/2}\mathbb{E}_{\mathcal{S}_m^{1/2}}\big[\,|\det X|\,\big]}_{=:C_m(\mathfrak{a})}.$$

Hence

$$\mathbb{E}\big[\,\mathfrak{C}^R_{\mathfrak{a}}[f]\,\big]=C_m(\mathfrak{a})R^m\int_{\mathbb{R}^m}f(\boldsymbol{y})d\boldsymbol{y}.. \tag{3.15}$$

**Remark 3.2.** One can prove that as $m\to\infty$

$$C_m(\mathfrak{a})\sim 2^{\frac{5}{2}}\Gamma\left(\frac{m+3}{2}\right)\left(\frac{h_m}{\pi d_m}\right)^{m/2}\left(\frac{1}{m+1}\right)^{1/2}.$$

Using (3.8) and (3.9) we deduce

$$\frac{h_m}{d_m}=\frac{\Gamma(1+m/2)}{\Gamma(2+m/2)}\times\frac{I_{m+3}(\mathfrak{a})}{I_{m+1}(\mathfrak{a})}=\frac{2I_{m+3}(\mathfrak{a})}{(m+2)I_{m+1}(\mathfrak{a})}.$$

Hence

$$\begin{aligned}C_m(\mathfrak{a})&\sim 2^{5/2}\left(\frac{h_m(\mathfrak{a})}{d_m(\mathfrak{a})}\right)^{m/2}\Gamma\left(\frac{m+3}{2}\right)m^{-1/2}\\&\sim 2^{\frac{m+5}{2}}\left(\frac{I_{m+3}(\mathfrak{a})}{(m+2)I_{m+1}(\mathfrak{a})}\right)^{m/2}\Gamma\left(\frac{m+3}{2}\right)m^{-1/2}\ \text{ as } m\to\infty.\end{aligned} \tag{3.16}$$

The constant $C_m(\mathfrak{a})$ tends to grow very fast[3] as $m\to\infty$, but its large $m$ behavior depends on the tail of the amplitude $\mathfrak{a}$. Roughly speaking, the slower the decay at $\infty$ of $\mathfrak{a}$ the faster the growth of $C_m(\mathfrak{a})$. □

3.2. **An integral formula for the variance.** We need to introduce some notation. Set

- Define
$$\widehat{\Phi}:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R},\ \ \widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})=\Phi_{\mathfrak{a}}(\boldsymbol{x})+\Phi_{\mathfrak{a}}(\boldsymbol{y}),\ \ \hat{\mathfrak{C}}=\mathfrak{C}_{\widehat{\Phi}},$$
$$\widehat{H}(\boldsymbol{x},\boldsymbol{y}):=\operatorname{Hess}_{\widehat{\Phi}}(\boldsymbol{x},\boldsymbol{y}),\ \ H(\boldsymbol{x}):=\operatorname{Hess}_{\Phi_{\mathfrak{a}}}(\boldsymbol{x}).$$
- Choose an independent copy $\Psi_{\mathfrak{a}}$ of $\Phi_{\mathfrak{a}}$ and set
$$\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y}):=\Phi_{\mathfrak{a}}(\boldsymbol{x})+\Psi_{\mathfrak{a}}(\boldsymbol{y}),\ \ \widetilde{H}(\boldsymbol{x},\boldsymbol{y}):=\operatorname{Hess}_{\widetilde{\Phi}}(\boldsymbol{x},\boldsymbol{y}),\ \ \tilde{\mathfrak{C}}=\mathfrak{C}_{\widetilde{\Phi}}.$$
- Set $\|f\|:=\|f\|_{C^0(\mathbb{R}^m)}$.
- Set
$$\mathfrak{X}=\mathbb{R}^m\times\mathbb{R}^m\setminus\Delta=\big\{\,(\boldsymbol{x},\boldsymbol{y})\in\mathbb{R}^m\times\mathbb{R}^m;\ \ \boldsymbol{x}\neq\boldsymbol{y}\,\big\}.$$

Observe that the random function on $\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})$ is *stationary* with respect to the action of $\mathbb{R}^m$ on $\mathbb{R}^m\times\mathbb{R}^m$ itself by translations

$$T_{\boldsymbol{v}}(\boldsymbol{x},\boldsymbol{y})=\big(\,\boldsymbol{x}+\boldsymbol{v},\boldsymbol{y}+\boldsymbol{v}\,\big),\ \ \boldsymbol{x},\boldsymbol{y},\boldsymbol{v}\in\mathbb{R}^m, \tag{3.17}$$

where as $\widetilde{\Phi}$ is stationary with respect to the action by translations of $\mathbb{R}^{2m}$ on itself.

We have

$$\hat{\mathfrak{C}}[\boldsymbol{I}_{\mathfrak{X}}f_R^{\boxtimes 2}]=\sum_{\substack{\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})=\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})=0,\\ \boldsymbol{x}\neq\boldsymbol{y}}}f_R(\boldsymbol{x})f_R(\boldsymbol{y})=\mathfrak{C}_{\mathfrak{a}}[f_R]^2-\mathfrak{C}_{\mathfrak{a}}[f_R^2].$$

[3]Think super factorial.

Bulinskaya's lemma implies that

$$\mathbb{P}\big[\,\exists \boldsymbol{x}:\;\; \nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}) = \nabla\Psi_{\mathfrak{a}}(\boldsymbol{x}) = 0\,\big] = 0$$

and we deduce

$$\tilde{\mathfrak{C}}[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}] = \sum_{\substack{\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})=\nabla\Psi_{\mathfrak{a}}(\boldsymbol{y})=0,\\ \boldsymbol{x}\neq\boldsymbol{y}}} f_R(\boldsymbol{x}) f_R(\boldsymbol{y})$$

$$= \sum_{\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})=\nabla\Psi_{\mathfrak{a}}(\boldsymbol{y})=0} f_R(\boldsymbol{x}) f_R(\boldsymbol{y}) = \mathfrak{C}[f,\Phi_{\mathfrak{a}}]\,]\mathfrak{C}[f,\Psi_{\mathfrak{a}}],\;\; \text{a.s.}.$$

Hence

$$\mathbb{E}\big[\,\mathfrak{C}[f_R,\Phi_{\mathfrak{a}}]\cdot\mathfrak{C}[f,\Psi_{\mathfrak{a}}]\,\big] = \mathbb{E}\big[\,\mathfrak{C}[f_R,\Phi_{\mathfrak{a}}]\,\big]\;\cdot\;\mathbb{E}\big[\,\mathfrak{C}[f_R,\Psi_{\mathfrak{a}}]\,\big] = \mathbb{E}\big[\,\mathfrak{C}[f_R,\Phi_{\mathfrak{a}}]\,\big]^2$$

so that

$$\mathbb{E}\big[\,\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] - \mathbb{E}\big[\,\tilde{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] = \underbrace{\mathbb{E}\big[\,\mathfrak{C}_{\mathfrak{a}}[f_R]^2\,\big] - \mathbb{E}\big[\,\mathfrak{C}_{\mathfrak{a}}[f_R]\,\big]^2}_{=\mathrm{Var}\,\big[\,\big[\,\mathfrak{C}_{\mathfrak{a}}[f_R]\,\big]} - \mathbb{E}\big[\,\mathfrak{C}_{\mathfrak{a}}[f_R^2]\,\big].$$

We have seen that

$$\mathbb{E}\big[\,\mathfrak{C}_{\mathfrak{a}}[f_R^2]\,\big] = R^m C_m(\mathfrak{a})\int_{\mathbb{R}^m} f^2(\boldsymbol{x})d\boldsymbol{x}$$

so we have to show that

$$I(R) := \mathbb{E}\big[\,\hat{\mathfrak{C}}[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] - \mathbb{E}\big[\,\tilde{\mathfrak{C}}[\boldsymbol{I}_{\mathfrak{X}} f_R^{\boxtimes 2}]\,\big] \sim Z_m(\mathfrak{a}) R^m \int_{\mathbb{R}^m} f(\boldsymbol{x})^2 d\boldsymbol{x}\;\; \text{as } R\to\infty \tag{3.18}$$

for some constant $Z_m(\mathfrak{a})\in\mathbb{R}$ that depends only on $m$ and $\mathfrak{a}$.

**Lemma 3.3.** *For any $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^m$, $\boldsymbol{x}\neq\boldsymbol{y}$, the Gaussian vector $\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate.*

*Proof.* We have

$$\mathrm{Var}\,\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big] = \left[\begin{array}{cc} \mathrm{Var}\,\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \mathrm{Cov}\,\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big] \\ \mathrm{Cov}\,\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \mathrm{Var}\,\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big] \end{array}\right].$$

As shown in (3.11), for any $\boldsymbol{x}\in\mathbb{R}^n$ we have

$$\mathrm{Var}\,\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] = d_m\mathbb{1}_m,\;\; d_m = \int_{\mathbb{R}^n}\xi_1^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big].$$

We have

$$\mathrm{Cov}\,\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big] = \big(\,\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y})\,\big)_{1\leq j,k\leq m}$$

and

$$\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y}) = \int_{\mathbb{R}^m} e^{-\boldsymbol{i}\langle\xi,\boldsymbol{x}-\boldsymbol{y}\rangle}\xi_j\xi_k\mu_{\mathfrak{a}}\big[\,d\xi\,\big]. \tag{3.19}$$

Since $\Phi_{\mathfrak{a}}$ is stationary it suffice to consider only the case $\boldsymbol{x}=0$. On the other hand, $\Phi_{\mathfrak{a}}$ is $O(m)$-invariant so, up to a rotation, we can assume that $\boldsymbol{x}-\boldsymbol{y}=-t\boldsymbol{e}_1$, $t\neq 0$, where $\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_m\}$ is the canonical basis of $\mathbb{R}^m$. Hence

$$\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x}-\boldsymbol{y}) = \int_{\mathbb{R}^m} e^{\boldsymbol{i}t\xi_1}\xi_j\xi_k\mu_{\mathfrak{a}}\big[\,d\xi\,\big].$$

Let us observe that if $j\neq k$, then either $j\neq 1$, or $k\neq 1$. Suppose $j\neq 1$. The function $e^{\boldsymbol{i}t\xi_1}\xi_j\xi_k$ is odd with respect to the reflection $\xi_j\mapsto -\xi_j$ so

$$\partial_{x_j}\partial_{y_k}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x},\boldsymbol{y})=\int_{\mathbb{R}^m} e^{\boldsymbol{i}t\xi_1}\xi_j\xi_k\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=0,\;\;\forall j\neq k.$$

If $j=k$, then

$$d_m(j):=\partial_{x_j}\partial_{y_j}\boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{x},\boldsymbol{y})=\int_{\mathbb{R}^m} e^{\boldsymbol{i}t\xi_1}\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=\int_{\mathbb{R}^m}\cos(t\xi_1)\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]$$

and we deduce[4]

$$|v_m(j)|\leq\int_{\mathbb{R}^m}\big|\cos(t\xi_1)\big|\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]<\int_{\mathbb{R}^m}\xi_j^2\mu_{\mathfrak{a}}\big[\,d\xi\,\big]=d_m.$$

After a reordering

$$\begin{gathered}\big(\,\partial_{x_1}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\dots,\partial_{x_m}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\partial_{y_1}\Phi_{\mathfrak{a}}(\boldsymbol{y}),\dots\partial_{y_m}\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big)\\ \downarrow\\ \big(\,\partial_{x_1}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\partial_{y_1}\Phi_{\mathfrak{a}}(\boldsymbol{y}),\dots,\partial_{x_m}\Phi_{\mathfrak{a}}(\boldsymbol{x}),\partial_{y_m}\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big)\end{gathered}$$

we see that

$$\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=\bigoplus_{j=1}^m\underbrace{\begin{bmatrix} d_m & d_m(j)\\ d_m(j) & d_m\end{bmatrix}}_{=:V_j}.$$

Note that, for each $j$, the symmetric matrix $V_j$ is positive definite since

$$\det V_j=d_m^2-d_m(j)^2>0.$$

□

Since $\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate for $\boldsymbol{x}\neq\boldsymbol{y}$ we can use the Kac-Rice formula to compute $\mathbb{E}\big[\,\hat{\mathfrak{C}}^R[\boldsymbol{I}_{\mathfrak{X}}f_R^{\boxtimes 2}]\,\big]$. We deduce that for any $R>0$

$$\begin{gathered}\mathbb{E}\big[\,\hat{\mathfrak{C}}[\boldsymbol{I}_{\mathfrak{X}}f_R^{\boxtimes 2}]\,\big]\\ =\int_{\mathbb{R}^m\times\mathbb{R}^m\setminus\Delta}\underbrace{\mathbb{E}\big[\,|\det\widehat{H}(\boldsymbol{x},\boldsymbol{y})|\,\big|\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})=0\,\big]p_{\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})}f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big].\end{gathered}\tag{3.20}$$

Since $\widehat{\Phi}$ is invariant under the translations (3.17) we deduce that $\widehat{\rho}$ depends only on $\boldsymbol{x}-\boldsymbol{y}$.

The gradient $\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})$ is nondegenerate for any $\boldsymbol{x},\boldsymbol{y}$ and invoking Kac-Rice again we obtain

$$\begin{gathered}\mathbb{E}\big[\,\tilde{\mathfrak{C}}[\boldsymbol{I}_{\mathfrak{X}}f_R^{\boxtimes 2}]\,\big]\\ =\int_{\mathbb{R}^m\times\mathbb{R}^m\setminus\Delta}\underbrace{\mathbb{E}\big[\,|\det\widetilde{H}(\boldsymbol{x},\boldsymbol{y})|\,\big|\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})=0\,\big]p_{\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)}_{=\widetilde{\rho}(\boldsymbol{x},\boldsymbol{y})}f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})\boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big].\end{gathered}\tag{3.21}$$

The function $\widetilde{\rho}_R(\boldsymbol{x},\boldsymbol{y})$ is independent of $\boldsymbol{x},\boldsymbol{y}$ since the random function $\widetilde{\Phi}$ is stationary. Thus

$$I(R)=\int_{\mathfrak{X}}\big(\,\underbrace{\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})-\widetilde{\rho}(\boldsymbol{x},\boldsymbol{y})}_{=:\Delta(\boldsymbol{x},\boldsymbol{y})}\,\big)f_R(\boldsymbol{x})f_R(\boldsymbol{y})\boldsymbol{\lambda}\big[\,d\boldsymbol{x}d\boldsymbol{y}\,\big].\tag{3.22}$$

[4]At this point we use the fact that $\mathfrak{a}\big(\,|\xi|\,\big)>0$ for $|\xi|$ sufficiently small.

There is a serious issue concerning $\widehat{\rho}(\boldsymbol{x},\boldsymbol{y})$ namely it blows up as $(\boldsymbol{x},\boldsymbol{y})$ approaches the diagonal so this Kac-Rice density may not be locally integrable.

3.3. **Off-diagonal behavior.** We first describe the behavior of $\Delta(\boldsymbol{x},\boldsymbol{y})$ away from the diagonal. Note that $\Delta$ depends only on the $\boldsymbol{x}-\boldsymbol{y}$.

For every $\boldsymbol{z}\in\mathbb{R}^m$ we set

$$T(z) := \sum_{|\alpha|\leq 4}\big|\,\partial^\alpha \boldsymbol{K}_{\mathfrak{a}}(\boldsymbol{z})\,\big|.$$

Since $\boldsymbol{K}_{\mathfrak{a}}$ is a Schwartz function we deduce that

$$T(\boldsymbol{z}) = O\big(\,|\boldsymbol{z}|^{-\infty}\,\big)\ \text{ as } |\boldsymbol{z}|\to\infty.$$

This means that

$$\forall p>0,\ \ T(\boldsymbol{z}) = O\big(\,|\boldsymbol{z}|^{-p}\,\big)\ \text{ as } |\boldsymbol{z}|\to\infty.$$

Observe that

$$\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big] = \begin{bmatrix}\operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\\ 0 & \operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{bmatrix} = d_m\mathbb{1}_{2m},$$

and

$$\begin{split}\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big] &= \begin{bmatrix}\operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Var}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{bmatrix}\\ &= \operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big] + \underbrace{\begin{bmatrix}0 & \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\end{bmatrix}}_{=:R_\nabla(\boldsymbol{x},\boldsymbol{y})}.\end{split}$$

Hence

$$\big\|\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]-\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\big\|_{\mathrm{op}} = \|R_\nabla(\boldsymbol{x},\boldsymbol{y})\|_{\mathrm{op}} = O\big(\,T_R(\boldsymbol{x}-\boldsymbol{y})\,\big), \tag{3.23}$$

The operators $\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]$ and $\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]$ are invertible for $\boldsymbol{x}\neq\boldsymbol{y}$. In particular

$$\begin{split}\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1} &= \Big(\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]+R_\nabla(\boldsymbol{x},\boldsymbol{y})\Big)^{-1}\\ &= \Big(\mathbb{1}+\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}R_\nabla(\boldsymbol{x},\boldsymbol{y})\Big)^{-1}\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1},\end{split} \tag{3.24}$$

$$\big\|\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}-\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\big\|_{\mathrm{op}} = O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big)\text{ as }|\boldsymbol{x}-\boldsymbol{y}|\to\infty. \tag{3.25}$$

Note that

$$\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big] = \begin{bmatrix}\operatorname{Var}\big[\,H(\boldsymbol{x})\,\big] & 0\\ 0 & \operatorname{Var}\big[\,H(\boldsymbol{y})\,\big]\end{bmatrix}.$$

Since $\Phi_{\mathfrak{a}}$ is stationary, $\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]$ is *independent* of $\boldsymbol{x}$ and $\boldsymbol{y}$. We have

$$\begin{split}\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big] &= \begin{bmatrix}\operatorname{Var}\big[\,H(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H(\boldsymbol{x}),H(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),H(\boldsymbol{x})\,\big] & \operatorname{Var}\big[\,H(\boldsymbol{y})\,\big]\end{bmatrix}\\ &= \operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big] + \underbrace{\begin{bmatrix}0 & \operatorname{Cov}\big[\,H(\boldsymbol{x}),H(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),H(\boldsymbol{x})\,\big] & 0\end{bmatrix}}_{=:R_H(\boldsymbol{x},\boldsymbol{y})}.\end{split}$$

We deduce that as $|\boldsymbol{x}-\boldsymbol{y}|\to\infty$ we have

$$\Big\|\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]-\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]\Big\|_{\mathrm{op}}=\|R_H(\boldsymbol{x},\boldsymbol{y})\|_{\mathrm{op}}=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big). \tag{3.26}$$

We denote by $\widetilde{H}(\boldsymbol{x},y)^\flat$ the Gaussian random matrix

$$\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat=\widetilde{H}(\boldsymbol{x},\boldsymbol{y})-\mathbb{E}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\|\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y}\,\big].$$

Similarly, we denote by $\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat$ the Gaussian random matrix

$$\widehat{H}(\boldsymbol{x},y)^\flat=\widehat{H}(\boldsymbol{x},\boldsymbol{y})-\mathbb{E}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\|\,,\nabla\widehat{\Phi}\,\big].$$

The distributions of $\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat$ and $\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat$ are determined by the Gaussian regression formula (2.3).

Since $\widetilde{H}(\boldsymbol{x},y)$ and $\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})$ are independent we deduce

$$\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]=\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big].$$

Using the regression formula we deduce that for $|\boldsymbol{x}-\boldsymbol{y}|>C_0$,

$$\begin{gathered}\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]=\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]\\ -\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}),\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]|\\ =\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]+R_H(\boldsymbol{x},\boldsymbol{y})\\ -\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}),\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big].\end{gathered}$$

We have

$$\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=\left[\begin{array}{cc}\operatorname{Cov}\big[\,H(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & \operatorname{Cov}\big[\,H(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\end{array}\right]$$

$$=\left[\begin{array}{cc}\operatorname{Cov}\big[\,0 & \operatorname{Cov}\big[\,H(\boldsymbol{x}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{y})\,\big]\\ \operatorname{Cov}\big[\,H(\boldsymbol{y}),\nabla\Phi_{\mathfrak{a}}(\boldsymbol{x})\,\big] & 0\end{array}\right].$$

This implies

$$\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big)\ \text{ as }|\boldsymbol{x}-\boldsymbol{y}|\to\infty.$$

We deduce from (3.25) that

$$\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}=\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}+O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big).$$

Since $\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]$ is independent of $\boldsymbol{x}$ and $\boldsymbol{y}$ we conclude that

$$\operatorname{Cov}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y}),\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\operatorname{Cov}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y}),\widehat{H}(\boldsymbol{x},\boldsymbol{y})\,\big]=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big), \tag{3.27}$$

Since $\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})\,\big]$ is independent of $\boldsymbol{x},\boldsymbol{y}$ we deduce that there exists $\mu_0>0$ such that

$$\operatorname{Var}\big[\,\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]\geq\mu_0\mathbb{1},\ \ \forall\boldsymbol{x}\neq\boldsymbol{y}.$$

We deduce from (3.27) and Lemma 2.3 that

$$\Big|\,\mathbb{E}\big[\,|\det\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat|\,\big]-\mathbb{E}\big[\,|\det\widetilde{H}(\boldsymbol{x},\boldsymbol{y})^\flat|\,\big]\,\Big|=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})^{1/2}\,\big). \tag{3.28}$$

Using (3.25) we deduce that as $|\boldsymbol{x}-\boldsymbol{y}|\to\infty$ we have

$$\begin{aligned}&\Big|\,p_{\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)-p_{\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})}(0)\,\Big|\\ &=(2\pi)^{-m/2}\Big|\,\det\operatorname{Var}\big[\,\nabla\widehat{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}-\det\operatorname{Var}\big[\,\nabla\widetilde{\Phi}(\boldsymbol{x},\boldsymbol{y})\,\big]^{-1}\,\Big|=O\big(\,T(\boldsymbol{x}-\boldsymbol{y})\,\big).\end{aligned}\tag{3.29}$$

Note also that (3.26) implies that

$$\sup_{|\boldsymbol{x}-\boldsymbol{y}|_\infty>1}\|\operatorname{Var}\big[\,\widehat{H}(\boldsymbol{x},\boldsymbol{y})^\flat\,\big]\|_{\mathrm{op}}<\infty\tag{3.30}$$

We can now estimate the right-hand-side of (3.22). Using (3.28), (3.29) and (3.30) we conclude that

$$\forall|\boldsymbol{x}-\boldsymbol{y}|>1,\ \ \big|\,\Delta(\boldsymbol{x},\boldsymbol{y})\,\big|=O\big(\,\big|\,\boldsymbol{x}-\boldsymbol{y}\,\big|^{-\infty}\,\big),\ \ \text{as } |\boldsymbol{x}-\boldsymbol{y}|\to\infty.\tag{3.31a}$$

$$\sup_{|\boldsymbol{x}-\boldsymbol{y}|>1}\big|\,\Delta(\boldsymbol{x}-\boldsymbol{y})\,\big|<\infty.\tag{3.31b}$$

3.4. **Conclusion.** Suppose that

$$\operatorname{supp} f\subset\{|\boldsymbol{x}|<r_0\}.$$

Denote by $\widehat{\mathfrak{X}}$ the radial-blowup of $\mathbb{R}^m\times\mathbb{R}^m$ along the diagonal. It is diffeomorphic to the product $\mathbb{R}^m\times S^{m-1}\times[0,\infty)$.

Choose new orthogonal coordinates $(\xi,\eta)$ given by

$$\xi=\boldsymbol{x}+\boldsymbol{y},\ \ \eta=\boldsymbol{x}-\boldsymbol{y}\Longleftrightarrow\boldsymbol{x}=\frac{1}{2}(\xi+\eta),\ \ \boldsymbol{y}=\frac{1}{2}(\xi-\eta)$$

then

$$|\boldsymbol{x}-\boldsymbol{y}|=|\eta|,\ \ d\boldsymbol{x}d\boldsymbol{y}=2^{-2m}d\xi d\eta.$$

Recall that $\widehat{\rho}$ depends only on $\eta$. Note that if $\boldsymbol{x},\boldsymbol{y}\in\operatorname{supp} f$, then $|\boldsymbol{x}|,|\boldsymbol{y}|<r_0$ and thus

$$\boldsymbol{x},\boldsymbol{y}\in\operatorname{supp} f\ \Rightarrow\ |\xi|,\ |\eta|<\frac{1}{2}|\xi+\eta|+\frac{1}{2}|\xi-\eta|\le r_0.\tag{3.32}$$

The natural projection $\pi:\widehat{\mathfrak{X}}\to\mathbb{R}^m\times\mathbb{R}^m$ can given the explicit description

$$\mathbb{R}^m\times S^{m-1}\times[0,\infty)\ni(\xi,\boldsymbol{\nu},r)\mapsto(\xi,\eta)=(\xi,r\boldsymbol{\nu})\in\mathbb{R}^m\times\mathbb{R}^m.$$

We set

$$w(\boldsymbol{x},\boldsymbol{y})=|\boldsymbol{x}-\boldsymbol{y}|^{m-2}\widehat{\rho}(\boldsymbol{x},\boldsymbol{y}).$$

We deduce from [5, Sec. 4.2] or [9, Appendix A.1]

$$\sup_{0<|\boldsymbol{x}-\boldsymbol{y}|\le 1}\big|\,w(\boldsymbol{x},\boldsymbol{y})\,\big|<\infty.\tag{3.33}$$

It is easy to see that $\widetilde{\rho}\circ\pi$ admits a continuous extension to the blow-up. Using (3.31a), (3.31b) and (3.33) we deduce that for any $p>0$ there exists a constant $K_p>0$, such that

$$|x-y|^{m-1}\big|\,\Delta(\boldsymbol{x},\boldsymbol{y})\,\big|\le K_p\big(\,1+|x-y|\,\big)^{-p+m-1},\ \ \forall\boldsymbol{x}\neq\boldsymbol{y}\tag{3.34}$$

Set

$$\delta(\xi,\eta)=\Delta\big(\,\pi(\xi,\eta)\,\big)$$

Since $\Delta(\boldsymbol{x},\boldsymbol{y})$ depends only on $\boldsymbol{y}-\boldsymbol{x}$ we deduce that $\delta(\xi,\eta)$ is independent of $\xi$. We have

$$I(R)=\int_{\mathfrak{X}}\Delta(\boldsymbol{x},\boldsymbol{y})f_R^{\boxtimes 2}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}$$

$$= \frac{1}{2^{2m}} \int_{\mathbb{R}^m} \int_{|\boldsymbol{\nu}|=1,\, r\in(0,\infty)} r^{m-1}\delta\big(\,\xi, r\boldsymbol{\nu}\,\big) f_R\Big(\,\frac{\xi+r\boldsymbol{\nu}}{2}\,\Big) f_R\Big(\,\frac{\xi-r\boldsymbol{\nu}}{2}\,\Big) dr\, \mathrm{vol}_{S^{m-1}}[d\boldsymbol{\nu}] d\xi$$

$(\xi = 2R\zeta)$

$$\stackrel{(3.32)}{=} R^m \underbrace{2^{-m} \int_{|\zeta\leq r_0} \left( \int_{\substack{|\boldsymbol{\nu}|=1\\ r>0}} r^{m-1}\delta\big(\,0, r\boldsymbol{\nu}\,\big) f\Big(\,\zeta+\frac{r\boldsymbol{\nu}}{2R}\,\Big) f(\zeta-\frac{r\boldsymbol{\nu}}{2R}\,\Big) dr\, \mathrm{vol}_{S^{m-1}}[d\boldsymbol{\nu}] \right) d\zeta}_{=:J(R)}.$$

We deduce from (3.32) and (3.33) that for any $p>0$ there exists $K_p>0$ such that for any $R>0$ , $|\boldsymbol{\nu}|=1$ and $r>0$ we have

$$\Big|\, r^{m-1}\delta\big(\,0, r\boldsymbol{\nu}\,\big) f\Big(\,\zeta+\frac{r\boldsymbol{\nu}}{2R}\,\Big) f\Big(\,\zeta-\frac{r\boldsymbol{\nu}}{2R}\,\Big)\,\Big| \leq K_p\|f\|^2\big(\,1+r\,\big)^{-p+m-1}.$$

For $p>m$ we have

$$\int_{|\zeta|\leq r_0} \left( \int_{(0,\infty\times S^{m-1}} \big(\,1+r\,\big)^{-p+m-1} dr\, \mathrm{vol}_{S^{m-1}}[d\boldsymbol{\nu}] \right) d\zeta < \infty.$$

The dominated convergence theorem implies that $J(R)$ has a finite limit as $R\to\infty$. More precisely

$$\lim_{R\to\infty} J(R) = \int_{|\zeta|\leq r_0} \underbrace{\left( 2^{-m} \int_{\substack{|\boldsymbol{\nu}|=1\\ r>0}} r^{m-1}\delta\big(\,0, r\boldsymbol{\nu}\,\big) dr\, \mathrm{vol}_{S^{m-1}}[d\boldsymbol{\nu}] \right)}_{=:Z_m(\mathfrak{a})} f(\zeta)^2 d\zeta.$$

This concludes the proof of Theorem 1.1 (ii) with $C_m^{(2)}(\mathfrak{a}) = Z_m(\mathfrak{a}) + C_m(\mathfrak{a})$.

## Appendix A. Random measures

The fact that $\Phi_{\mathfrak{a}}$ is a stationary random function implies that the random measure $\mathfrak{C}_{\mathfrak{a}}$ is stationary. This in itself has deep consequences. Let us elaborate.

Denote by $\widehat{\mathrm{Meas}}(\mathbb{R}^m)$ the space locally finite of Borel measures $\mu$ on $\mathbb{R}^m$, i.e., $\mu\big[\,B\,\big]<\infty$ for any bounded Borel subset $B\subset\mathbb{R}^m$. Each $f\in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$ defines a map

$$I_f : \widehat{\mathrm{Meas}}(\mathbb{R}^m) \to \mathbb{R},\ \ \mu\mapsto I_f(\mu) = \mu\big[\,f\,\big] := \int_{\mathbb{R}^m} f(\boldsymbol{x})\mu\big[\,dx\,\big].$$

The *vague topology* on $\widehat{\mathrm{Meas}}(\mathbb{R}^m)$ is the smallest topology such that all the functions $I_f$, $f\in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$ are continuous. As shown in [14, Thm. 4.2], this topology is Polish, i.e., it is induced by a complete and separable metric. We denote by $\mathcal{M}$ is metric space. T We denote by $\mathrm{Prob}\big(\,\mathcal{M}\,\big)$ the space of Borel probability measures on $\mathcal{M}$.

A sequence $(\mu_n)$ in $\mathcal{M}$ converges vaguely to $\mu\in\mathcal{M}$, and we indicate this as $\mu_n\xrightarrow{v}\mu$, if and only if

$$\mu_n\big[\,f\,\big]\to\mu\big[\,f\,\big],\ \ \forall f\in C^0_{\mathrm{cpt}}(\mathbb{R}^m).$$

A random locally finite measure on $\mathbb{R}^m$ is a Borel measurable map

$$\mathfrak{M} : \big(\,\Omega, \mathcal{S}, \mathbb{P}\,\big) \to \mathcal{M}.$$

Its distribution is a Borel probability measure $\mathbb{P}_{\mathfrak{M}}$ on $\mathcal{M}$.

Recall that a sequence of probability measures $\mu_n \in \operatorname{Prob}(\mathcal{M})$ is said to converge weakly to $\mu \in \operatorname{Prob}(\mathcal{M})$, and we indicate this $\mu_n \to \mu$ if

$$\lim_{n\to\infty} \int_{\mathfrak{X}} F d\mu_n = \int_{\mathfrak{X}} F d\mu,$$

for any bounded and continuous function $F : \mathcal{M} \to \mathbb{R}$. A sequence of random measures $\mathfrak{M}_n$ is said to converge weakly to the random measure $\mathfrak{M}$ if the distributions $\mathbb{P}_{\mathfrak{M}_n}$ converge weakly in $\operatorname{Prob}(\mathcal{M})$ to $\mathbb{P}_{\mathfrak{M}}$. We use the notation $\mathfrak{M}_n \to \mathfrak{M}$ to indicate this.

A subset $Q \subset \mathbb{R}^m$ is a *quasi-box* if it is a product of finite intervals

$$Q = I_1 \times \cdots \times I_m.$$

The intervals $I_k$ need not be closed and could have length zero. A quasi-box $Q$ is called a box if all the intervals $I_k$ are closed and have nonzero lengths.

We have the following result, [8, Prop.11.1.VIII], [14, Thm. 4.11].

**Theorem A.1.** *Consider a sequence* $\big(\mathfrak{M}_n\big)_{n\in\mathbb{N}}$ *of random locally finite measures* $\operatorname{Prob}\big(\mathcal{M}\big)$. *The following are equivalent.*

(i) *The sequence $\mathfrak{M}_n$ converges weakly to the random locally finite measure $\mathfrak{M}$.*
(ii) *For any $f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$, the random variables $\mathfrak{M}_n\big[\, f\,\big]$ converge in distribution to $\mathfrak{M}\big[\, f\,\big]$.*
(iii) *For any quasi-box $Q \subset \mathbb{R}^m$ the random variables $\mathfrak{M}_n\big[\, Q\,\big]$ converge in distribution to $\mathfrak{M}\big[\, Q\,\big]$.*
(iv) *For any bounded Borel subset $S \subset \mathbb{R}^m$ random variables $\mathfrak{M}_n\big[\, S\,\big]$ converge in distribution to $\mathfrak{M}\big[\, S\,\big]$*

□

There are other modes of convergence of random measures corresponding to the various modes of convergence of random variables. Suppose that

$$\mathfrak{M}_n,\ \mathfrak{M} : \big(\Omega, \mathcal{S}, \mathbb{P}\big) \to \operatorname{Prob}\big(\mathfrak{X}\big),\ \ n \in \mathbb{N}$$

are random locally finite measures. We say that $\mathfrak{M}_n$ converges almost surely to $\mathfrak{M}$ and we indicate this $\mathfrak{M}_n \stackrel{\text{a.s.}}{\to} \mathfrak{M}$ if there exists a $\mathbb{P}$-negligible set $\mathcal{N} \in \Omega$ such that

$$\mathfrak{M}_n(\omega) \stackrel{v}{\to} \mathfrak{M}(\omega),\ \ \forall \omega \in \Omega \setminus \mathcal{N},$$

i.e.,

$$\mathfrak{M}_n \xrightarrow{\text{a.s.}} \mathfrak{M} \Longleftrightarrow \mathfrak{M}_n\big[\, f\,\big] \xrightarrow{\text{a.s.}} \mathfrak{M}\big[\, f\,\big],\ \forall f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m), .$$

The convergence $\mathfrak{M}_n \xrightarrow{L^p} \mathfrak{M}$ is defined in a similar fashion namely

$$\mathfrak{M}_n \xrightarrow{L^p} \mathfrak{M} \Longleftrightarrow \mathfrak{M}_n\big[\, f\,\big] \xrightarrow{L^p} \mathfrak{M}\big[\, f\,\big],\ \forall f \in C^0_{\mathrm{cpt}}(\mathbb{R}^m), .$$

One can show (see [14, Lemma 4.8]) that

$$\mathfrak{M}_n \xrightarrow{\text{a.s.}} \mathfrak{M} \Longleftrightarrow \mathfrak{M}_n\big[\, S\,\big] \xrightarrow{\text{a.s.}} \mathfrak{M}\big[\, S\,\big],\ \text{ for any bounded Borel set } S \subset \mathbb{R}^m, \tag{A.1a}$$

$$\mathfrak{M}_n \xrightarrow{L^p} \mathfrak{M} \Longleftrightarrow \mathfrak{M}_n\big[\, S\,\big] \xrightarrow{L^p} \mathfrak{M}\big[\, S\,\big],\ \text{ for any bounded Borel set } S \subset \mathbb{R}^m. \tag{A.1b}$$

The action of $\mathbb{R}^m$ on itself by translations induces an action on $\mathcal{M}$. We denote by $\mathcal{I}$ the sigma-subalgebra of $\mathcal{B}_{\mathcal{M}}$ consisting of translation invariant Borel subsets of $\mathcal{M}$. A measure $\mathbb{P} \in \operatorname{Prob}\big(\mathcal{M}\big)$ is called stationary if its invariant with respect to this action. A random measure $\mathfrak{M}$ is called stationary if its distribution $\mathbb{P}_{\mathfrak{M}}$ is stationary.

As discussed in [8, Chap.12] or [14, Chap.5], every stationary random locally finite measure $\mathfrak{M}$ on $\mathbb{R}^m$ has an asymptotic intensity $\hat{\mathfrak{M}} \in L^1\big(\mathcal{M}, \mathbb{P}_{\mathcal{M}}\big)$. More precisely

$$\widehat{\mathfrak{M}} := \mathbb{E}\big[\,\mathfrak{M}[C_1]\,\big|\,\mathcal{I}\,\big],\;\; C_1 = [0,1]^m.$$

This is an integrable random variable $\widehat{\mathfrak{M}}$. It has an ergodic interpretation. Wiener's ergodic theorem shows that for any compact convex subset $C \subset \mathbb{R}^m$ containing the origin in the interior we have

$$\widehat{\mathfrak{M}} = \lim_{N\to\infty} \frac{1}{N^m \operatorname{vol}\big[\,C\,\big]} \int_{NC} \mathfrak{M}[C_1 - \boldsymbol{x}\,]\,d\boldsymbol{x} \;\text{ a.s.}.$$

Thus, if the action of $\mathbb{R}^m$ on $(\mathcal{M}, \mathbb{P}_{\mathfrak{M}})$ is ergodic, then $\widehat{\mathfrak{M}}$ is $\mathbb{P}_{\mathfrak{M}}$-a.s. constant.

The intensity $\widehat{\mathfrak{M}}$ has another ergodic interpretation; see [8, Thm. 12.2.IV] or [14, Th. 5.23]. More precisely, for $C$ as above we have

$$\frac{1}{N^m \operatorname{vol}\big[\,C\,\big]} \mathfrak{M}\big[\,NC\,\big] \to \widehat{\mathfrak{M}}$$

a.s. and $L^1$. Moreover, if $\mathfrak{M}[C_1] \in L^p$, then the convergence holds also in $L^p$. This fact admits the following generalization.

A sequence $\varphi_N \in C^0_{\mathrm{cpt}}(\mathbb{R}^m)$, $N \in \mathbb{N}$ is called *asymptotically* stationary if there exists $C > 0$ such that

$$\varphi_N \geq 0,\;\; \int_{\mathbb{R}^m} \varphi_N(\boldsymbol{x})d\boldsymbol{x} = C,\;\; \forall N,$$

and

$$\lim_{N\to\infty} \int_{\mathbb{R}^m} \big|\,\varphi_N(\boldsymbol{x}) - \varphi_N(\boldsymbol{x}-\boldsymbol{y})\,\big|d\boldsymbol{x} = 0,\;\; \forall \boldsymbol{y} \in \mathbb{R}^m.$$

We have the following result, [14, Thm. 5.24]

**Theorem A.2.** *If* $(\varphi_N)_{N\in\mathbb{N}}$ *is asymptotically stationary, then*

$$\mathfrak{M}\big[\,\varphi_N\,\big] \to C\widehat{\mathfrak{M}},$$

*in* $L^1$ *and* a.s.. □

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