Multiscale Cohesive Model

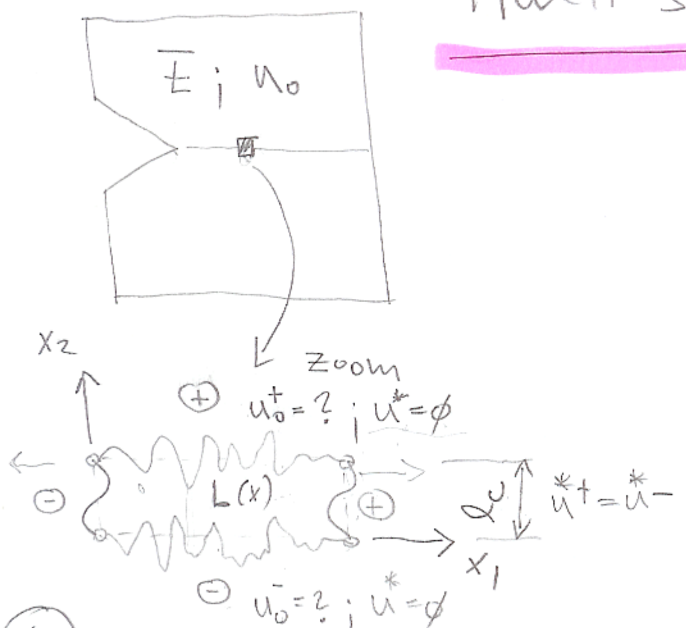
Derived by Dr. Karel Matouš

These are my original hand written notes, I derived sometimes in June 2004.

2D - plane strain

Multi-scale cohesive law

(A)



Displacement decomposition

$$u(x) = u_0 + \overset{*}{u}$$

$$\varepsilon(x) = \varepsilon_0 + \varepsilon^* = \nabla^s u_0 + \nabla^s \overset{*}{u}$$

Hill's Lema

$\overset{*}{u}$ Periodic in x_1 and $\overset{*}{u} = \emptyset$ on $\partial\Omega_{x_2}$

$$\delta W = \frac{1}{A}\int_\Omega \sigma \cdot \delta\varepsilon(u)\, d\Omega = \Sigma \cdot [\![\delta u_0]\!]$$

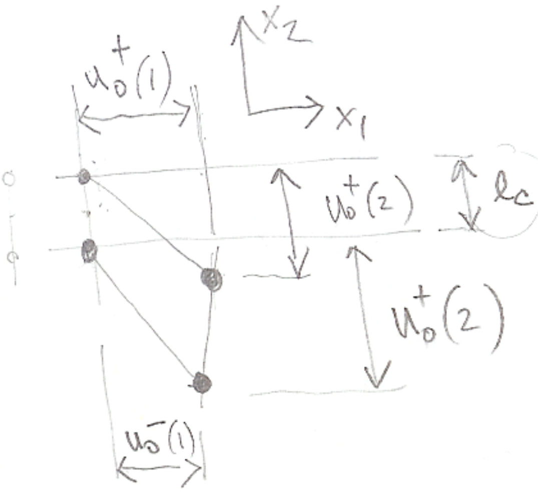
$$[\![u_0]\!] = u_0^+ - u_0^-$$

Average of fluctuation strain

$$\frac{1}{A}\int_\Omega \delta\overset{*}{\varepsilon}\, d\Omega = \frac{1}{A}\int_{\partial\Omega} x \otimes N\, d\Gamma = \emptyset$$

Cohesive layer kinematics

and average strain



$$\varepsilon^0_{11} = \frac{\partial u_0^A}{\partial x_1} = \frac{1}{2}\left[\frac{\partial u_0^+(1)}{\partial x_1} + \frac{\partial u_0^-(1)}{\partial x_1}\right]$$

$$\varepsilon^0_{22} = \frac{u_0^+(2) - u_0^-(2)}{l_c}$$

$$u_0^{Average} = \frac{1}{2}\left(u_0^+(1) + u_0^-(1)\right)$$

$$\varepsilon^0_{12} = \frac{u_0^+(1) - u_0^-(1)}{l_c}$$

l_c - intrinsic length scale

It can be assumed that $\underline{\varepsilon_{11}^{0} = \emptyset}$ (B)

$\varepsilon_0 = \frac{1}{\ell c} [\![u_0]\!]$ $\Rightarrow$ ε_0 in term of displacement jump

$\delta \varepsilon_0 = \frac{1}{\ell c} [\![\delta u_0]\!]$

δu_0 is an arbitrary function

$\Rightarrow \delta \varepsilon_0 = [\![\delta u_0]\!]$! NOT CORRECT

$[-]$ $\quad$ $[m]$

$\delta \varepsilon_0 = \frac{1}{\ell c} N \otimes [\![\delta^0 u]\!]$

Cohesive layer constitutive response

$\Psi(\omega; \varepsilon(u)) = (1-\omega) \Psi_V = (1-\omega) \bar{\sigma} : \varepsilon \frac{1}{2}$

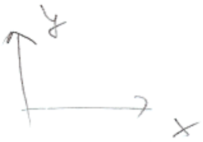
↳ effective stress

Clausius-Duhem inequality

$-\dot{\Psi} + \sigma : \dot{\varepsilon} \geqslant \emptyset$

$\Psi_v \rightarrow$ virgin free stored energy function

$\Psi_V = \frac{1}{2} \varepsilon : L : \varepsilon$



$$\underline{\varepsilon_0} = \begin{Bmatrix} \emptyset \\ \frac{1}{\ell c} [\![u_0^2]\!] \\ \frac{1}{\ell c} [\![u_0^1]\!] \end{Bmatrix}$$

$$\sigma = \frac{\partial \Psi}{\partial \varepsilon} = (1-\omega)\frac{\partial \Psi^0}{\partial \varepsilon} = (1-\omega)\, L : \varepsilon$$ c

---

ω – isotropic damage parameter

$$\dot{\Psi} = -\dot{\omega}\,\frac{1}{2}\,\varepsilon : L : \varepsilon + (1-\omega)\frac{1}{2}\left[\dot{\varepsilon} : L : \varepsilon + \varepsilon : L : \dot{\varepsilon}\right]$$

$$\dot{\Psi} = -\dot{\omega}\,\underbrace{\frac{1}{2}\,\varepsilon : L : \varepsilon}_{-Y} + \underbrace{(1-\omega)\,\bar{\sigma}}_{\sigma} \cdot \dot{\varepsilon}$$

C–D $\quad -\dot{\Psi} + \sigma : \dot{\varepsilon} \geq \phi$

$$-\dot{\omega} Y - \sigma \cdot \dot{\varepsilon} + \sigma \cdot \dot{\varepsilon} \geq \phi$$

$\mathcal{D} = \dot{\omega}\Psi_0 = -\dot{\omega} Y \Rightarrow$ Damage disipation

$Y = -\frac{1}{2}\,\varepsilon : L : \varepsilon \Rightarrow$ Thermodynamic force

---

$$\dot{\sigma} = (1-\omega)\, L : \dot{\varepsilon} - \dot{\omega}\,\underbrace{L : \varepsilon}_{\bar{\sigma}}$$

Damage function

$$g(Y, \chi) = G(Y) - \chi^t \leq \phi \quad , \quad t \in R^+$$

$G(Y)$ – damage function

χ^t – damage treshold

D

Damage evolution equation

$$\dot{\omega} = \dot{\kappa}\frac{\partial g}{\partial Y} = \dot{\kappa} H \; ; \quad H = \frac{\partial G(Y)}{\partial Y}$$

we define that $\quad \dot{\kappa} = \mu \langle \phi \rangle$

$$\dot{\chi}^t = \dot{\kappa} H$$

and from consistency condition

$$\dot{\kappa} = \dot{Y}$$

Simo – Ju

$$g(\zeta; \chi) = -Y - \chi^t \leq \phi$$
$$= \zeta - \chi^t \leq \phi$$

$$\dot{\kappa} = \dot{\zeta} = \varepsilon : L : \dot{\varepsilon} = \bar{\sigma} \cdot \dot{\varepsilon}$$

$$\dot{\omega} = H \bar{\sigma} \cdot \dot{\varepsilon}$$

Const – equation

$$\dot{\sigma} = (1-\omega) L : \dot{\varepsilon} - H \bar{\sigma} \otimes \bar{\sigma} : \dot{\varepsilon}$$

$$\dot{\sigma} = \underbrace{\left[(1-\omega) L - H \bar{\sigma} \otimes \bar{\sigma}\right] : \dot{\varepsilon}}_{\mathcal{L} : \dot{\varepsilon}}$$

Hill's Lema

Da

$$\int_{\Omega} \frac{1}{|\theta|} \int_{\theta} \left[\underline{\delta^{\circ} \varepsilon} + \delta \varepsilon^{*} \right] : \tau \, d\theta \, d\Omega = \int_{\Gamma_c} t \cdot [\delta^{\circ} u] \, d\Gamma$$

$Pa\,m^3$

$Pa \cdot m \cdot m^2 = Pa\,m^3$

$\frac{N}{m^2} m^3 = \underline{Nm}$

$$l_c \int_{\Gamma_c} \frac{1}{|\theta|} \int_{\theta} \delta^{\circ} \varepsilon : \tau \, d\theta \, d\Gamma = \int_{\Gamma_c} t \cdot [\delta^{\circ} u] \, d\Gamma$$

$\frac{1}{l_c} N \otimes [\delta^{\circ} u]$

$$l_c \frac{1}{|\theta|} \int_{\theta} \delta^{\circ} \varepsilon \cdot \tau \, d\theta = t \cdot [\delta^{\circ} u]$$

$m\,Pa = Pa\,m$

$$[\delta^{\circ} u] \cdot \left[\frac{1}{|\theta|} \int \tau \, d\theta \right] \cdot N = [\delta^{\circ} u] \cdot t$$

m· Pa

Hill's Lema

(E)

$$l_c \frac{1}{A}\int_\Omega \left[\frac{1}{l_c}[\![\delta u_0]\!] + \delta\overset{*}{\varepsilon}\right]\cdot\left[\underbrace{(1-\omega)L(x):\left(\frac{1}{l_c}[\![u_0]\!]+\overset{*}{\varepsilon}\right)}_{\sigma}\right]d\Omega = \underset{\text{m}\cdot\text{Pa}}{[\![\delta u_0]\!]\cdot \bar{t}}$$

$$\frac{1}{A}\int_\Omega\left\{(1-\omega)[\![\delta u_0]\!]:L(x):\left(\frac{1}{l_c}[\![u_0]\!]+\overset{*}{\varepsilon}\right)\right\}d\Omega + \frac{1}{A}\int_\Omega\left\{\frac{(1-\omega)}{l_c}\delta\overset{*}{\varepsilon}:L(x):[\![u_0]\!] + \delta\overset{*}{\varepsilon}\underbrace{(1-\omega)L(x)}_{\text{secant stiffness}}\overset{*}{\varepsilon}\right\}d\Omega$$

δu_0 and $\delta\overset{*}{\varepsilon}$ are independent

(1) $$\frac{1}{A}\frac{1}{l_c}[\![\delta u_0]\!]:\left\{\int_\Omega (1-\omega(x))L(x)\,d\Omega:[\![u_0]\!] + l_c\int_\Omega(1-\omega)L(x):\overset{*}{\varepsilon}\,d\Omega\right\} = [\![\delta u_0]\!]\cdot\bar{t}$$

$$\Rightarrow \bar{t} = \frac{1}{l_c}\frac{1}{A}\int_\Omega(1-\omega)L\,d\Omega:[\![u_0]\!] + \frac{1}{A}\int_\Omega(1-\omega)L:\overset{*}{\varepsilon}\,d\Omega$$

(2) $$\frac{1}{l_c}\frac{1}{A}\int_\Omega\delta\overset{*}{\varepsilon}:(1-\omega)L\,d\Omega:[\![u_0]\!] + \frac{1}{A}\int_\Omega\delta\overset{*}{\varepsilon}:(1-\omega)L\overset{*}{\varepsilon}\,d\Omega = \emptyset$$

From ① for $L(x) = L$ and $\omega = \phi \Rightarrow$ no damage

$$\frac{1}{l_c} [\![\delta u_0]\!] : \left\{ L : [\![u_0]\!] + l_c L \underbrace{\frac{1}{A} \int_\Omega \overset{*}{\varepsilon} \, d\Omega}_{=\phi} \right\} = [\![\delta u_0]\!] \cdot \bar{t}$$

$$\Rightarrow \quad \bar{t} = \frac{1}{l_c} L : [\![u_0]\!] \rightarrow \text{initial cohesive tractions}$$

$\bar{t} = \frac{1}{l_c} L : [\![u_0]\!]$

limit case $l_c \rightarrow \phi \quad \bar{t} \rightarrow \infty$

$\bar{t} = \sigma_0$ $\quad \bar{t}_2 = \sigma_{22} \quad \bar{t}_1 = \phi$

$\bar{t}_{12} = \sigma_{12}$

FE implementation

$\overset{*}{\varepsilon} = B \overset{*}{u}$ grad operator

First solve ②

$$\frac{1}{A} \int_\Omega B^T (1-\omega) L B \overset{*}{u} \, d\Omega = -\frac{1}{l_c} \frac{1}{A} \int_\Omega B^T (1-\omega) L \, d\Omega \, [\![u_0]\!]$$

From ① evaulate tractions (macroscopic tractions)

$$\bar{t} = \frac{1}{l_c} \cdot \frac{1}{A} \int_\Omega (1-\omega) L \, d\Omega : [\![u_0]\!] + \frac{1}{A} \int_\Omega (1-\omega) L : (B \overset{*}{u}) \, d\Omega$$