

DELINEATING THE AVERAGE RATE OF CHANGE AND
CONSEQUENCES OF FITTING AN INCORRECT GROWTH MODEL

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Abstract

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The average rate of change is a key concept in longitudinal analyses that examine change over time. However, this concept has been misunderstood both implicitly and explicitly in the literature. The present work attempts to clarify the concept and show unequivocally the mathematical definition and meaning of the average rate of change. Oftentimes the slope from the straight-line growth model is interpreted as though it were the average rate of change. It is shown, however, that this is generally not the case and holds true in only a limited number of situations. General equations are presented for the bias and discrepancy factor when the slope from the straight-line growth model is used to estimate the average rate of change. The importance of fitting an appropriate individual growth model is discussed, as are the benefits provided by nonlinear models for longitudinal data.

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INTRODUCTION

The goal of data analytic techniques and procedures applied within the context of longitudinal data analysis should be to help researchers systematically and quantitatively document *what* is changing (The attribute(s) should be operationally defined.), *who* is changing (What type of individual changes?), *how* they change (What sort of trend describes transition over time?), *when* they change (Does change occur only after a specified time or after a certain event?), with the ultimate goal of understanding *why* the attribute of interest is in a state of transition. The what, who, how, when, and why of longitudinal research is important for developing predictions and explanations of the changing phenomena in order for the research community to make sense of, and potentially using, the findings and conclusions educed from longitudinal investigations.

The study of change has long been a central topic in the sciences. A philosophical account of the process of change was given more than 100 years ago by Davies (1900). Davies realized even then that “change is by no means a simple affair” (1900, p. 506). Davies identified the mind as being “in no sense Eleatic,” but rather that such “perpetual change” is what “differentiates and constitutes the unique problem of psychology” when compared to other disciplines (p. 508). Rather than being a “problem” with psychology per se, it seems that such “perpetual change” is what leads to such a dynamic field of study. In fact, studying such non-static relationships in the behavioral sciences dates back at least to the early part of the twentieth century, where Billings and Shepard (1910) attempted to measure change in heart

rate as a function of attention level. Of course the study of change is much older in the physical and mathematical sciences, where Newton (1643-1727) and Leibniz (1646-1716) laid the foundation for modern calculus, the branch of mathematics concerned with the study of change, in the late seventeenth century.

In the behavioral sciences the analysis of change is generally studied over time by way of longitudinal research designs. The general requirement for longitudinal research is that one or more variables are repeatedly measured over time on the same unit of analysis. It is through longitudinal research that inferences regarding *intraindividual* change, *interindividual* change, and group change can be examined. A formal set of rationales for longitudinal research has been given by Baltes and Nesselroade (1979, pp. 23 – 27):

1. direct identification of intraindividual change;
2. direct identification of interindividual differences (similarity) in intraindividual change;
3. analysis of interrelationships in behavioral change;
4. analysis of causes (determinants) of intraindividual change;
5. analysis of causes (determinants) of interindividual differences in intraindividual change.

While understanding various descriptions of change (Rationale 1 & 2) is important before attempting to understand correlates (Rationale 3) and causes (Rationale 4 & 5) of change, measuring change has proved to be no less than a daunting task. The methodological literature in the behavioral sciences has long attempted to address problems associated with the conceptualization and measurement of change. Since the design, analysis, and the interpretation of longitudinal research is the driving force behind many areas of inquiry, it is important when utilizing current methods or developing novel ones that they lead

to accurate and meaningful descriptions of the transition over time exhibited by the phenomena of interest.

From the 1950s until the 1980s, different conceptualizations of change, its measurement, and the design of studies examining change led to serious questions in the methodological literature of the behavioral sciences about the appropriateness of the analysis of change. An extreme view on the measurement of change by Cronbach and Furby (1970) questioned if the measurement and analysis of change should even be attempted. Their assessment of the “problems” of measuring and analyzing change, particularly with difference scores, along with similar “problems” documented by others (e.g., the works contained in Harris, 1963; Lord, 1956 & 1958; Linn & Slinde, 1977), left those who worked within the longitudinal framework in a disconcerted position regarding the measurement and analysis of change. The publication of these works suggested that researchers who were interested in the examination of change “frame their questions in other ways” (Cronbach & Furby, 1970, p. 80). Because a major goal in the behavioral sciences is to understand transition over time, framing such questions in “other ways” detours and potentially wreaks havoc on the scientific goals and inferences of the investigator.¹

However, hope was not lost for the analysis of change. In recent times it has been realized that there were both implicit and explicit problems in the vintage arguments against the analysis of change. The major problem with the works that criticized the analysis of change is that the focus of such critiques examined cases where emphasis was placed on distinguishing interindividual differences in change between measurement occasions rather than distinguishing intraindividual differences in change among individuals over time. This misconception in the

¹Sometimes interest is not in transition over time, but rather in the lack of transition. This lack of transition is known as stability. Stability is thus a special case of change, specifically implying the lack of change (i.e., constancy).

measurement of change is detrimental to the conceptualization and analysis of change (Bryk & Raudenbush, 1992, pp. 130-131). In general, methods and strategies for distinguishing interindividual differences at a particular time are ill equipped for the analysis of intraindividual change over time (Collins, 1996).

More recently, however, the analysis of change has been reconceptualized where “individual time paths are the proper focus for the analysis of change” (Rogosa, Brandt, & Zimowski, 1982, p. 744; see also the methodological works of Raudenbush, 2001; Mehta & West, 2000; Rogosa & Willett, 1985; Collins, 1996; Bryk & Raudenbush, 1987; Willett, 1988; with applications of these strategies in Francis, Fletcher, Stuebing, Davidson, & Thompson, 1991; Francis, Schatschneider, & Carlson, 2000; Karney & Bradbury, 1995). By focusing on individuals *over* time, rather than at a *specific* time, researchers can develop and test precise hypotheses regarding change using reliable and sophisticated models. From these models a better understanding of the phenomenon of interest as it exists, changes, and evolves over time can be realized. Generally speaking, most behavioral phenomena seem to change in a continuous fashion over time rather than at discrete steps or stages. The process of this continuous change is important and can be quite informative in understanding the underlying system(s) responsible for transition. An extension of examining how individuals change is to examine whether there are differences in rates of change for the overall trends among individuals or groups. However, the first step in understanding interindividual differences in change is, by logical necessity, the precise and valid measurement of intraindividual change (Collins, 1996, p. 38), and the foundation of intraindividual change is a statistical model for individual time paths (Rogosa et al., 1982, p. 726).

In large part the analysis of change has been facilitated over the last 20 years by the realization that longitudinal data are hierarchical in nature, where observations

over time are nested within the entity under study (e.g., the individual), which in turn may be nested within an organizational structure (e.g., a group) at a higher level (Shadish, 2002). Given this realization, new methods for the analysis of change have been developed that explicitly model the hierarchical structure of longitudinal data in a class of statistical models known as hierarchical linear models (HLM) and hierarchical nonlinear models (HNLM; Laird & Ware, 1982; Bryk & Raudenbush, 1987; Bryk & Raudenbush, 1992; Davidian & Giltinan, 1995; Goldstein, 1995; chapters 6 & 8 of Vonesh & Chinchilli, 1997).² Given the new conceptualization, measurement, and design of studies involving issues of change, some of the long held beliefs about “problems” measuring change have been dispelled, as many of the previous criticisms of the analysis of change were misguided and based on inappropriate assumptions (See Rogosa et al., 1982, Rogosa, 1995, and Willett, 1988, for their critiques of works that criticized and questioned the measurement and analysis of change.).

The concept of intraindividual change should be the starting point for longitudinal research. It is by first focusing on the individual that broad generalizations over individuals can (or cannot) be made. The description of intraindividual change can be given in numerous ways, and is limited only by the research design and the researcher’s creativity in forming and testing models. For example, by focusing on one individual trajectory, the unknown functional form of growth can be described as any combination of linear, quadratic, exponential, or even as a dampened or undampened sinusoidal function. The adequacy of the particular model chosen, however, depends in large part on the true functional form of growth and the number of timepoints that measurements are obtained. Given

²Because of the simultaneous interdisciplinary development of HLMs and HNLMs, such models are also termed multilevel, mixed-effects, random-effects, covariance components, and random coefficient models.

that such a vast array of possibilities exists for describing intraindividual change, a measure of change that can describe all possible functional forms of growth by way of a single descriptive statistic would have great practical value for the numerical description that it could provide. A measure known as the *average rate of change* is such a value.

The major purpose of the present work is to delineate the meaning and interpretation of the average rate of change (ARC), as well as what the ARC is not, in order for this measure of overall change to be better used by researchers attempting to understand the *what, who, how, when* and *why* of longitudinal research as well as the rationales laid out by Baltes and Nesselroade (1979, pp. 23–27), such that the potentially complicated and problematic process(es) of intraindividual change can be better understood. The delineation of the ARC begins at an intuitive level and progresses to a mathematical description. Most of the emphasis throughout the present work is concerned foremost with a single trajectory, as the individual trajectory is the appropriate starting point for understanding change.

Implicitly or explicitly the ARC is often a central focus for many longitudinal research projects. Attempts are often made to succinctly describe the *average* or *typical* amount of change that occurs within some time interval of interest. Rather than describing change in several dimensions simultaneously (e.g., linear, quadratic, and cubic), researchers often wish to describe change parsimoniously. The regression coefficient from the straight-line growth model has often been the medium in which a succinct description of change over time has been attempted. Apparently a widely held belief is that the regression coefficient from the straight-line growth model provides a measure of the typical, or average, amount of change occurring within an interval of time for an individual trend. The idea of using a single value as a descriptor of a potentially complicated process of change has great intuitive appeal.

However, as the remainder of the work shows, the regression coefficient from the straight-line growth model is generally not equal to the mean rate of change over time for a given trajectory. A major purpose of this work is to illustrate that making use of the regression coefficient from the straight-line growth model as if it was the ARC can yield biased estimates, potentially leading to incorrect conclusions about the underlying process of change. This work contends that the ARC should be used more often in applied research, not to supplant other analyses, but to supplement the information gained from modeling and examining growth over time.

Because the ARC provides the average or typical rate of change, not only a single dimension of change (such as the linear, quadratic, or exponential growth component does), it is a parsimonious measure that describes the overall trend of a growth trajectory, regardless of the functional form of growth. The ARC is itself a mean and thus the well-known and generally desirable properties of the mean are also properties of the ARC. Although the concept of the ARC is appealing and seems to be straightforward, the technical underpinnings have not received much formal attention. The attention that the ARC has received, however, is often misguided and surrounded by confusion and misinterpretation. It is believed that the ARC will help researchers and the research community in general in the quest of understanding the dynamic and static relationships among a set of variables as they exist over time.

DERIVATION OF THE AVERAGE RATE OF CHANGE

The present section details the precise mathematical definition of the ARC by a set of mathematical derivations. The derivations presented define the ARC without ambiguity and illustrate in a longitudinal data analytic context a well-known fact of the mathematical sciences. It will be shown that regardless of whether the functional form of growth is known or unknown, and regardless of whether time is measured continuously or at discrete occasions, the same definition of the ARC holds. To ease the transition between the mathematical statements given in this section and their application to growth models in future sections, the symbolic representations within this section will make use of the corresponding analysis of change notation.

The rate of change of a nonvertical line that passes through two sets of points, (T_t, Y_t) and $(T_{t+\tau}, Y_{t+\tau})$, is the slope of the line, where T represents time, t the measurement occasion ($t = 1, \dots, F$), τ is some arbitrary, yet constant once defined, length of time between measurement occasions, and Y_t is the dependent variable at the t th measurement occasion. The slope of the line connecting two points is the change in Y divided by the change in time, and can be represented by the following expression:

$$\text{Slope} = \frac{f(T_{t+\tau}) - f(T_t)}{T_{t+\tau} - T_t} = \frac{Y_{t+\tau} - Y_t}{T_{t+\tau} - T_t} = \frac{\Delta Y}{\Delta T}, \quad (1)$$

where $f(T)$ is the dependent variable Y , which is some function of time, and Δx is the change in the variable x (where x represents some random variable).

Equation 1 is closely related to the derivative. In the limit as τ approaches zero, Equation 1 yields the *instantaneous* rate of change when evaluated at a specific time value. The instantaneous rate of change can also be conceptualized as the slope of

the tangent line to the function at the specified value of time. A tangent line is a line that adjoins, but does not intersect, the curve of the function at the specified value of time. The instantaneous rate of change, or slope of the tangent line, at a specified value of time is literally the derivative of the function evaluated at the particular time value. The derivative of a function can be written as follows:

$$\frac{dY}{dT} = \lim_{\tau \rightarrow 0} \frac{f(T_{t+\tau}) - f(T_t)}{\tau} = f'(T), \quad (2)$$

where $\frac{dY}{dT}$ is read as the derivative of Y with respect to time, which can be represented as $f'(T)$. The notation used to represent derivatives throughout the work will be $f'(T)$, as this notation shows explicitly that the derivative of the function is contingent upon time. The ARC can thus be conceptualized as the mean instantaneous rate of change (i.e., derivative), or as the mean slope of all possible tangent lines, over some time interval of interest.

Figure 1 illustrates visually the content of the preceding paragraph. The dark line represents a hypothetical growth trajectory over time. The midpoint of the 21 tangent lines adjoins the growth curve at the point identified by the circles. The slope of each tangent line is equal to the derivative of the function evaluated at the particular time value indicated by the circle. Thus, the slope of each tangent line is the instantaneous rate of change of the growth curve at the specified time value. The ARC is literally the mean slope of all possible tangent lines. For continuous functions that are differentiable there are an infinite number of tangent lines adjoining the curve. Figure 1 uses a small number of the infinitely many tangent lines in order to facilitate the understanding of the ARC.

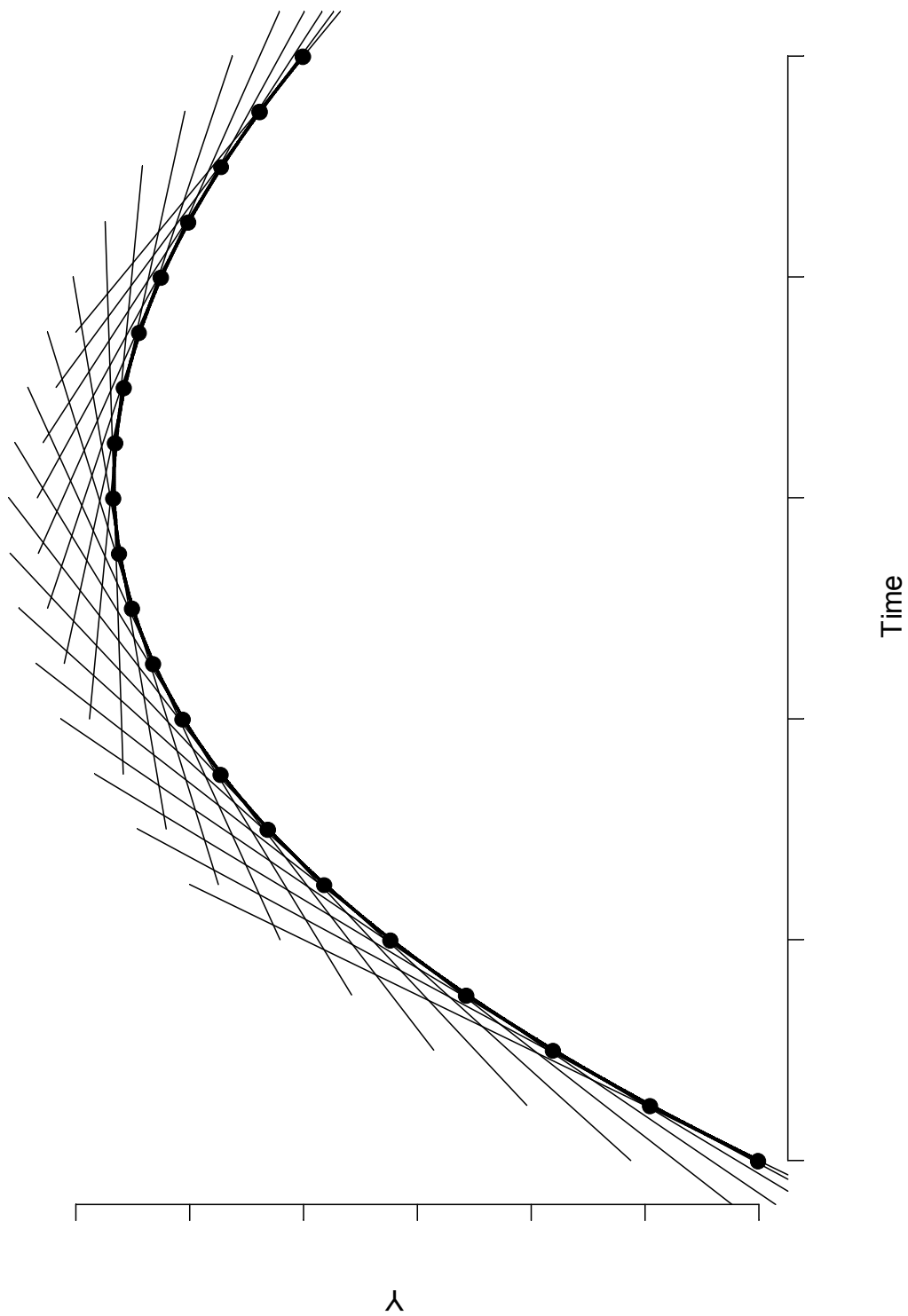


Figure 1. Illustration of tangent lines to the growth curve at specified points. The ARC is the mean slope of all theoretically possible tangent lines to the function within a particular interval of time.

Since the derivative evaluated at a specific timepoint is the instantaneous rate of change at that timepoint, and interest lies in finding the mean of the set of derivatives, accomplishing such a task seems to require evaluating Equation 2 at each of the values of time and then dividing the sum of each of the derivatives by F . However, there are at least two seemingly unavoidable problems with attempting to calculate the ARC in such a manner. The first problem is that derivatives are defined only for continuous functions that are differentiable, that is, where $\tau \rightarrow 0$ and $F \rightarrow \infty$ (F approaches infinity). The second problem is that in practice the functional form of growth is generally unknown. Because the derivative itself is based on the functional form, the derivative is also unknown. Although these complications seem unavoidable, the following section illustrates that they are overcome with what turns out to be a simple algebraic solution.

The Mathematical Definition of the Average Rate of Change

Because the true functional form of growth is generally assumed to exist continuously over time, deriving the mean of an infinite number of derivatives requires integral calculus. The integral of a continuous function is a limit of summations. After finding the infinite sum of derivatives for some function, obtained by integrating the derivative of the functional form of growth, the infinite sum must be divided by the width of the time interval in order to obtain the mean of the derivatives. The rationale for such a procedure is due to the Mean Value Theorem for Integrals (See section 4.5 of Finney, Weir, & Giordano, 2001 or section 6.5 of Stewart, 1998, for a thorough treatment of the Mean Value Theorem for Integrals.).

The Mean Value Theorem for Integrals states that over a closed interval a continuous function assumes its average (i.e., the mean) value at least once within the interval. The particular mean value for a continuous function that is

differentiable over the interval T_t to $T_{t+\tau}$ is given by an application of the Mean Value Theorem for Integrals:

$$\bar{f}_c = \frac{1}{T_{t+\tau} - T_t} \int_{T_t}^{T_{t+\tau}} f(T) dT, \quad (3)$$

where $\bar{x}$ represents the mean of x , in this case a continuous function that is differentiable, and $\int_{T_t}^{T_{t+\tau}} f(T) dT$ is read as the integral of $f(T)$ with respect to time from T_t to $T_{t+\tau}$ (Finney et al., 2001, p. 352; Stewart, 1998, p. 470). Thus, after the function has been integrated, the value of the integral is divided by the length of the time interval in order to obtain the mean value of the function.³ Since Equation 3 yields the mean of a continuous differentiable function, and Equation 2 is a special case of a continuous function, combining the two equations will yield the mean instantaneous rate of change (i.e., mean mean derivative or mean slope of all possible tangent lines) of the function from T_t to $T_{t+\tau}$. Thus, when Equations 2 and 3 are combined, the resultant value is the ARC.

However, to integrate a function is to find its antiderivative. Finding the derivative of a function (as in Equation 2) and then integrating the function (to find its antiderivative) yields the original function itself and is a corollary of the fundamental theorem of calculus (Kline, 1977, p. 258). An example of an integral of a derivative is given as,

$$\int f'(T) dT = f(T) + C, \quad (4)$$

³In the present situation the Mean Value Theorem for Integrals guarantees that for integrable functions there is at least one value of time where the instantaneous rate of change equals the mean of the instantaneous rates of change within the time interval of interest. Furthermore, Equation 3 follows the same formulation as the expected value of a uniform probability distribution. In fact, a uniform probability distribution can be thought of as a special case of a continuous function, where all values on the abscissa have the same corresponding value of the ordinate (i.e., occur with the same probability).

where C is the constant of integration and the resultant value is the function itself.⁴ Therefore, when Equations 2 and 3 are combined, the mean of the derivatives, which is literally the ARC, can be written as the following:

$$\overline{f'(T)} = \frac{1}{T_{t+\tau} - T_t} \int_{T_t}^{T_{t+\tau}} f'(T) dT \quad (5a)$$

$$= \frac{f(T_{t+\tau}) - f(T_t)}{T_{t+\tau} - T_t} \quad (5b)$$

$$= \frac{\Delta Y}{\Delta T}. \quad (5c)$$

As can be seen in Equation 5c, the ARC is simply the change in Y divided by the change in time. The resultant formulation of Equation 5c is a well-known mathematical fact of analytic calculus (e.g., Stewart, 1998, pp. 146-147 and p. 208; Finney et al., 2001, pp. 86-88), where it is known that regardless of the function, the mean of all of the derivatives evaluated over a specified interval must equal $\frac{\Delta Y}{\Delta T}$.

While the definition of the ARC may at first seem simplistic, it is important to understand why this is the mathematical definition. The Total Change Theorem states that “the integral of a rate of change is the total change” (Stewart, 1998, p. 377). Thus, when the integral of the derivative is evaluated in Equation 5a, it leads to the numerator of Equation 5c, which is $(Y_{t+\tau} - Y_t)$, the total change. This total change is then divided by the length of the time interval, as specified by Equation 3, in order to obtain the mean. Thus, the mean of the infinitely many instantaneous rates of change (i.e., the ARC) is the change in the dependent variable divided by the change in time. Notice that the true functional form of growth was never specified. Equation 5c holds regardless of whether the functional form is known or unknown, as only the initial and final pairs of points are required. Although the ARC was

⁴Finding the integral (antiderivative) of a function actually defines a family of functions that differ by at most a constant value (Finney et al., 2001, p. 314). Further, Equation 4 is technically an indefinite integral because it yields another equation, not a numeric value. An integral with real valued limits that yields a numeric value (not another equation) is a definite integral.

defined in the case where time was continuous, the same formulation holds true in the more typical case where the occasions of measurement are discrete, regardless of whether the occasions of measurement are equally spaced.

Although Equation 5c is a well-known fact of analytic calculus, in the context of longitudinal data analysis the mathematics underlying the ARC have not been well delineated. Because of the lack of attention to the ARC, but yet its intuitive appeal as the mean instantaneous rate of change, the ARC has often been misunderstood in practice. A major purpose of this work is to clarify misconceptions that persist both implicitly and explicitly throughout the methodological and applied longitudinal literature regarding the ARC.

STATISTICAL MODELS OF INDIVIDUAL GROWTH

Before further delineation of the ARC in the context of longitudinal data analysis, a necessary digression provides an overview of statistical models useful for describing individual growth curves. This digression provides a broad context for the ARC as well as elucidating a variety of growth models not often discussed or considered in applications of longitudinal data analyses within the behavioral sciences.

Statistical models that examine growth or change explicitly as a function of time are known as growth curve models. In addition to one or more time-varying covariates, growth curve models may or may not include fixed covariates. Throughout the remainder of the work the only time-varying covariate explicitly included in the growth model will be time itself (However, the discussion is equally applicable to other time-varying covariates, such as age, grade-level, etc.). A variety of methods can be used for modeling growth over time, with an important common theme among them being that the individual is explicitly incorporated into the model. In the following two sections, two classes of intraindividual growth will be described. The first class of growth models are those commonly employed in the behavioral sciences, where the growth models are linear in their parameters and consist of a limited set of polynomial trends. The second class of growth models illustrates three nonlinear models that are likely a better approximation to reality in some situations than a limited set of polynomial trends, as these models allow asymptotic values to be explicitly included in the model.

Throughout the work, Y_{it} is the dependent variable for the i th individual ($i = 1, \dots, N$) at the t th timepoint ($t = 1, \dots, F$). Unless otherwise specified, it is

assumed that the observed data are completely balanced and that the occasions of measurement are equally spaced with the same starting point and constant time-lag (τ) within and across individuals. Such a data set implies that all N individuals have the same starting value, no missing data, F measurement occasions, and constant τ both within and across individuals. Thus, all of the N individuals have a common set (i.e., vector) of time values.

Polynomial Models for the Analysis of Change

Polynomial growth models are the most common and straightforward statistical models for fitting growth curves to data. Growth models in the polynomial family can be conceptualized as a function of a systematic growth trajectory and random error. An example of a polynomial growth curve of degree K is given as,

$$Y_{it} = \pi_{0i} + \pi_{1i}T_{it} + \pi_{2i}T_{it}^2 + \cdots + \pi_{Ki}T_{it}^K + \varepsilon_{it}, \quad (6)$$

where π_{ki} is the k th growth rate parameter ($k = 0, \dots, K$) for individual i , and ε_{it} is the individual's random error, generally assumed normally distributed about zero with a constant variance across time.

While Equation 6 illustrates the intraindividual model of growth, each of the K polynomial growth parameters can themselves be modeled in an interindividual (between individual) fashion. That is, each of the parameters in Equation 6, known as the Level 1 model, are the dependent variables in the interindividual Level 2 model. For example, a growth parameter in Equation 6 can be modeled as,

$$\pi_{ki} = \beta_{k0} + \beta_{k1}X_{k1i} + \beta_{k2}X_{k2i} + \cdots + \beta_{kP}X_{kPi} + u_{ki}, \quad (7)$$

where there is one constant, β_{k0} (when $p = 0$), with p ($p = 0, \dots, P$) representing the particular X variable, β_{kp} being the overall effect of X_{kp} on the k th growth

parameter, and u_{ki} is the unique effect for the i th individual's k th trend, generally assumed normally distributed about zero (Bryk & Raudenbush, 1987).

Equation 6 is a special case of a linear model. A linear model is one that is linear in its parameters, not necessarily in the predictors (“predictors” can be thought of as covariates or independent variables). The π s in Equation 6 and the β s in Equation 7 are the parameters (each of which is raised only to the first power in linear models) while $T_{it}, T_{it}^2, \dots, T_{it}^K$ and $X_{k1i}, X_{k2i}, \dots, X_{KPi}$ represent the predictors. Notice that there are various powers of time included in the model. When nonzero weights are given to powers of time other than one, the predicted growth curve will not be a straight-line. Thus, a linear model and a linear trend (straight-line) are not synonymous. In theory, a linear model of growth can be of any shape as long as there are enough additive effects of powers of time included in the model. The next section extends linear models in order to allow non-additive effect parameters to be included in the growth model.

Nonlinear Growth Models for the Analysis of Change

Statistical models that are linear in their parameters are generally straightforward to fit given a set of observed data. As the phenomenon under study grows increasingly more complex, the order of the polynomial growth model can be increased accordingly until the predicted scores reasonably correspond with the observed scores. Nonlinear models of the same complex phenomenon can oftentimes be more interpretable, parsimonious, and are generally more valid beyond the observed range of data when compared to linear models (Pinheiro & Bates, 2000, p. 273). Furthermore, it is often the case that the parameters in nonlinear models can be easily interpreted, while once a polynomial model is beyond quadratic, the meaning of the set of polynomial trends typically offers little physical or behavioral

interpretation. An example of such a difference between nonlinear and linear models relates to asymptotes.

In polynomial growth models, limiting asymptotic values cannot generally be modeled in order for the asymptotic value to hold beyond the range of the observed data. Thus, researchers who make use of polynomial trends must accept the fact that their model will likely fail beyond the range of the data actually collected.⁵ This is not to say that researchers who use nonlinear models can haphazardly extrapolate beyond their data, however, if the phenomenon truly asymptotes linear models will generally fail to take this into consideration outside the range of observed values. Such scenarios can potentially lead to inadequate models where impossible values occur (e.g., probabilities larger than 1 or smaller than 0, negative reaction times, predicted scores higher than the admissible range, unlimited growth or decay, etc.).

In order to demonstrate problems that arise when data truly follow nonlinear trends but yet are modeled by straight-line growth models, three nonlinear growth models will be illustrated. The selected nonlinear models are the asymptotic regression growth curve, the Gompertz growth curve, and the logistic growth curve. Although a wide variety of nonlinear models exist, the asymptotic regression, Gompertz, and logistic growth curves were chosen because they seemed most useful for behavioral science research. A brief introduction to each of these models is given followed by some potential applications of nonlinear models in the behavioral sciences.

The Asymptotic Regression Growth Curve

The general asymptotic regression growth curve (often referred to as exponential growth or decay) describes a family of potential regression models where the

⁵However, the point at which the polynomial growth model eventually fails may be beyond the range of theoretical interest. In such situations there may be a less compelling argument in favor of nonlinear models because of the benefits they provide regarding asymptotic values.

dependent variable asymptotes to some limiting value as time increases. A general asymptotic regression equation for a single trajectory is given by Stevens (1951) as:

$$Y_t = \alpha + \beta \rho^{T_t} + \varepsilon_t, \quad (8)$$

where α is the asymptotic value approached as $T \rightarrow \infty$, β is the change in Y_t from $T = 0$ to $T \rightarrow \infty$ (i.e., β represents total change in Y), and ρ ($0 < \rho < 1$) is a scaler that defines the factor by which the deviation between Y_t and α is reduced for each unit change of time, thus reflecting the rate at which $Y_t \rightarrow \alpha$.⁶ Equation 8 can be equivalently rewritten, such that it is explicitly expressed as an exponential equation:

$$Y_t = \alpha + \beta \exp(-\gamma T_t) + \varepsilon_t, \quad (9)$$

where $\gamma = -\log(\rho)$ ($0 < \gamma < \infty$) and can be thought of as a scaling parameter (Stevens, 1951).

In order to facilitate the discussion of the asymptotic regression model, a graphical depiction of a “typical” asymptotic regression curve is presented in Figure 2. The graphical depiction is said to be typical, even though there are an infinite number of asymptotic growth curves, because the overall shape of the curve presented in Figure 2 shows the general characteristic of asymptotic growth.⁷ The dark line represents the asymptotic growth function over the time interval zero to one ($T \in [0, 1]$) with parameter values $\alpha = 5$, $\beta = 4$, and $\gamma = 8$.

⁶The deviation between Y_t and α can be expressed as $\beta \rho^{T_t}$. When time changes in unit steps, ρ is literally the factor by which the deviation is reduced from one step of time to the next step.

⁷Note that the particular asymptotic regression model plotted in Figure 2 illustrates asymptotic decay. Holding everything else constant, changing the sign of the β in Figure 2 to negative would illustrate asymptotic growth.

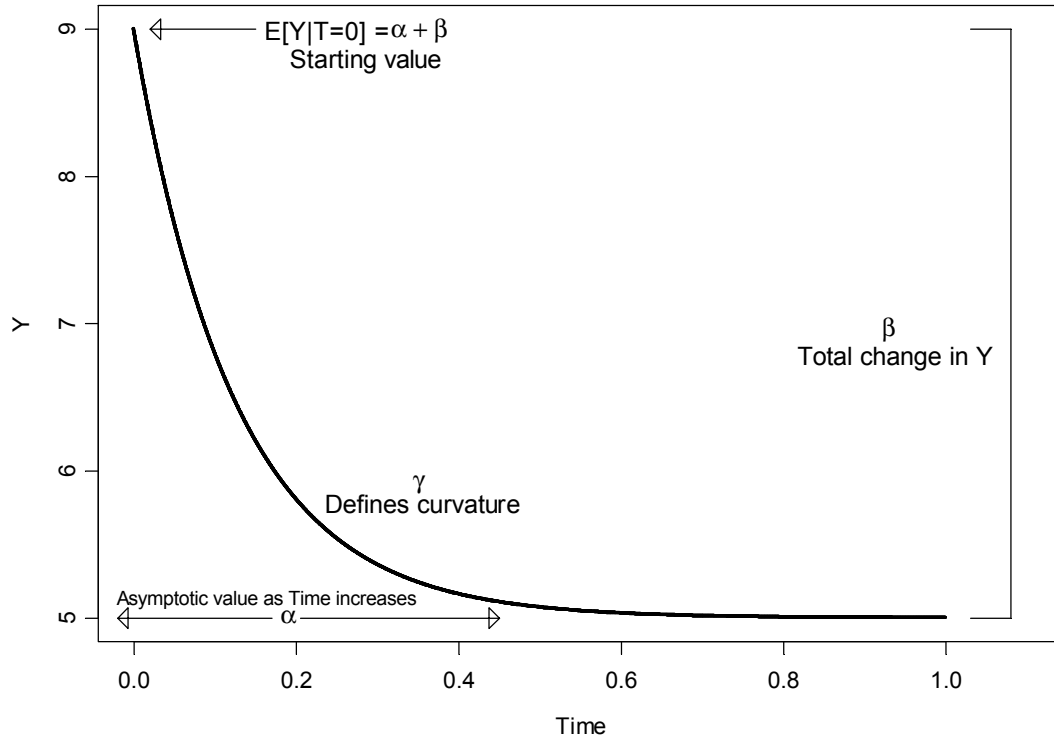


Figure 2. Illustration of a typical asymptotic regression model where the meaningfulness and direct interpretation of the parameters is illustrated

Although not of direct interest in the present work is the fact that the basic structure of Equation 9 is general and can be reparameterized into a variety of other models (Ratkowsky, 1983, chapter 5) leading to other families of growth models (Stevens, 1951). Given that “we meet [special cases of Equation 9] repeatedly in every branch of science” (Stevens, 1951, p. 248), it is important to understand how the asymptotic regression model provides a viable option for growth modeling and how common growth models may misrepresent asymptotic growth.

The Gompertz Growth Curve

The Gompertz growth model is a nonlinear model that is often used in the biological sciences. The asymmetric sigmoidal (“S” shape) form of the Gompertz growth offers an interesting option for those who seek to model certain types of nonlinear trends. The general three parameter Gompertz growth model for a single trajectory at time t can be written as:

$$Y_t = \alpha \exp(-\exp(\beta - \gamma T_t)) + \varepsilon_t, \quad (10)$$

where α is the asymptote as $T \rightarrow \infty$ ($Y \rightarrow 0$ as $T \rightarrow -\infty$). The parameters β and γ define the point of inflection on the abscissa at $T = \frac{\beta}{\gamma}$. The point of inflection on the ordinate is at $Y = \frac{\alpha}{\exp(1)}$, which is approximately 37 percent of the asymptotic growth (Ratkowsky, 1983, chapter 4 and pp. 163-167; Winsor, 1932).

In order to facilitate the discussion of the Gompertz growth model, as was done with the asymptotic regression model, a graphical depiction of a typical Gompertz curve is presented in Figure 3. The dark line represents the Gompertz growth function for the same parameters specified for the asymptotic regression model, where $T \in [0, 1]$ for parameters $\alpha = 5$, $\beta = 4$, and $\gamma = 8$.

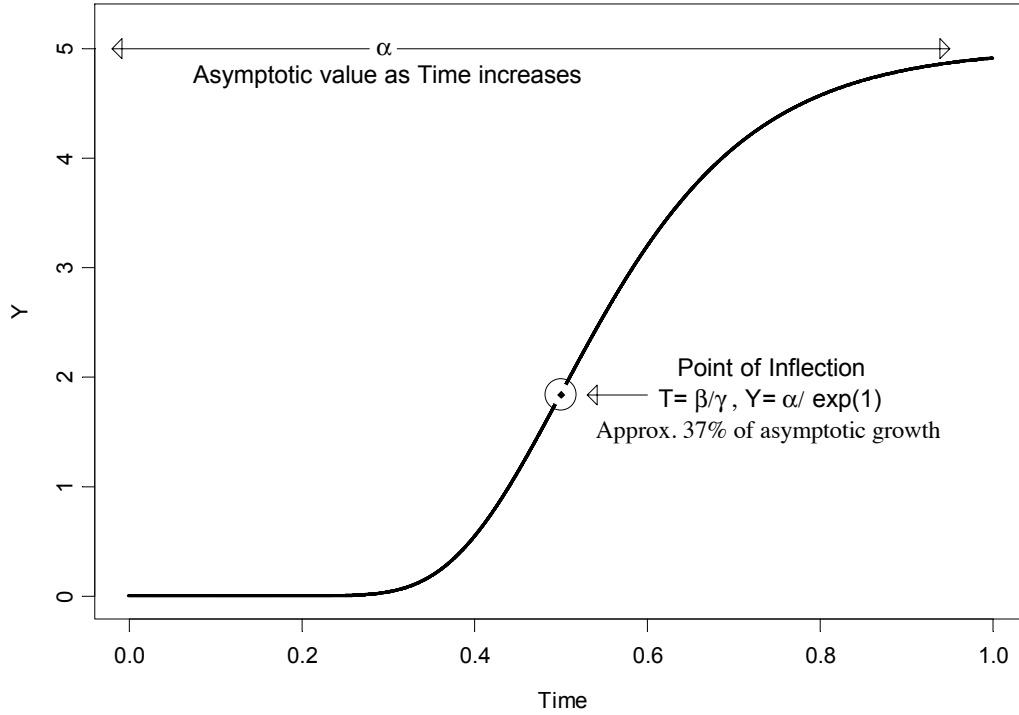


Figure 3. Illustration of a typical Gompertz growth model where the meaningfulness and interpretation of the parameters is illustrated

Although not of interest for the present work, Equation 10 can be transformed such that the transformed dependent variable can be expressed as a linear equation:

$$\log \left(-\log \left(\frac{Y_t}{\alpha} \right) \right) = \beta - \gamma T_t + \varepsilon_t. \quad (11)$$

The parameters β and γ can then be conceptualized as the intercept and the slope of the transformed dependent variable respectively. Even though the Gompertz growth model can be transformed to a simple linear model, doing so generally does not lead to any meaningful interpretation. The reason transformations such as that given in Equation 11 should not be used in place of an appropriate nonlinear model is because the dependent variable (i.e., $\log \left(-\log \left(\frac{Y_t}{\alpha} \right) \right)$) generally does not represent any real-world phenomenon. Thus, obtaining an estimate of the slope and intercept

is oftentimes of little or no interest. It is interesting to note, however, that by taking the exponential of Equation 8, the resultant model is a form of Gompertz growth (Stevens, 1951, p. 249).

The Logistic Growth Curve

The logistic growth model is another nonlinear sigmoidal model that shows promise for modeling growth over time in the behavioral sciences. The general three parameter logistic growth model for a single trajectory at time t can be written as:

$$Y_t = \frac{\alpha}{1 + \exp(\beta - \gamma T_t)} + \varepsilon_t, \quad (12)$$

where α is the asymptote as $T \rightarrow \infty$ ($Y \rightarrow 0$ as $T \rightarrow -\infty$). The parameters β and γ define the point of inflection on the abscissa at $T = \frac{\beta}{\gamma}$. The point of inflection on the ordinate is at $Y = \frac{\alpha}{2}$, 50 percent of the asymptotic growth (chapter 4 and pp. 167-169 of Ratkowsky, 1983 and Winsor, 1932).

In order to facilitate the discussion of the logistic growth model, as was done with the asymptotic regression and Gompertz growth models, a graphical depiction of a typical logistic curve is presented in Figure 4. The dark line represents the logistic growth function for the same parameters specified for the asymptotic regression and Gompertz growth models, where the $T \in [0, 1]$ for parameters $\alpha = 5$, $\beta = 4$, and $\gamma = 8$.

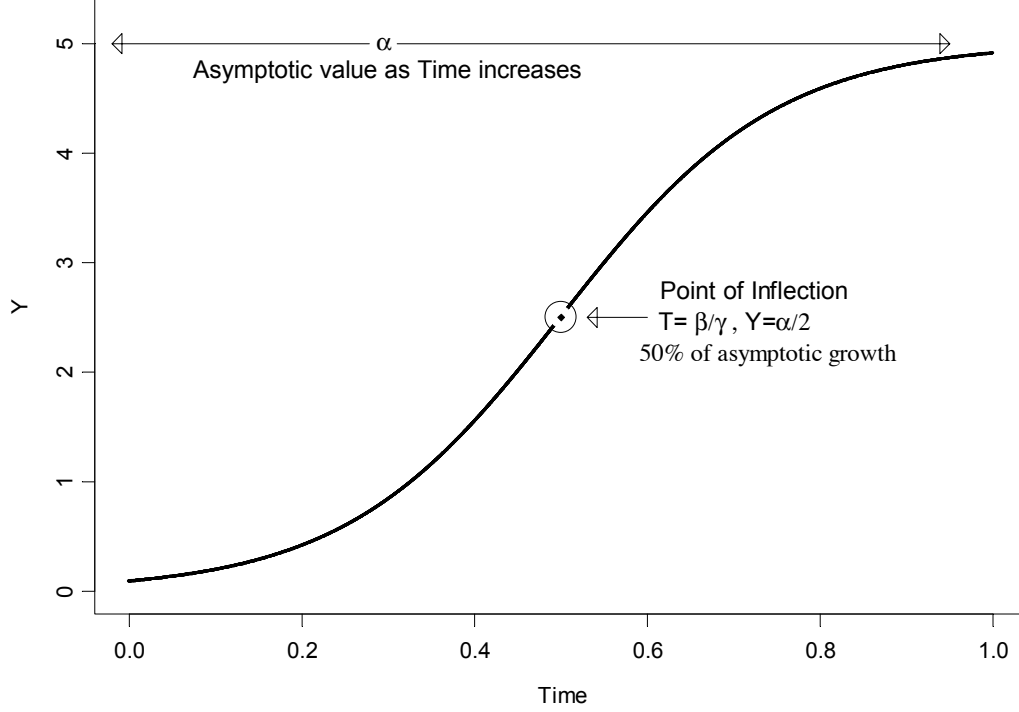


Figure 4. Illustration of a typical logistic growth model where the meaningfulness and interpretation of the parameters is illustrated

Equation 12 can be transformed such that the dependent variable can be expressed as a linear equation:

$$\log \left(\frac{\alpha}{Y_t} - 1 \right) = \beta - \gamma T_t + \varepsilon_t. \quad (13)$$

The parameters β and γ can then be conceptualized as the intercept and slope of the transformed dependent variable respectively. However, as was the case with the Gompertz growth model, such transformations should generally not be used in place of the appropriate nonlinear model, as the transformed dependent variable for the logistic growth curve leads to a dependent variable (i.e., $\log \left(\frac{\alpha}{Y_t} - 1 \right)$) that is not likely to represent any real-world phenomenon. The major difference between

the Gompertz and the logistic growth curves is that the logistic is symmetric about its point of inflection (50% of the asymptotic value) whereas the Gompertz is not (approximately 37% of the asymptotic value). It is interesting to note that by taking the inverse of Equation 8, the resultant model is a form of logistic growth (Stevens, 1951, p. 249).

Nonlinear Models in the Behavioral Sciences

Given the three classes of nonlinear models that have been introduced, it is beneficial to relate their functional forms to phenomena encountered in the behavioral sciences. When limits on some behavior or action exist, ability for example, nonlinear models will likely offer more realistic representations of reality than do linear models. As Cudeck (1996) states referring to human behavior, “many responses are inherently nonlinear and cannot be treated by a linear mixed [i.e., hierarchical linear] model” (p. 372). For example, in a methodological paper Hartz, Ben-Shahar, and Tyler (2001) illustrated the benefits of using a logistic growth curve for modeling associative learning data. From a classical conditioning paradigm of learning and memory, interest was in modeling the proportion of times each trained honey bee would associate a reward of sugar with a specific training odor. The proportion of bees correctly responding to their respective training odor was originally small, increased rapidly, and then reached an asymptote near 100 percent as the study continued.

The use of sigmoidal curves applied to animal research is not uncommon. In fact, modeling biological processes has played a large part in the motivation, development, application, and use of nonlinear models (See Davidian & Giltinan, 1995, and Pinheiro & Bates, 2000, for examples of applications of nonlinear models in the biological and medical sciences.). The behavioral sciences, however, have

been slower to make the transition from various extensions of the general linear model to models that are explicitly nonlinear in their parameters. With the recent emphasis some psychologists have placed on prescribing medication, particularly in New Mexico where a law was recently passed (New Mexico House Bill 170 took effect in July of 2002) allowing qualified psychologists to prescribe medication for certain mental illnesses, dose response curves will likely be a new area of interest and research activity for psychologists. Because such dose response curves are generally sigmoidal in nature, it seems likely that psychologists will soon make better use of sigmoidal models in both practice and research, as such sigmoidal relationships cannot generally be satisfactorily modeled by linear models.

Van Gert (1991) provides a powerful argument for taking seriously the notion of applying nonlinear models. Van Gert argues that a variation of the logistic growth function “applies to all—or at least a very significant majority—of the variables involved in cognitive growth processes” (p. 45).⁸ Van Gert contends that cognitive processes occur under the constraints of limited resources and that these constraints need to be explicit in models of growth. Models that are linear in their parameters, typically the ones used to model growth, are usually untenable realizations of the phenomenon of interest because there is no constraint on growth. For example, as time increases the model may predict erratic and unrealistic growth. Such untenable characteristics of unconstrained models are evidenced by linear models that continuously “grow” or “decay” as time increases.

In the context of latent variable models, Browne and Du Toit (1991) present three different model formulations for data on learning with the goal of isolating interindividual differences in intraindividual learning characteristics and to discern

⁸Van Gert speaks of a logistic growth curve where there is a “forgetting” parameter included (his Equation 17a') and then goes on to generalize the logistic growth function such that it has the potential to allow dynamic components to be considered.

the effects of a covariate on this relationship. Browne and Du Toit make use of the Gompertz growth curve of Equation 10 for each of the model formulations but state that the exponential (a special case of Equation 8) and logistic curves (Equation 12) may also be suitable (p. 56). Using such nonlinear models seems reasonable in the sense that learning is not an unlimited cognitive process (Van Gert, 1991), but yet one that changes little after the task has been mastered and tends to level off at some asymptotic value (Browne & Du Toit, 1991, pp. 57–59).

Psychophysics is one area within behavioral science that commonly makes use of nonlinear models. For example, Fechner’s Law uses logarithmic functions for models of indirect scaling, while Steven’s power law uses exponential functions to model magnitude estimation (Coren, Ward, & Enns, 1994, p. 51). While there may well be a plethora of phenomena that follow nonlinear patterns, which may exhibit one or two asymptotic values, some examples that may follow nonlinear functional forms are offered: therapy progression as a function of the number of sessions, performance over time on a vigilance (sustained attention) task, auditory sensitivity as a function of age (or distance from the source), group productivity as a function of time spent on task (or group cohesion), visual adaptation as a function of time in a dark environment, marital satisfaction as a function of time married (or since the relationship began), motivation as a function of time spent on task (or complexity), and perceptual ability as a function of age. As researchers’ questions become more sophisticated, it is likely that growth over time for some phenomena mentioned will be modeled and tested with functional forms nonlinear in their parameters, such that asymptomatic values can be explicitly considered.

Although nonlinear models offer a valuable tool for modeling behavioral phenomena, in some instances, a researcher’s question may pertain to an area within a range of a nonlinear function where the relationship between time and

the dependent variable is essentially linear.⁹ For example, in Figure 2 there appears to be a near linear relationship within the interval $T \in [0, .2]$ and $T \in [.6, 1]$. Thus, had a researcher been interested in a phenomenon whose functional form over time is governed by the parameters of Figure 2, he or she would likely conclude that the straight-line growth model provides a satisfactory representation of reality if data were collected within the interval $T \in [0, .2]$ or $T \in [.6, 1]$. In fact, it could be argued that fitting an asymptotic regression model (i.e., the true functional form) over the near linear intervals, but not both simultaneously, would be unnecessary as over such a limited range the trajectory does not markedly diverge from a straight-line.

When an approximate linear relationship exists over a limited range within a model whose true functional form is something other than a straight-line, such a range is said to be locally linear. Researchers who study relationships over time within a range that is seemingly linear may not realize that beyond the range of interest the functional form is governed by a function more sophisticated than a straight-line. Researchers in such situations would likely make use of the straight-line growth model, even though the true (but yet unknown) model is actually something other than a straight-line. The straight-line growth model used within a limited range of time for some nonlinear relationship where local linearity persists will generally yield meaningful results. Although a straight-line model may technically be incorrect, “all models are wrong but some are useful” (Box, 1979, p. 202). In general, fitting the straight-line growth model to a limited range within some nonlinear growth, specifically where the relationship is approximately locally

⁹While the difference between linear and nonlinear models was previously made clear, it should be noted that a linear relationship implies a straight-line model. A model that is linear in its parameters does not necessitate a straight-line model. However, when a straight-line model is used, it implies a linear relationship. Thus, the terms linear relationship (i.e., a straight-line model) and linear model (where parameters are linear) should not be confused.

linear, likely provides “useful” information about the phenomenon under study.¹⁰ Local linearity is a topic that nicely illustrates potential problems that can arise when extrapolation beyond the range of data is carried out. While the relationship between time and the dependent variable may be locally linear, values of the dependent variable may diverge sharply from such an apparent linear relationship just beyond the range of collected data. In situations where local linearity persists, provided the entire time interval of interest exhibits local linearity, the need for nonlinear models may be less pressing.¹¹

Although there may be numerous behavioral phenomena that follow nonlinear functional forms, the straight-line growth model seems to be used more than any other growth model. One reason the straight-line growth model is so often used is because researchers would like to describe growth in a parsimonious way, often by talking about the “average” rate or amount of change. The “average” descriptor is often the regression coefficient from the straight line growth model. For this reason the next section explores the relationship between the straight-line growth model and the ARC.

¹⁰However, such a statement rests on the assumption that there truly is a near linear relationship between time and Y . To the extent that this is not true, making use of the straight-line growth model may provide misleading information about change.

¹¹The concept of local linearity can be extended to local quadrature. In such a case, the relationship between time and the dependent variable may be nonlinear, but the relationship between time and the dependent variable may be essentially quadratic. Of course local quadrature can be extended to local cubature. However, once a relationship is beyond quadratic, the interpretation of such a polynomial growth model is difficult. Thus, rather than thinking of a relationship as locally cubic, for example, a nonlinear model should be considered.

RELATIONSHIP BETWEEN STRAIGHT-LINE GROWTH MODELS AND THE AVERAGE RATE OF CHANGE

Due to the hierarchical structure of longitudinal data (scores over time nested within person, who in turn may be nested within group), special statistical models are required that take into consideration the nonindependence of the hierarchically structured data. HLMs and HNLMs explicitly model the hierarchical structure of nested data and allow for nonconstant time-lags within and across individuals, and some types of missing data (i.e., the design need not be completely balanced nor occasions of measurement equally spaced within or across individuals; See Raudenbush & Bryk, 2002, Davidian & Giltinan, 1995, or Goldstein, 1995, for a thorough treatment of these issues.).

The most common method of analyzing an individual's trajectory is through the generalization of the polynomial growth model (illustrated in Equation 6) into a HLM. Growth models linear in their parameters allow various polynomial trends to be specified and then tested against other competing models. Given an observed set of data, provided a sufficient number of polynomial trends are specified, (at the expense of degrees of freedom) the growth model can be made to accurately represent the data. This desirable property, combined with the relative ease of calculation, has made the HLM of polynomial growth essentially the model of choice for analyzing individual change over time in the behavioral sciences. However, a caution is warranted because by adding additional polynomial trends to a growth model, the sum of squared deviations between the predicted scores and the observed scores will necessarily decrease (or at the very least stay the same). In fact, as the

number of polynomial trends approaches the number of timepoints, the sum of squared deviations between predicted and observed scores approaches zero. One wants to avoid overparameterization in growth models, otherwise the model will account for measurement error in addition to the true relationship (Box, 1984).

The general HLM for the i th individual's set of scores can be given as,

$$\mathbf{Y}_i = \mathbf{X}_i\boldsymbol{\beta} + \mathbf{Z}_i\mathbf{U}_i + \boldsymbol{\varepsilon}_i, \quad (14)$$

where $\boldsymbol{\beta}$ is the vector of unknown population parameters linked to the vector $\mathbf{Y}_i$ by the design matrix $\mathbf{X}_i$, $\mathbf{U}_i$ is a matrix of unknown unique individual effects linked to $\mathbf{Y}_i$ by the design matrix $\mathbf{Z}_i$, and $\boldsymbol{\varepsilon}_i$ is a vector of errors generally assumed to be normally distributed about a mean of zero with a constant variance across time (Laird & Ware, 1982). This general HLM formulation allows for the desired polynomial function(s) of time to be included in the model, as well as other time-varying and fixed covariates. Furthermore, models having the form of Equation 14 offer great flexibility in terms of model testing and model comparisons.

A straight-line HLM of growth for individual i , a special case of Equations 6 and 14, can be represented by the following growth model:

$$Y_{it} = \pi_{0i} + \pi_{1i}T_{it} + \varepsilon_{it}. \quad (15)$$

As illustrated in Equation 7, the parameters of Equation 15 are themselves modeled as dependent variables in the following manner:

$$\pi_{0i} = \beta_{00} + u_{0i}, \quad (16a)$$

$$\pi_{1i} = \beta_{10} + u_{1i}, \quad (16b)$$

where β_{00} is the mean of the individual intercepts (i.e., $E[Y|T = 0]$), β_{10} is the mean of the individual slopes (rate of change) across the N individuals, and u_{0i} and u_{1i}

represent the unique effects associated with the i th individual's intercept and slope parameter respectively.¹² By combining Equations 15, 16a, and 16b, the full HLM model for the i th individual's straight-line growth model can be rewritten as follows:

$$Y_{it} = \beta_{00} + u_{0i} + (\beta_{10} + u_{1i})T_{it} + \varepsilon_{it}. \quad (17)$$

While the fixed effect parameters (β_{00} & β_{10}) of Equation 17 define the overall growth model, the unique effects (u_{0i} & u_{1i}) lead directly to the population covariance matrix of the unique components and the reliability of the sample estimates. In the special case of zero variance for the unique effects of a particular fixed effect parameter (equivalent to a set of parameter estimates with zero reliability), the parameter is said to be constant across individuals and a unique term need not be included in the model. For example, if the variance of u_{0i} in Equation 16a was zero, the implication is that all N individuals have the same intercept (and thus are indistinguishable from one another, which is why the reliability would be zero in the case of zero variance for a random component).

In the context of growth models where each of the N individuals share a common design matrix for the unique effects ($\mathbf{Z}_i$ from Equation 14 equals $\mathbf{Z}$ for all N individuals), the mean of the ordinary least squares (OLS) regression coefficients are equivalent to the fixed effects of the HLM model (see Laird & Ware, 1982, p. 966, for technical details). In the context of the straight-line growth model of Equation 15, a common design matrix of the unique effects implies a common set (i.e., vector) of time values across each of the N individuals. When the previously stated assumptions of the work are satisfied (i.e., those outlined on pages 15 & 16), each of the N individuals will share a common vector for the time values. Thus, the β s in Equation 17 will be equivalent to the mean of the OLS estimates across

¹²Equation 15 is an unconditional model. An unconditional model implies that there are no Level 2 predictors, and thus the dependent variable is not *conditioned* upon any Level 2 predictors.

individuals. That is, if OLS regression analyses were performed for each of the N individuals, the mean of the estimated intercepts and slopes would correspond to the estimated fixed effects calculated via the HLM growth model. Thus, the mean of the N intercepts and the mean of the N slopes would estimate β_{00} and β_{10} in Equation 17. Because the estimated fixed effects of the HLM model are equal to the mean OLS estimates in the present context, in order to make the discussion more comprehensible and generalizable, the remainder of the work focuses specifically on the OLS estimates of a single trajectory. The HLM regression model of straight-line growth for a specific individual thus simplifies to the following OLS formulation:

$$Y_t = \beta_0 + \beta_1 T_t + \varepsilon_t, \quad (18)$$

where the intercept is

$$\beta_0 = \mu_Y - \beta_1 \mu_T \quad (19)$$

and the slope from the straight-line growth model is

$$\beta_1 = \frac{\sum_{t=1}^F (Y_t - \mu_Y)(T_t - \mu_T)}{\sum_{t=1}^F (T_t - \mu_T)^2} = \beta_{\text{SLGM}}, \quad (20)$$

where μ_Y and μ_T represent the population means of the dependent variable (Y) and time (T) respectively, and β_{SLGM} is the regression coefficient for the straight-line growth model. Notice that no i subscripts are needed in Equation 18 (and thus Equations 19 and 20) because $N = 1$.

When straight-line growth models are used in the context of the analysis of change, an implicit assumption for descriptive and inferential purposes is that Equation 20 provides a meaningful measure of change. If the relationship between time and the dependent variable of interest is something other than linear, making use of β_{SLGM} for individual trajectories may lead to incorrect conclusions. When

making use of statistical methods that treat β_{SLGM} as a dependent variable, such as HLMs or two-stage analyses, the results of such statistical procedures may be misleading, as the chosen measure of change (β_{SLGM}) may not accurately reflect the particular phenomenon under study as it changes and evolves over time. The forthcoming sections detail the relationship that exists between β_{SLGM} and ARC as well as the importance of correctly specifying an appropriate Level 1 HLM model. It is illustrated in the remainder of the work that conceptualizing β_{SLGM} as a measure of the ARC potentially leads to incorrect conclusions, not only for an individual trend, but such a misconception also has implications when looking across individuals and when examining group differences.

The Regression Coefficient from the Straight-Line Growth Model and the Average Rate of Change

Recall that it is assumed data are completely balanced with equally spaced occasions of measurement with the same starting point and constant time-lag throughout the work (i.e., $\mathbf{Z}_i$ from Equation 14 is constant across individuals; see also pages 15 & 16). The slope from the straight-line growth model implied by Equation 20 is how some researchers (often incorrectly) label and/or interpret the ARC for an individual trajectory. For example, in a methodological work Kraemer and Thiemann (1989) recommended using the regression coefficient from the straight-line growth model calculated separately on each individual trajectory as the dependent variable in the analysis of group differences over time in applications of the intensive design. The authors state and illustrate a proof that supposedly shows over the time interval zero to one “the slope from the usual, ordinary least-squares regression measures the average rate of change” (p. 150). However, this statement is generally not correct, and it will be shown momentarily that the slope

from the straight-line growth model and the ARC are equivalent only in a limited number of circumstances. The fact that Kraemer and Thiemann define the slope as the ARC and then go on to state that “no assumption of linearity [straight-line growth] is made” when using the slope from OLS regression as a measure of the ARC (p. 150) is unfortunate. Such statements are troublesome because they potentially lead researchers to believe they are examining the ARC (or the mean of the ARC across individuals) when in fact they are examining an OLS regression coefficient (or the mean OLS regression coefficient), a measure designed to minimize the sum of squared deviations between the predicted and observed scores, not to measure the ARC over some time interval.¹³

An example that shows how the regression coefficient from the straight-line growth model can be incorrectly conceptualized in applied applications of longitudinal data analysis is taken from Svartberg (1999). Svartberg states that it is a “fact” that the “linear component provides a good estimate of average change even when the growth pattern is complicated” (p. 1315). Svartberg goes on to state that “even when the underlying trajectory is curved, the straight-line model is a reasonable option since the linear rate of change is equal to the average rate of change of the curved function” (p. 1318).¹⁴ Svartberg is not alone

¹³Kraemer and Thiemann go on to state that the regression coefficient from the straight-line growth model is the “average rate of change over all pairs of time points” (p. 150). This is also misleading as the reader is first told that the slope is the ARC regardless of the true function and then the reader is told that it is actually the ARC over all *pairs* of time points. In fact, the regression coefficient is not the average over pairs of timepoints, but a weighted average over all pairs of timepoints, where the weights are defined as $(T_t - T_{t'})^2 / \sum_{t \neq t'} (T_t - T_{t'})^2$, and are simply the OLS regression weights obtained by rewriting Equation 20.

¹⁴Svartberg (1999) cites Willett (1989) after he claims “the linear rate of change is equal to the average rate of change of the curved function” (p. 1918). However, Willett states that “even when the underlying trajectory is quadratic, the use of the straight-line model is equal to the average slope of the quadratic function over the same time interval” (p. 590). Thus, Svartberg has overgeneralized Willett’s summary of Seigel (1975). Furthermore, it should be noted that Willett’s statement is only true when the occasions of measurement are equally spaced, which he failed to acknowledge.

in his use of the straight-line growth model for seemingly complicated growth functions, as it supposedly provides an “overall average” for potentially complicated functional forms of growth. However, he is very explicit about why he made use of the straight-line growth model, potentially leading other researchers astray when analyzing and attempting to understand their own potentially complicated longitudinal data. Since Svartberg’s clear exposition on the rationale for making use of the straight-line growth model is, like some of Kraemer and Thiemann’s statements, flawed, the works have the potential to: (a) encourage others to ignore searching for the true functional form of growth, (b) “fall-back” on the straight-line growth model, and (c) lead to interpretations based on biased estimates of the mean ARC across individuals.

A commonly used but potentially confusing statement regarding the ARC occurs when the “average rate of change” is presented and interpreted in HLMs. This “average rate of change,” however, is generally not the ARC examined in the present work. When fitting the straight-line growth model in the context of HLM, each individual is typically allowed a unique value for their slope over time, as well as a unique intercept. As previously stated, the expected value (i.e., mean) of each parameter across all individuals are known as fixed effects. Recall the fixed effect for the slope is represented in Equation 17 by β_{10} (see p. 32). In straight-line growth models this parameter is often referred to as the “average rate of change” (examples in methodological works include Laird and Wang, 1990, p. 405, and Raudenbush and Xiao-Feng, 2001, p. 387) because it is literally the mean of all individual slope (i.e., rate of change) estimates. Authors who use the term “average rate of change” when referring to the fixed effect are not literally wrong, *provided* the “average rate of change” is *not* interpreted as the grand mean (mean of the individual means) of the instantaneous rate of change for the individual trajectories over time, but as the

mean of the individual slopes. However, using the term “average rate of change” to describe the fixed effect value provides a poor description of its meaning, as it gives the impression that it measures the ARC across a set of individuals. As is will be shown momentarily, in general $\beta_{10} \neq \overline{\text{ARC}}$ and $\beta_{\text{SLGM}} \neq \text{ARC}$. In summary, the value of the fixed effect for the slope is the *average of the individual rates of change* across individuals (i.e., the average of the individual slopes), whereas the ARC is literally the *average rate of change* for an individual. Although the phrases may be subtly different, the two convey very different concepts. When individuals’ average rates of change are biased, by conceptualizing them as if they were the slope from the straight-line growth model for example, the mean of the biased estimates is itself a biased estimate of the group ARC. Confusion surrounding the use and interpretation of the ARC persists, in part, because of the labels commonly employed for the fixed effect parameter estimates in the context of HLMs.

In summary, β_{10} (Equation 17) is the mean slope across all individuals in some population, however, β_{10} generally does not represent the mean ARC across individuals, nor does β_{SLGM} (Equation 20) generally represent the ARC for an individual. The belief that the slope from the straight-line growth model is always equal to the ARC is explicit in some work and implicit in the interpretations of many others. The overall group effect for the rate of change, although it is an averaged value, is not generally a measure of the overall ARC across individuals. Although it is a measure of the average of the individual slopes, the individuals’ slopes do not generally measure the average rate of change. As it will be shown, the slope from the straight-line growth model is equal to the ARC in a limited number of situations. A major goal of this work is to examine the potential bias that develops when interpreting the ARC as if it was the slope from the straight-line growth model.

Fitting a Quadratic Growth Model To Estimate the Average Rate of Change

When curvature is, or at least seems to be, present in longitudinal data, it is often suggested that the straight-line growth model be extended to a quadratic growth model. The Level 1 (intraindividual) model for a quadratic growth model is extended by incorporating the square of the time-varying covariate as another predictor variable in addition to the intercept and the time-varying covariate itself. The Level 1 quadratic growth model for a single individual can be represented as,

$$Y_t = \beta_0 + \beta_1 T_t + \beta_2 T_t^2 + \varepsilon_t, \quad (21)$$

where β_2 represents the quadratic component of the growth curve. Note that all three parameter estimates (β_0, β_1 , and β_2) can be modeled via a Level 2 (interindividual) model. Furthermore, because the parameter estimates are solved for simultaneously, β_0 and β_1 of Equation 18 are generally not equal to β_0 and β_1 of Equation 21, as adding one term generally changes all terms.

Making the assumption that a particular phenomenon is a function of linear and quadratic components over time, a question arises as to how the ARC may be estimated given the parameter estimates from a quadratic growth model. Assuming that the true underlying model follows a second degree polynomial, Seigel (1975) shows that the ARC can be estimated by evaluating the derivative of Equation 21 (with respect to T) at the mean value of T . Thus, for a quadratic growth model where growth is truly governed by a second order polynomial equation, the ARC is equal to the following:

$$\text{ARC}_{\text{QGM}} = \beta_1 + 2\beta_2\mu_T. \quad (22)$$

However, Seigel shows that Equation 22 reduces to Equation 20. Thus, the first derivative (with respect to time) of a quadratic growth model evaluated at the mean value of time is equivalent to the slope from the straight-line growth model. Thus,

in an attempt to obtain a “better estimate” of the ARC when curvature is present, Equation 22 yields exactly the same value as the slope from the straight-line growth model when there is equal spacing between measurement occasions. In this respect, conceptualizing the ARC via the parameter estimates from a quadratic growth model is equivalent to conceptualizing it as the ARC using the regression coefficient from the straight-line growth model. Thus, in this narrow sense, using a more elaborate model provides no better estimate of the ARC than does the slope from the straight-line growth model, regardless of the true functional form of growth. As Seigel points out (and which will be proved momentarily) if the true functional form of growth is a first or second degree polynomial, the ARC is equal to Equation 22 or Equation 20. However, as the next section shows, when the true functional form of growth is something other than a first or second degree polynomial, the slope from the straight-line growth model (or Equation 22) is generally a biased estimate of the ARC.

THE DISCREPANCY BETWEEN THE REGRESSION
COEFFICIENT FROM THE STRAIGHT-LINE GROWTH
MODEL AND THE AVERAGE RATE OF CHANGE

The potential discrepancy between the regression coefficient and the ARC is described by two parameters. For fixed values of time the first parameter that describes the discrepancy is the bias and is operationally defined as,

$$B = E[\beta_{\text{SLGM}}|f(T)] - \frac{E[(Y_F - Y_1)|f(T)]}{T_F - T_1} = \beta_{\text{SLGM}} - \text{ARC}, \quad (23)$$

where Y is conditional on the true functional form of growth and $E[x]$ represents the expectation value of the random variable x . For fixed values of time the second parameter that describes the discrepancy is the discrepancy factor and is operationally defined as,

$$\Psi = \frac{E[\beta_{\text{SLGM}}|f(T)]}{\frac{E[(Y_F - Y_1)|f(T)]}{T_F - T_1}} = \frac{\beta_{\text{SLGM}}}{\text{ARC}}, \quad (24)$$

where again Y is conditional on the true functional form of growth. When B equals zero Ψ must equal one; the converse is also true. In this special case β_{SLGM} is exactly equal to the ARC. Thus, in situations where $B=0$ ($\Psi=1$) interpreting β_{SLGM} as the ARC yields no inconsistency in research conclusions or interpretation. However, when $B \neq 0$ (by implication $\Psi \neq 1$), conceptualizing β_{SLGM} as the ARC is problematic and can potentially lead to misinformed conclusions regarding

intraindividual change, interindividual change, as well as group differences in change.¹⁵

Before examining B and Ψ , it is first helpful to realize that any functional form can generally be represented by a power series, such that the sum of squared deviations between values of the true function and the values approximated by the power series can be made to be infinitesimally small by adding enough polynomial powers and coefficients (Stewart, 1998, section 8.6; Finney et al., 2001, chapter 8). A power series in the longitudinal context is a limiting sum of coefficients multiplied by integer powers of time. Such a power series is given as,

$$f(T) = \lim_{M \rightarrow \infty} \sum_{m=0}^M (\lambda_m T^m), \quad (25)$$

where λ_m is the coefficient ($-\infty < \lambda_m < \infty$) for the m th power ($m = 0, \dots, M$).

Although a power series is infinite by definition, known functional forms can be represented by finite sums. In general, the following finite sum can be used to impose or approximate some known or unknown functional form of growth and is more general than the power series, as the powers of time are not limited to nonnegative integers, but can take on any real values,

$$f(T) = \sum_{k=1}^K (\lambda_k T^{\Upsilon_k}), \quad (26)$$

where Υ_k ($-\infty < \Upsilon < \infty$) represents the k th ($k = 1, \dots, K; 1 \leq K < \infty$) power constant.¹⁶ General results emerge for B and Ψ by realizing that functional forms of

¹⁵It should be noted that Ψ can be reformulated into another measure of discrepancy, such that it represents the proportion of bias relative to the ARC. The proportion of bias is thus given as: $\frac{\beta_{\text{SLGM}} - \text{ARC}}{\text{ARC}} = \frac{\beta_{\text{SLGM}}}{\text{ARC}} - \frac{\text{ARC}}{\text{ARC}} = \Psi - 1$. Because the proportion of bias and Ψ provide the same information, only one needs to be used as a supplement for B . The second discrepancy measure that will be used throughout the remainder of the work is Ψ .

¹⁶The intercept of a particular growth curve is the sum of the λ_k s whose Υ_k is zero. In the special case where $T \in [0, T_F]$ the intercept is the $\sum \lambda_k 0^{\Upsilon_k}$, which strictly speaking is an indeterminate form when $\Upsilon_k = 0$. However, due to l'Hôpital's Rule which uses derivatives to evaluate the converging limit of a function that would otherwise be indeterminate under standard algebraic rules, the quantity $0^0 \equiv 1$ by standard conventions (Stewart, 1998, section 4.5; Finney et al., 2001, section 7.6). When evaluating the equations given in this section by computer, care should be taken to ensure the particular program defines 0^0 as 1 (rather than returning an error message).

growth can generally be represented by Equation 26.¹⁷ The following section makes use of this fact when examining B and Ψ when time is continuous for any linear model as well as the nonlinear models presented in the *Nonlinear Growth Models for the Analysis of Change* section.

Examining the Bias in the Average Rate of Change:

The Limiting Case when Time Is Continuous

Often times in statistics, and mathematically based disciplines in general, limiting cases are of most interest and utility. It is in this spirit that the examination of B and Ψ will first proceed. For the limiting case where time is continuous [$F \rightarrow \infty$ as $(T_{t+1} - T_t) \rightarrow 0$], deriving B and Ψ will provide insight into problems associated with using β_{SLGM} as a measure of the ARC.

Regardless of the functional form of growth, B and Ψ can both be obtained by making use of the equations for β_{SLGM} and ARC. The equation for the slope presented in Equation 20 is for a finite number of timepoints. In the case of continuous data, Equation 20 generalizes (by integrating rather than summing) to the following:

$$\beta_{\text{SLGM}_C} = \frac{\int_{T_1}^{T_F} (T_t - \mu_T)(Y_t - \mu_Y) dT}{\int_{T_1}^{T_F} (T_t - \mu_T)^2 dT}, \quad (27)$$

where β_{SLGM_C} is the regression coefficient for the straight-line growth model when time is continuous.

¹⁷If the functional form of growth is a known linear equation, the values of λ_k and Υ_k can be specified and written via Equation 26. If the functional form is known but yet follows some nonlinear equation, an exact linearized analog to a nonlinear function can be derived by specifying a sequence of Υ s from zero to F (if F is large or even infinite, the function can be approximated by specifying a large number of Υ s) by one and solving for the λ s via multiple regression. The nonlinear equation can then be written as Equation 26 with the specified λ and Υ values such that the nonlinear function is represented as a linear equation.

Equation 27 can be rewritten as the integral of a sum after expanding the numerator and denominator,

$$\beta_{\text{SLGM}_C} = \frac{\int_{T_1}^{T_F} (Y_t T_t - Y_t \mu_T - T_t \mu_Y + \mu_T \mu_Y) dT}{\int_{T_1}^{T_F} (T_t^2 - 2T_t \mu_T + \mu_T^2) dT}. \quad (28)$$

Because the integral of a sum is the sum of the integrals, Equation 28 can be rewritten as the following:

$$\beta_{\text{SLGM}_C} = \frac{\int_{T_1}^{T_F} (Y_t T_t) dT - \mu_T \int_{T_1}^{T_F} (Y_t) dT - \mu_Y \int_{T_1}^{T_F} (T_t) dT + \mu_T \mu_Y (T_F - T_1)}{\int_{T_1}^{T_F} (T_t^2) dT - 2\mu_T \int_{T_1}^{T_F} (T_t) dT + \mu_T^2 (T_F - T_1)}. \quad (29)$$

Realizing that $\mu_T = (T_F + T_1)/2$ and that $\mu_Y \int_{T_1}^{T_F} (T_t) dT$ equals $\mu_Y (T_F^2 - T_1^2)/2$, the last two components in the numerator of Equation 29 are equal and of opposite sign, leading to a reduction of the numerator because the two components cancel. Alternatively, a second perspective to understand why the two components cancel in the numerator of Equation 29 can be seen by rewriting the last two components as $\mu_Y \int_{T_1}^{T_F} (T_t - \mu_T) dT$. Because $\int_{T_1}^{T_F} (T_t - \mu_T) dT$ is the first moment about the mean, this quantity must always equal zero (Stuart and Ord, 1994, chapter 3). In the following sections where time is continuous, the reduced form of Equation 29,

$$\beta_{\text{SLGM}_C} = \frac{\int_{T_1}^{T_F} (Y_t T_t) dT - \mu_T \int_{T_1}^{T_F} (Y_t) dT}{\int_{T_1}^{T_F} (T_t^2) dT - 2\mu_T \int_{T_1}^{T_F} (T_t) dT + \mu_T^2 (T_F - T_1)}, \quad (30)$$

will be applied to linear and then to nonlinear models.

When Y Can Be Written As a Linear Function of Time

Recall from Equations 25 and 26 that Y can be written as a sum of coefficients multiplied by powers of time. This sum can be in the form of a limiting sum (Equation 25) of some unknown functional form of growth or in the form of a finite sum (Equation 26) of a known functional form of growth written as a linear equation. In the remainder of this section, Y will be represented by Equation 26, in order for general results to be derived for B and Ψ .

Replacing Y in Equation 30 with the finite sum of Equation 26 yields:

$$\beta_{\text{SLGM}_C} = \frac{\int_{T_1}^{T_F} \left(\sum_{k=1}^K \lambda_k T_t^{\Upsilon_k+1} \right) dT - \mu_T \int_{T_1}^{T_F} \left(\sum_{k=1}^K \lambda_k T_t^{\Upsilon_k} \right) dT}{\int_{T_1}^{T_F} (T_t^2) dT - 2\mu_T \int_{T_1}^{T_F} (T_t) dT + \mu_T^2 (T_F - T_1)}. \quad (31)$$

Carrying out the integration and replacing μ_T by its definition $[\mu_T = (T_F + T_1)/2]$ yields:

$$\beta_{\text{SLGM}_C} = \frac{\sum_{k=1}^K \frac{\lambda_k (T_F^{\Upsilon_k+2} - T_1^{\Upsilon_k+2})}{\Upsilon_k+2} - \frac{T_F+T_1}{2} \sum_{k=1}^K \frac{\lambda_k (T_F^{\Upsilon_k+1} - T_1^{\Upsilon_k+1})}{\Upsilon_k+1}}{\frac{T_F^3 - T_1^3}{3} - \frac{(T_F+T_1)(T_F^2 - T_1^2)}{2} + \frac{(T_F+T_1)^2 (T_F - T_1)}{4}}. \quad (32)$$

After simplifying both the numerator and the denominator, details of which are given in Appendix A, the general equation for the regression coefficient from the straight-line growth model when Y can be written in the form of Equation 26 and when time is continuous is given as follows:

$$\beta_{\text{SLGM}_C} = \frac{6}{(T_F - T_1)^3} \sum_{k=1}^K \left(\frac{\lambda_k \left[\Upsilon_k (T_F^{\Upsilon_k+2} - T_1^{\Upsilon_k+2} + T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1}) + 2 (T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1}) \right]}{(\Upsilon_k + 2)(\Upsilon_k + 1)} \right). \quad (33)$$

It is useful to note that Equation 33 does not constrain the values of T_1 or T_F , the number of components defining Y (i.e., K), or the values of Υ and λ .¹⁸

¹⁸Implicit in the derivations presented is the fact that the equations are only valid for legitimate sets of parameter values. As with any mathematic model, evaluating the model for indeterminate forms will not result in meaningful solutions. Thus, care must be taken to ensure that a chosen set of parameters provides a mathematically meaningful set of parameters.

The ARC when Y is defined as a sum of K coefficients multiplied by powers of time can be written as the following:

$$\text{ARC} = \frac{\sum_{k=1}^K \lambda_k (T_F^{\Upsilon_k} - T_1^{\Upsilon_k})}{T_F - T_1}. \quad (34)$$

Because the slope (Equation 33) and the ARC (Equation 34) have been defined when Y is expressed as a special case of Equation 26, general equations emerge for B and Ψ .

The general bias for the present situation is given as,

$$B = \frac{6}{(T_F - T_1)^3} \sum_{k=1}^K \left(\frac{\lambda_k \left[\Upsilon_k \left(T_F^{\Upsilon_k+2} - T_1^{\Upsilon_k+2} + T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1} \right) + 2 \left(T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1} \right) \right]}{(\Upsilon_k + 2)(\Upsilon_k + 1)} \right) - \frac{\sum_{k=1}^K \lambda_k (T_F^{\Upsilon_k} - T_1^{\Upsilon_k})}{T_F - T_1}, \quad (35)$$

which is Equation 34 subtracted from Equation 33, as specified by the definition of B given in Equation 23. The general discrepancy factor is then given as

$$\Psi = \frac{6}{(T_F - T_1)^2 \sum_{k=1}^K \lambda_k (T_F^{\Upsilon_k} - T_1^{\Upsilon_k})} \sum_{k=1}^K \left(\frac{\lambda_k \left[\Upsilon_k \left(T_F^{\Upsilon_k+2} - T_1^{\Upsilon_k+2} + T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1} \right) + 2 \left(T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1} \right) \right]}{(\Upsilon_k + 2)(\Upsilon_k + 1)} \right) \quad (36)$$

which is Equation 33 divided by Equation 34, as specified by the definition of Ψ given in Equation 24.

Oftentimes in behavioral research the initial value of time is represented as zero ($T_1 = 0$). This is especially true in experimental studies when Y_1 represents a baseline measure of some attribute (pretest) before treatment begins. Another reason why T_1 often equals zero is because time is often scaled such that the intercept represents the initial (starting) value. In the special case where T_1 is replaced by zero, Equations 35 and 36 can be simplified. The simplified slope when the initial value of time (or scaled time) is zero can be written as the following:

$$\beta_{\text{SLGM}_C} = 6 \sum_{k=1}^K \left(\frac{\lambda_k T_F^{\Upsilon_k-1} \Upsilon_k}{(\Upsilon_k + 2)(\Upsilon_k + 1)} \right). \quad (37)$$

The ARC for such a series defined by Equation 26 can be written as,

$$\text{ARC} = \frac{\left(\sum_{k=1}^K \lambda_k T_F^{\Upsilon_k} \right) - \beta_0}{T_F}, \quad (38)$$

where β_0 is the intercept of the particular growth curve. Recall that the intercept is simply the sum of the coefficients whose Υ_k equals zero. If no Υ_k equals zero when $T = 0$, then β_0 itself equals zero and the growth curve goes through the origin.

The general expression for B when time is contained in the interval zero to some arbitrary endpoint, $T \in [0, T_F]$, is obtained by subtracting Equation 38 from Equation 37:

$$B = 6 \sum_{k=1}^K \left(\frac{\lambda_k T_F^{\Upsilon_k-1} \Upsilon_k}{(\Upsilon_k + 2)(\Upsilon_k + 1)} \right) - \frac{\left(\sum_{k=1}^K \lambda_k T_F^{\Upsilon_k} \right) - \beta_0}{T_F}. \quad (39)$$

The general expression for Ψ in this situation is obtained by dividing Equation 38 into Equation 37:

$$\Psi = \frac{6}{\left(\sum_{k=1}^K \lambda_k T_F^{\Upsilon_k} \right) - \beta_0} \sum_{k=1}^K \left(\frac{\lambda_k T_F^{\Upsilon_k} \Upsilon_k}{(\Upsilon_k + 2)(\Upsilon_k + 1)} \right). \quad (40)$$

An example making use of Equations 39 and 40 will be given. Suppose one is interested in the functional form of growth governed by the equation: $Y = 10 - 6T + 4.5T^2 - .5T^3$ when $T \in [0, 5]$. The λ_k values in this example are (10, -6, 4.5, & -0.5), while the Υ_k values are (0, 1, 2, & 3). The slope in this example is given by $6(.875)=5.25$ while the ARC is 4 (Note that the intercept β_0 is 10 because it is the only λ_k whose Υ equals 0.). Thus, B in this situation is 1.25 while Ψ is 1.3125.

When Y Conforms to Certain Nonlinear Functions of Time

As previously stated, it seems as though the behavioral sciences have not yet fully recognized the potential benefits offered by nonlinear growth models. Part of the benefit of nonlinear models is that the parameters of nonlinear models often have meaningful real world interpretations (recall Figures 5, 6, and 7), while the parameters for various powers of time are generally less directly interpretable (especially beyond quadratic growth models). Although any function can be approximated by Equation 26, using such an equation can potentially lead to a large number of parameters included in the model in order for the true functional form of growth to be adequately represented and well approximated by a linear model. Such a situation can be avoided if an appropriate nonlinear model can be found.

In this section the β_{SLGM_C} and the ARC are derived for the asymptotic growth curve, the Gompertz growth curve, and the logistic growth curve. General equations are presented for β_{SLGM_C} and ARC for these nonlinear models, thus allowing one to compute B by subtraction and/or Ψ by division. The derivations proceeded in an analogous manner as they did (albeit not as detailed) for the derivations presented in the previous section for models linear in their parameters. Appendix B provides computer syntax for the program Maple (Version 7.00, 2001) that allows one to compute the regression coefficient from the straight-line growth model, ARC, B, and Ψ for (essentially) any functional form of growth where Y_t is some function of time. Details of the derivations are presented for each of the three nonlinear models in Appendix C by making use of the Maple syntax given in Appendix B. Note that the difficulties in deriving the desired parameters for the nonlinear models in this section arise, in large part, because Y cannot (parsimoniously) be written as a sum of coefficients multiplied by powers of time (Equation 26) for these nonlinear

functions.

The regression coefficient for the straight-line growth model applied to growth that follows an asymptotic regression model is given by,

$$\beta_{\text{SLGM}_{\text{CAR}}} = 6\beta \exp[-\gamma(T_F + T_1)] \frac{T_1 [\exp(\gamma T_1) + \exp(\gamma T_F)] - T_F [\exp(\gamma T_1) + \exp(\gamma T_F)] + 2[\exp(\gamma T_F) - \exp(\gamma T_1)]/\gamma}{\gamma(T_F - T_1)^3}, \quad (41)$$

where subscripts will be used, AR in this case, to identify the particular nonlinear growth model. The ARC for the asymptotic regression model is given as the following:

$$\text{ARC}_{\text{AR}} = \frac{\beta [\exp(-\gamma T_F) - \exp(-\gamma T_1)]}{T_F - T_1}. \quad (42)$$

The value of B for the asymptotic regression model is thus obtained by subtracting Equation 42 from Equation 41, while Ψ is obtained by dividing Equation 41 by Equation 42.

The regression coefficient for the straight-line growth model applied to growth that follows a Gompertz growth model is obtained by first letting G equal the following:

$$G = - \int_{T_1}^{T_F} [TF + T_1 - 2T] [\exp(-\exp(\beta - \gamma T))\gamma(T_F - T_1) + \text{Ei}(1, \exp(\beta - \gamma T_F)) - \text{Ei}(1, \exp(\beta - \gamma T_1))] dT, \quad (43)$$

where Ei is the exponential integral. The exponential integral is defined as,

$$\text{Ei}(a, x) = \int_{g=1}^{\infty} \frac{\exp(-xg)}{g^a} dg, \quad (44)$$

with a being a nonnegative integer and x some algebraic expression (Abramowitz & Stegun, 1965). Given G , the slope for the Gompertz growth model is equal to the following:

$$\beta_{\text{SLGM}_{\text{CGG}}} = G \frac{6\alpha}{\gamma(T_F - T_1)^4}. \quad (45)$$

The ARC for the Gompertz growth model is given as the following:

$$\text{ARC}_{\text{GG}} = \frac{\alpha [\exp(-\exp(\beta - T_F \gamma)) - \exp(-\exp(\beta - T_1 \gamma))]}{T_F - T_1}. \quad (46)$$

The value of B for the Gompertz growth model is thus obtained by subtracting Equation 46 from Equation 45, while Ψ is obtained by dividing Equation 45 by Equation 46.

The regression coefficient for the straight-line growth model applied to growth that follows a logistic growth model is obtained by first letting L_1 equal,

$$L_1 = T_F [\log(1 + \exp(\beta - T_1 \gamma)) - \log(\exp(\beta - T_1 \gamma)) + \log(\exp(\beta - T_F \gamma)) + \log(1 + \exp(\beta - T_F \gamma)) - 2\beta + T_F \gamma], \quad (47)$$

L_2 equal,

$$L_2 = T_1 [\log(1 + \exp(\beta - T_F \gamma)) - \log(\exp(\beta - T_F \gamma)) + \log(\exp(\beta - T_1 \gamma)) + \log(1 + \exp(\beta - T_1 \gamma)) - 2\beta + T_1 \gamma], \quad (48)$$

L_3 equal,

$$L_3 = \text{dilog}[(\exp(T_1 \gamma) + \exp(\beta)) \exp(-T_1 \gamma)] - \text{dilog}[(\exp(T_F \gamma) + \exp(\beta)) \exp(-T_F \gamma)], \quad (49)$$

and finally by letting L_4 equal,

$$L_4 = \beta [\log(\exp(\beta - T_1 \gamma)) - \log(\exp(\beta - T_F \gamma))]. \quad (50)$$

In L_3 the dilogarithm function is required. The function dilog (Lewin, 1981) is defined as the following:

$$\text{dilog}(x) = \int_{g=1}^x \frac{\log(g)}{1-g} dg. \quad (51)$$

The four logistic components are then combined with the other necessary parameters in the following manner:

$$\beta_{\text{SLGM}_{\text{C}_{\text{LG}}}} = 6\alpha \frac{L_1 - L_2 + 2(L_3 + L_4)/\gamma}{\gamma(T_F - T_1)^3}. \quad (52)$$

The ARC for the logistic growth model is given as follows:

$$\text{ARC}_{\text{LG}} = \frac{\alpha[\exp(\beta - T_F\gamma) - \exp(\beta - T_1\gamma)]}{[1 + \exp(\beta - T_F\gamma)][1 + \exp(\beta - T_1\gamma)](T_1 - T_F)}. \quad (53)$$

The value of B for the logistic growth model is thus obtained by subtracting Equation 53 from Equation 52, while Ψ is obtained by dividing Equation 52 by Equation 53.

Although Equations 41 through 53 are very general expression, examples will be given for specific cases of the three nonlinear models examined. Figures 5, 6, and 7 show plots of asymptotic regression, Gompertz, and logistic growth models respectively, for 15 different combinations of β and γ values when $T \in [0, 1]$ and α is fixed at 5 in the case of continuous time. The purpose of the figures is to show the reader a variety of nonlinear functional forms of growth governed by the nonlinear growth models included in the present work. An added benefit is to illustrate that B is sometimes positive while in other situations it is negative, and that Ψ is less than one in some situations while it is greater than one in others. It is important to note that B and Ψ for the 45 different scenarios examined are specific to the selected parameters and the chosen time interval. Because the parameter values are linked to the time interval, holding time constant and changing the parameter or changing the parameters and holding time constant would lead to different results. Thus, the scaling of time and the parameters are interrelated and the results given are not general, but specific to the particular combinations of time and the values of the parameters chosen.

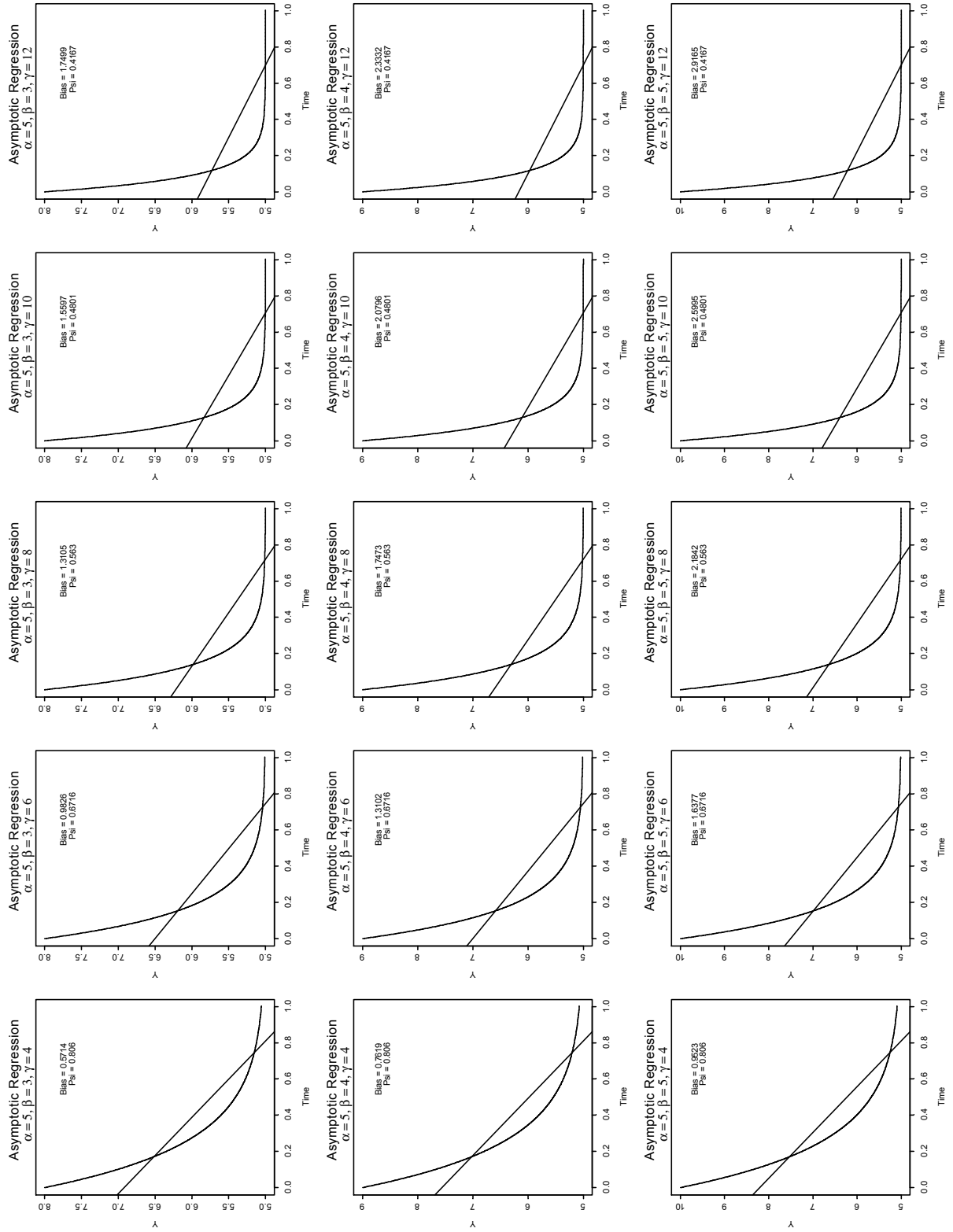


Figure 5. Illustration of the straight-line growth model fit to a variety of asymptotic growth curves along with B and Ψ , given the parameters that are specified.

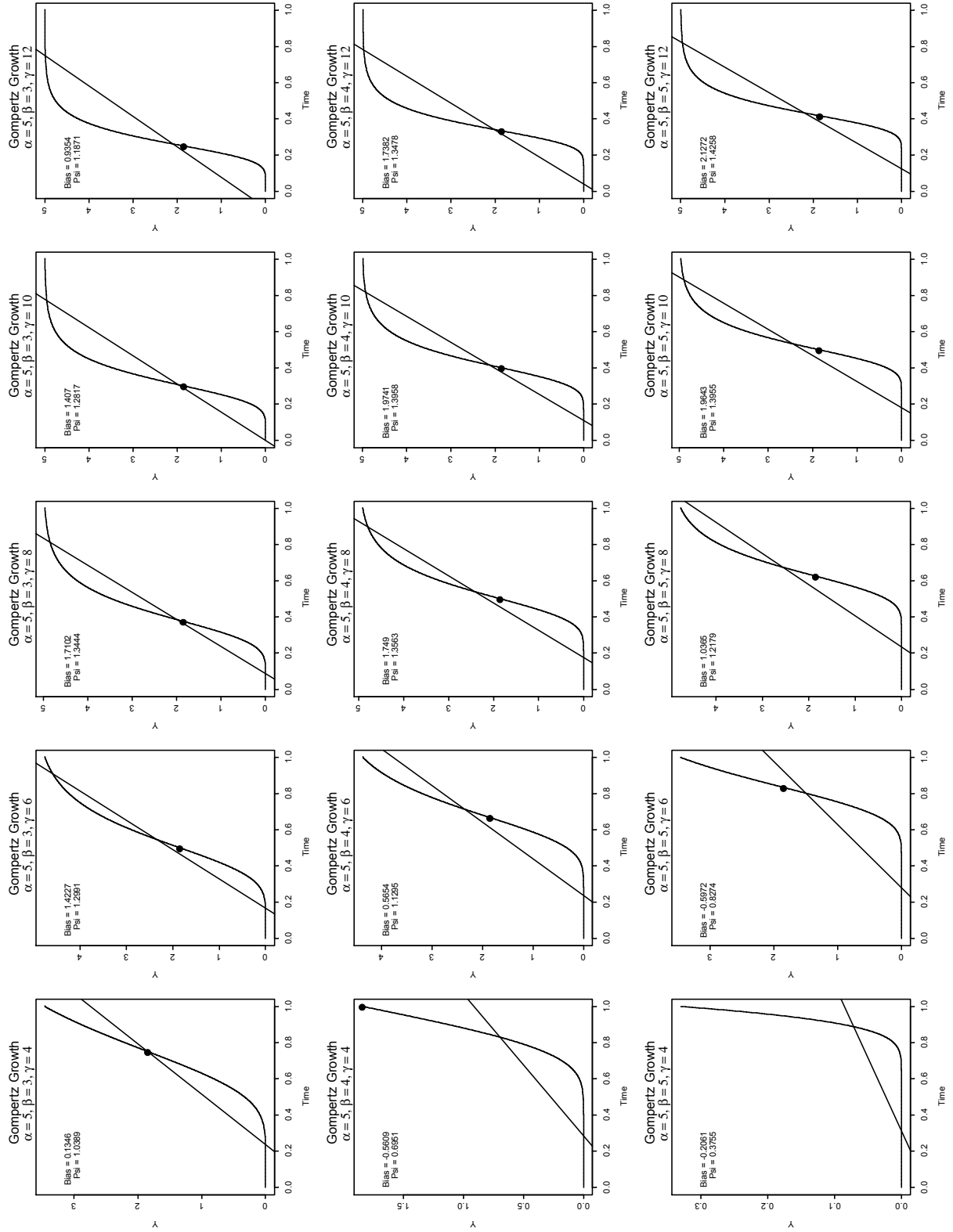


Figure 6. Illustration of the straight-line growth model fit to a variety of Gompertz growth curves along with B and Ψ , given the parameters that are specified.

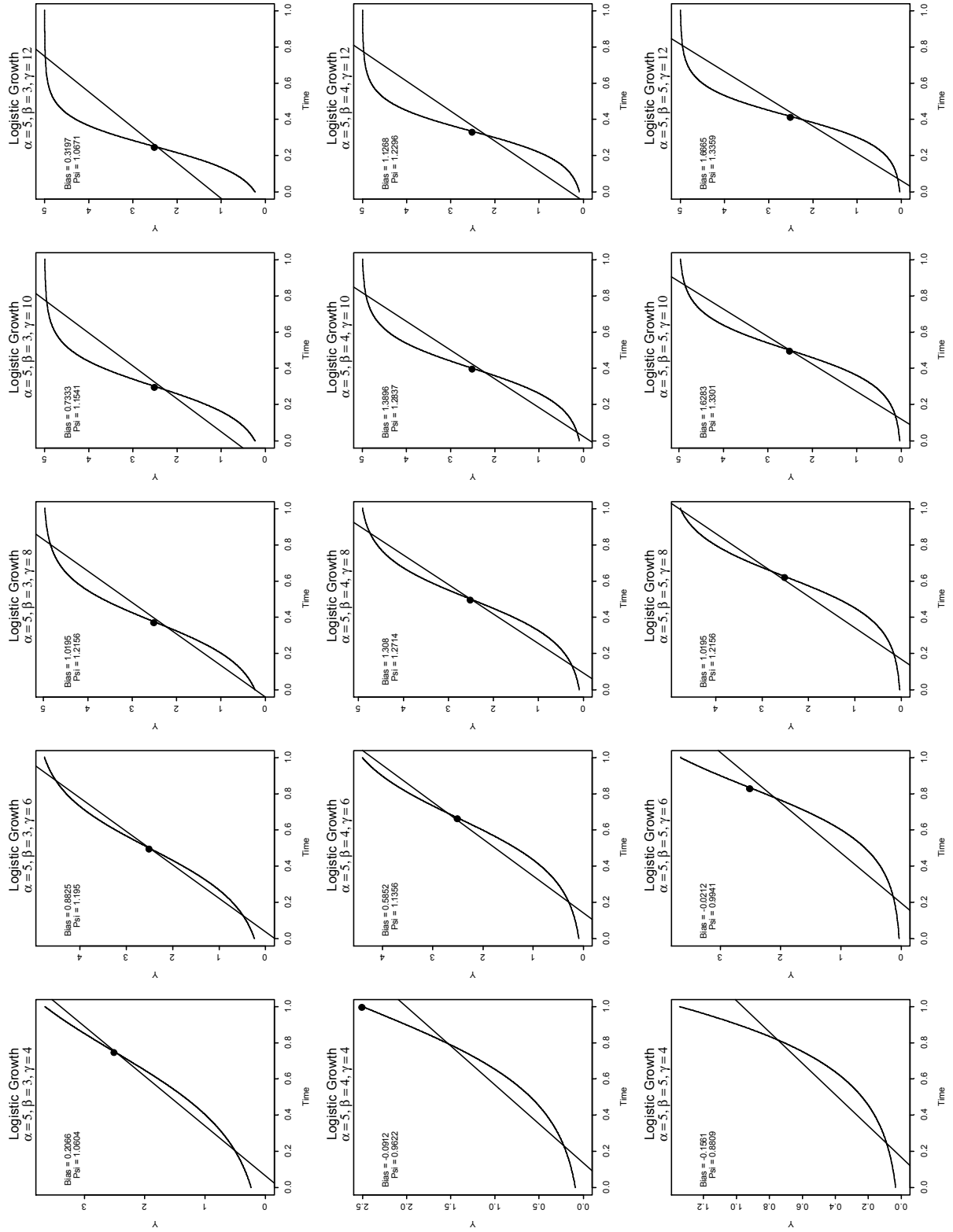


Figure 7. Illustration of the straight-line growth model fit to a variety of logistic growth curves along with B and Ψ given, given the parameters that are specified.

The straight-line within each plot represents the predicted Y scores given time (i.e., the regression line) for the straight-line growth model whereas the nonlinear trend represents the true growth for the particular situation. Notice the dark circles on the Gompertz and logistic plots. The circle represents the point of inflection for the particular nonlinear growth model, that is, the point where the concavity of the curvature changes.¹⁹ There is no point of inflection for the asymptotic regression model, as the concavity does not change but is concave upward in each of the 15 scenarios in Figure 5.

Examining the Bias in the Average Rate of Change:

The Case when Time Is Discrete

From a methodological perspective the previous section may be of most interest, however, in applied behavioral science research, time is nearly always measured at discrete occasions. For this reason it is important to examine B and Ψ when time is limited to a finite number of measurement occasions. For finite occasions of measurement the general equation for the bias can be written as the following:

$$B = \frac{\sum_{t=1}^F (Y_t - \mu_Y)(T_t - \mu_T)}{\sum_{t=1}^F (T_t - \mu_T)^2} - \frac{Y_F - Y_1}{T_F - T_1}. \quad (54)$$

The equation for Ψ for the general case of discrete time is given by the following:

$$\Psi = \frac{(T_F - T_1) \sum_{t=1}^F (Y_t - \mu_Y)(T_t - \mu_T)}{(Y_F - Y_1) \sum_{t=1}^F (T_t - \mu_T)^2}. \quad (55)$$

¹⁹Notice that for the bottom right plot on Figures 6 and 7 there is no circle indicating the point of inflection. This is because the point of inflection occurs at a value of time greater than 1 and thus occurs outside of the plotting region. The point of inflection for the condition where $\alpha = 5$, $\beta = 5$, and $\gamma = 4$ is $[T = 1.25, Y = 1.839]$ for the Gompertz growth model and $[T = 1.25, Y = 2.5]$ for the logistic growth model.

For equally spaced occasions of measurement, Equations 54 and 55 can be simplified by realizing that all of the values of time (T_t) can be written in terms of T_1 and T_F .²⁰ By making use of T_1 and T_F , the remaining $F - 2$ values of time can be expressed as

$$T_t = T_1 + (t - 1) \frac{T_F - T_1}{F - 1}. \quad (56)$$

Combining Equation 56 with Equations 54 and Equations 55 allows B and Ψ to be derived for arbitrary values of F . The general expression for B , given F equally spaced occasions of measurement, can be written as follows:

$$B_F = \frac{\sum_{t=1}^F Y_t \left[\left(T_1 + (t - 1) \frac{T_F - T_1}{F - 1} \right) - \mu_T \right]}{\sum_{t=1}^F \left[\left(T_1 + (t - 1) \frac{T_F - T_1}{F - 1} \right) - \mu_T \right]^2} - \frac{(Y_F - Y_1)}{(T_F - T_1)}. \quad (57)$$

The general expression for Ψ , given F equally spaced occasions of measurement, can be written as follows:

$$\Psi_F = \frac{(T_F - T_1) \sum_{t=1}^F Y_t \left[\left(T_1 + (t - 1) \frac{T_F - T_1}{F - 1} \right) - \mu_T \right]}{(Y_F - Y_1) \sum_{t=1}^F \left[\left(T_1 + (t - 1) \frac{T_F - T_1}{F - 1} \right) - \mu_T \right]^2} \quad (58)$$

General results for Equations 57 and 58 can be derived for special cases of F . Table 1 shows the general B and Ψ for two to 12 equally spaced measurements with arbitrary T_1 and T_F values. Notice that regardless of the true functional form of growth, for two or three equally spaced values of time, the slope from the straight-line growth model and the ARC are always equivalent. Thus, for $F = 2$ or $F = 3$, one need not worry about any discrepancy that may arise if β_{SLGM} is labelled and interpreted as the ARC. However, for $F \geq 4$, the ARC generally does not equal β_{SLGM} .

²⁰In the case of discrete time Y can still be represented as a finite sum (Equation 26) as was done in the previous section where time was considered continuous. In the present section Y is written generally, such that it can be conceptualized as any linear or nonlinear function of time.

Table 1. GENERAL EQUATIONS FOR B AND Ψ FOR TWO TO 12 EQUALLY SPACED TIMEPOINTS WITH ARBITRARY INITIAL AND END POINTS

F	Bias	Ψ
2	0	1
3	0	1
4	$\frac{(Y_1 - Y_4) + 3(Y_3 - Y_2)}{10(T_4 - T_1)}$	$\frac{3}{10} \frac{3(Y_1 - Y_4) + (Y_2 - Y_3)}{(Y_1 - Y_4)}$
5	$\frac{(Y_1 - Y_5) + 2(Y_4 - Y_2)}{5(T_5 - T_1)}$	$\frac{2}{5} \frac{2(Y_5 - Y_1) + (Y_4 - Y_2)}{(Y_5 - Y_1)}$
6	$\frac{2(Y_1 - Y_6) + 3(Y_5 - Y_2) + (Y_4 - Y_3)}{7(T_6 - T_1)}$	$\frac{5(Y_6 - Y_1) + 3(Y_5 - Y_2) + (Y_4 - Y_3)}{7(Y_6 - Y_1)}$
7	$\frac{5(Y_1 - Y_7) + 6(Y_6 - Y_2) + 3(Y_5 - Y_3)}{14(T_7 - T_1)}$	$\frac{3}{14} \frac{3(Y_7 - Y_1) + 2(Y_6 - Y_2) + (Y_5 - Y_3)}{(Y_7 - Y_1)}$
8	$\frac{5(Y_1 - Y_8) + Y_7 - Y_2 + 3(Y_6 - Y_3) + (Y_5 - Y_4)}{12(T_8 - T_1)}$	$\frac{7(Y_8 - Y_1) + 5(Y_7 - Y_2) + 3(Y_6 - Y_3) + (Y_5 - Y_4)}{12(Y_8 - Y_1)}$
9	$\frac{7(Y_1 - Y_9) + 6(Y_8 - Y_2) + 4(Y_7 - Y_3) + 2(Y_6 - Y_4)}{15(T_9 - T_1)}$	$\frac{2}{15} \frac{4(Y_9 - Y_1) + 3(Y_8 - Y_2) + 2(Y_7 - Y_3) + (Y_6 - Y_4)}{(Y_9 - Y_1)}$
10	$\frac{28(Y_1 - Y_{10}) + 21(Y_9 - Y_2) + 15(Y_8 - Y_3) + 9(Y_7 - Y_4) + 3(Y_6 - Y_5)}{55(T_{10} - T_1)}$	$\frac{3}{55} \frac{9(Y_{10} - Y_1) + 7(Y_9 - Y_2) + 5(Y_8 - Y_3) + 3(Y_7 - Y_4) + (Y_6 - Y_5)}{(Y_{10} - Y_1)}$
11	$\frac{6(Y_1 - Y_{11}) + 4(Y_{10} - Y_2) + 3(Y_9 - Y_3) + 2(Y_8 - Y_4) + (Y_7 - Y_5)}{11(T_{11} - T_1)}$	$\frac{5(Y_{11} - Y_1) + 4(Y_{10} - Y_2) + 3(Y_9 - Y_3) + 2(Y_8 - Y_4) + (Y_7 - Y_5)}{11(Y_{11} - Y_1)}$
12	$\frac{15(Y_1 - Y_{12}) + 9(Y_{11} - Y_2) + 7(Y_{10} - Y_3) + 5(Y_9 - Y_4) + 3(Y_8 - Y_5) + (Y_7 - Y_6)}{26(T_{12} - T_1)}$	$\frac{11(Y_{12} - Y_1) + 9(Y_{11} - Y_2) + 7(Y_{10} - Y_3) + 5(Y_9 - Y_4) + 3(Y_8 - Y_5) + (Y_7 - Y_6)}{26(Y_{12} - Y_1)}$

Table 1 can be used in at least two ways. Suppose that some functional form of growth, the initial value of time, and F are known for equally spaced values of time. The value of B and Ψ can then be determined by making use of the expressions given in Table 1 ($2 \leq F \leq 12$). For any case where B does not equal zero (implying $\Psi \neq 1$), the extent of the discrepancy will be known for the functional form of interest. In the event that the true functional form of growth is known exactly, the expected value for β_{SLGM} could be “corrected” by scaling the expected β_{SLGM} in order for B to equal zero (equivalently making $\Psi = 1$), such that the scaled regression coefficient could be used as an unbiased estimate of the ARC.²¹ Furthermore, for a given F , B could be set to zero (or Ψ set to one) in order to discern under what circumstances the expected β_{SLGM} will equal ARC. For example, when $F = 4$, $\beta_{\text{SLGM}} = \text{ARC}$ whenever $Y_1 - Y_4 = 3(Y_2 - Y_3)$. Thus, there is zero bias whenever $-Y_1 + 3Y_2 - 3Y_3 + Y_4 = 0$.

Note that the coefficients in the bias equation when $F = 4$ (-1, 3, -3, and 1 for Y_1 through Y_4 respectively) have an interpretation beyond that of the present context. In fact, the coefficients (or the coefficients scaled by a constant) correspond to orthogonal polynomial coefficients in the context of trend analysis. Specifically for four levels of a quantitative factor in an analysis of variance context, the coefficients in the bias equation for $F = 4$ correspond to the coefficients for the test of the cubic trend (Tables of orthogonal polynomial can be found in Pearson & Hartley, 1970, specifically Table 47). When $F = 4$, fitting a third degree polynomial growth model ensures a perfect fit and fitting a fourth degree polynomial equation cannot be carried out as the degrees of freedom become negative. In order for the expected value of β_{SLGM} to be equal to the ARC when $F = 4$, there must be no evidence

By writing Y generally, B and Ψ can themselves be expressed in the most general general form for discrete data.

²¹Of course, this second potential use for Table 1 would imply that the true functional form of growth was known exactly. If this were true, it would make more sense to fit the correct functional form of growth to begin with rather than fitting the straight-line growth model for “interpretational ease.”

of a cubic trend. That is, the coefficients for the bias from Table 1 multiplied by the appropriate expected dependent variables must equal zero. Because any strictly linear trend or a combination of linear and/or quadratic trends (when measurement occasions are equally spaced) yields an unbiased estimate of the ARC as measured from β_{SLGM} , the sum of the coefficients multiplied by the appropriate Y_t will equal zero. In cases where $F = 4$ (and indeed when $F > 4$) and there is evidence of a cubic trend, β_{SLGM} will not be an unbiased estimate of the ARC.

A similar interpretation for values of $F > 4$ also exists, namely that when a strictly linear trend exists, when occasions of measurement are equally spaced and a quadratic trend exists, or when occasions of measurement are equally spaced and a combination of linear and quadratic trends exists, the sum of the orthogonal polynomials multiplied by the appropriate Y_t will be zero for all trends greater than quadratic. Thus, when measurement occasions are equally spaced, the sum of the orthogonal polynomials multiplied by the appropriate Y_t is zero for trends greater than quadratic, implying that only a linear and/or a quadratic trend exists, β_{SLGM} will exactly equal ARC. The point is that when a strictly linear trend exists, when measurement occasions are equally spaced and a strictly quadratic trend exists, or some combination of linear and quadratic trends exists for equally spaced measurement occasions, the bias in Table 1 will be zero. When there are trends greater than quadratic, the equations in Table 1 will not be zero implying that β_{SLGM} will differ from the ARC.

Rather than examining Table 1 for general cases of discrete time (for two to 12 timepoints), it is also helpful to examine B and Ψ for specific functional forms of growth. Tables 2, 3, 4, and 5 illustrate B and Ψ for a variety of scenarios (9 conditions within each of 57 conditions = 513 total scenarios). Specifically Table 2 examines various linear models when $T \in [0, 1]$, as well as additional linear models

(and two logarithmic) when $T \in [1, 5]$.²² Tables 3, 4, and 5 give the value of B and Ψ for four to 12 timepoints for the asymptotic regression, Gompertz, and logistic growth models previously illustrated in Figures 5, 6, and 7 respectively.

²²Note that the $ARC=1$ in each condition where $T \in [0, 1]$. Further note that the interval $T \in [0, 5]$ was used for the linear models not defined when $T = 0$.

Table 2. B AND Ψ VALUES FOR FOUR TO 12 TIMEPOINTS FOR SELECTED LINEAR AND NONLINEAR MODELS

	$F = 4$		$F = 5$		$F = 6$		$F = 7$		$F = 8$	
$T \in [0, 1]$										
$GrowthModel$	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ
$Y = T^3$	-.02	.98	-.04	.96	-.05	.95	-.06	.94	-.06	0.94
$Y = T^4$	-.04	.96	-.07	.92	-.10	.90	-.11	.89	-.12	.88
$Y = T^5$	-.06	.94	-.11	.89	-.14	.86	-.16	.84	-.17	.83
$Y = .5T^2 + .5T^3$	-.01	.99	-.02	.98	-.02	.98	-.03	.97	-.03	.97
$Y = 3.5T - .5T^2 - 2T^3$	.04	1.04	.08	1.08	.10	1.1	.11	1.11	.12	1.12
$Y = .5\sqrt{T} + .1T + .4T^3$	-.02	.98	-.04	.96	-.06	.94	-.07	.93	-.08	.92
$T \in [1, 5]$										
$Y = \sqrt{(T)}$	< .01	.99	< .01	.99	< .01	.98	-.01	.98	-.01	.98
$Y = \frac{1}{T}$	.01	.96	.01	.92	.02	.90	.02	.88	.03	.87
$Y = \frac{1}{\sqrt{T}}$	<.01	.97	.01	.95	.01	.93	.01	.92	.01	.91
$Y = .25\sqrt{(T)} + .25T + .25T^2 + .25T^3$	-.09	.99	-.15	.98	-.19	.98	-.22	.98	-.25	.97
$Y = \log(T)$	-.01	.98	-.01	.97	-.01	.96	-.02	.96	-.02	.95
$Y = 2\log(T)$	-.01	.98	-.02	.97	-.03	.96	-.03	.96	-.04	.95
$F = 9$										
$F = 10$										
$F = 11$										
$F = 12$										
$T \in [0, 1]$										
$GrowthModel$	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ
$Y = T^3$	-.07	.93	-.07	.93	-.07	.93	-.07	.93	-.07	.93
$Y = T^4$	-.13	.87	-.14	.86	-.14	.86	-.15	.85	-.15	.85
$Y = T^5$	-.19	.81	-.2	.80	-.21	.79	-.21	.79	-.21	.79
$Y = .5T^2 + .5T^3$	-.03	.97	-.03	.97	-.04	.96	-.04	.96	-.04	.96
$Y = 3.5T - .5T^2 - 2T^3$	.13	1.13	.14	1.14	.14	1.14	.15	1.15	.15	1.15
$Y = .5\sqrt{T} + .1T + .4T^3$	-.08	.92	-.09	.91	-.09	.91	-.10	.90	-.10	.90
$T \in [1, 5]$										
$Y = \sqrt{(T)}$	-.01	.98	-.01	.98	-.01	.98	-.01	.98	-.01	.98
$Y = \frac{1}{T}$	.03	.86	.03	.85	.03	.84	.03	.84	.03	.84
$Y = \frac{1}{\sqrt{T}}$	.01	.91	.01	.90	.01	.90	.01	.89	.01	.89
$Y = .25\sqrt{(T)} + .25T + .25T^2 + .25T^3$	-.26	.97	-.28	.97	-.29	.97	-.30	.97	-.30	.97
$Y = \log(T)$	-.02	.95	-.02	.95	-.02	.94	-.02	.94	-.02	.94
$Y = 2\log(T)$	-.04	.95	-.04	.95	-.05	.94	-.05	.94	-.05	.94

Tables 2, 3, 4, and 5 helps to illustrate the relationship between B and Ψ as a function of F and the functional form of growth. Interestingly, as F increases, so too does the amount of bias. Generally speaking, the more information available the better an estimate performs. However, this convention does not hold true when the ARC is conceptualized as β_{SLGM} . Notice that the straight-line growth model and the quadratic growth model are not illustrated in Table 2. The reason for their exclusion is because β_{SLGM} is an unbiased ($B = 0, \Psi = 1$) estimate of the ARC when the relationship between time and Y is linear or when occasions of measurement are equally spaced and growth is quadratic or a combination of linear and quadratic.

In several of the functional forms of growth examined over the selected time intervals, the B s are near zero and thus many of the Ψ s are near one. However, even when $F = 4$, for some functional forms of growth the B can be large *relative* to the ARC, thus leading values of Ψ to be markedly different from 1. For example, when growth follows a Gompertz growth model with parameters $\alpha = 5$, $\beta = 5$, and $\gamma = 12$ where $T \in [0, 1]$ for $F = 4$, β_{SLGM} is 17 percent larger than the ARC. Such a large discrepancy could translate into incorrect conclusions regarding change. Although β_{SLGM} can be a reasonable estimate for some functional forms of growth for the values of the time interval used in the present work (e.g., $Y = \frac{1}{\sqrt{T}}$, $Y = \sqrt{T}$ $T \in [0, 5]$), a universal statement regarding the appropriateness of β_{SLGM} for estimating the ARC cannot be made, other than to say that, in general, the two quantities are not equal to one another.

Table 3. B AND Ψ VALUES FOR FOUR TO 12 TIMEPOINTS WHEN GROWTH FOLLOWS AN ASYMPTOTIC REGRESSION GROWTH CURVE WITH $\alpha = 5$, AND COMBINATIONS OF β (3, 4, & 5) AND γ (4, 6, 8, 10, & 12) VALUES, $T \in [0, 1]$

Parameters	F = 4		F = 5		F = 6		F = 7		F = 8		F = 9		F = 10		F = 11		F = 12	
	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ
$\beta = 3, \gamma = 4$	.12	.96	.21	.93	.27	.91	.31	.89	.35	.88	.37	.87	.39	.87	.41	.86	.42	.86
$\beta = 3, \gamma = 6$	.19	.94	.34	.89	.45	.85	.53	.82	.59	.80	.63	.79	.67	.78	.70	.77	.72	.76
$\beta = 3, \gamma = 8$	.24	.92	.44	.85	.59	.80	.69	.77	.77	.74	.84	.72	.89	.70	.93	.69	.96	.68
$\beta = 3, \gamma = 10$	.27	.91	.50	.83	.68	.77	.81	.73	.91	.70	.98	.67	1.04	.65	1.09	.64	1.13	.62
$\beta = 3, \gamma = 12$	.28	.91	.54	.82	.74	.75	.89	.70	1.0	.67	1.09	.64	1.16	.61	1.22	.59	1.26	.58
$\beta = 4, \gamma = 4$	.16	.96	.28	.93	.36	.91	.42	.89	.46	.88	.50	.87	.52	.87	.55	.86	.56	.86
$\beta = 4, \gamma = 6$	.26	.94	.46	.89	.60	.85	.71	.82	.78	.80	.84	.79	.89	.78	.93	.77	.97	.76
$\beta = 4, \gamma = 8$	.32	.92	.59	.85	.78	.80	.92	.77	1.03	.74	1.11	.72	1.18	.70	1.24	.69	1.28	.68
$\beta = 4, \gamma = 10$	.36	.91	.67	.83	.90	.77	1.08	.73	1.21	.70	1.31	.67	1.39	.65	1.46	.64	1.51	.62
$\beta = 4, \gamma = 12$	.38	.91	.72	.82	.98	.75	1.18	.70	1.33	.67	1.45	.64	1.55	.61	1.62	.59	1.69	.58
$\beta = 5, \gamma = 4$	.20	.96	.35	.93	.45	.91	.52	.89	.58	.88	.62	.87	.65	.87	.68	.86	.71	.86
$\beta = 5, \gamma = 6$	.32	.94	.57	.89	.75	.85	.88	.82	.98	.80	1.06	.79	1.12	.78	1.17	.77	1.21	.76
$\beta = 5, \gamma = 8$	.40	.92	.73	.85	.98	.80	1.15	.77	1.29	.74	1.39	.72	1.48	.70	1.54	.69	1.60	.68
$\beta = 5, \gamma = 10$	.45	.91	.84	.83	1.13	.77	1.34	.73	1.51	.70	1.64	.67	1.74	.65	1.82	.64	1.89	.62
$\beta = 5, \gamma = 12$	.47	.91	.90	.82	1.23	.75	1.48	.70	1.67	.67	1.81	.64	1.93	.61	2.03	.59	2.11	.58

Table 4. B AND Ψ VALUES FOR FOUR TO 12 TIMEPOINTS WHEN GROWTH FOLLOWS A GOMPERTZ GROWTH CURVE WITH $\alpha = 5$, AND COMBINATIONS OF β (3, 4, & 5) AND γ (4, 6, 8, 10, & 12) VALUES, $T \in [0, 1]$

<i>Parameters</i>	$F = 4$		$F = 5$		$F = 6$		$F = 7$		$F = 8$		$F = 9$		$F = 10$		$F = 11$		$F = 12$	
	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ
$\beta = 3, \gamma = 4$	.02	1.01	.04	1.01	.06	1.02	.07	1.02	.08	1.02	.09	1.02	.09	1.03	.10	1.03	.10	1.03
$\beta = 3, \gamma = 6$	.46	1.10	.63	1.13	.75	1.16	.84	1.18	.91	1.19	0.97	1.20	1.01	1.21	1.05	1.22	1.08	1.23
$\beta = 3, \gamma = 8$	.49	1.10	.78	1.16	.91	1.18	1.01	1.20	1.10	1.22	1.16	1.23	1.22	1.24	1.26	1.25	1.30	1.26
$\beta = 3, \gamma = 10$	.23	1.05	.59	1.12	.74	1.15	.82	1.16	.89	1.18	.94	1.19	.99	1.20	1.03	1.21	1.06	1.21
$\beta = 4, \gamma = 12$	-.05	.99	.26	1.05	.46	1.09	.54	1.11	.57	1.11	.61	1.12	.64	1.13	.67	1.13	.69	1.14
$\beta = 4, \gamma = 4$	-.15	.92	-.24	.87	-.29	.84	-.33	.82	-.36	.81	-.38	.79	-.40	.78	-.41	.78	-.42	.77
$\beta = 4, \gamma = 6$	.11	1.03	.22	1.05	.28	1.06	.32	1.07	.35	1.08	.37	1.09	.39	1.09	.41	1.09	.42	1.10
$\beta = 4, \gamma = 8$	.63	1.13	.76	1.16	.93	1.19	1.04	1.21	1.13	1.23	1.19	1.24	1.25	1.25	1.29	1.26	1.33	1.27
$\beta = 4, \gamma = 10$	.69	1.14	.92	1.18	1.04	1.21	1.18	1.24	1.28	1.26	1.35	1.27	1.41	1.28	1.46	1.29	1.50	1.30
$\beta = 4, \gamma = 12$	.42	1.08	.85	1.17	.92	1.18	1.01	1.20	1.11	1.22	1.18	1.24	1.24	1.25	1.28	1.26	1.32	1.26
$\beta = 5, \gamma = 4$	-.03	.90	-.06	.80	-.09	.73	-.11	.68	-.12	.64	-.13	.60	-.14	.58	-.15	.56	-.15	.54
$\beta = 5, \gamma = 6$	-.25	.93	-.31	.91	-.34	.90	-.38	.89	-.40	.88	-.42	.88	-.44	.87	-.45	.87	-.46	.87
$\beta = 5, \gamma = 8$	.26	1.05	.43	1.09	.52	1.11	.60	1.13	.65	1.14	.69	1.15	.73	1.15	.76	1.16	.78	1.16
$\beta = 5, \gamma = 10$	.74	1.15	.85	1.17	1.07	1.21	1.18	1.24	1.27	1.26	1.34	1.27	1.40	1.28	1.45	1.29	1.49	1.30
$\beta = 5, \gamma = 12$	.83	1.17	.96	1.19	1.12	1.22	1.29	1.26	1.38	1.28	1.45	1.29	1.52	1.30	1.57	1.32	1.62	1.32

Table 5. B AND Ψ VALUES FOR FOUR TO 12 TIMEPOINTS WHEN GROWTH FOLLOWS A LOGISTIC GROWTH CURVE WITH $\alpha = 5$, AND COMBINATIONS OF β (3, 4, & 5) AND γ (4, 6, 8, 10, & 12) VALUES, $T \in [0, 1]$

<i>Parameters</i>	$F = 4$		$F = 5$		$F = 6$		$F = 7$		$F = 8$		$F = 9$		$F = 10$		$F = 11$		$F = 12$	
	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ	B	Ψ
$\beta = 3, \gamma = 4$	.05	1.01	.08	1.02	.10	1.03	.12	1.03	.13	1.04	.14	1.04	.14	1.04	.15	1.04	.15	1.05
$\beta = 3, \gamma = 6$	.24	1.05	.37	1.08	.45	1.10	.51	1.11	.56	1.12	.59	1.13	.62	1.14	.64	1.14	.66	1.15
$\beta = 3, \gamma = 8$	.27	1.06	.42	1.09	.52	1.11	.59	1.12	.64	1.14	.68	1.14	.72	1.15	.74	1.16	.77	1.16
$\beta = 3, \gamma = 10$	.11	1.02	.27	1.06	.35	1.07	.41	1.09	.45	1.09	.48	1.10	.51	1.11	.53	1.11	.55	1.11
$\beta = 4, \gamma = 12$	-.08	.98	.04	1.01	.11	1.02	.14	1.03	.17	1.04	.19	1.04	.20	1.04	.22	1.05	.23	1.05
$\beta = 4, \gamma = 4$	-.03	.99	-.04	.98	-.05	.98	-.05	.98	-.06	.98	-.06	.97	-.06	.97	-.07	.97	-.07	.97
$\beta = 4, \gamma = 6$	.14	1.03	.23	1.05	.29	1.07	.33	1.08	.36	1.08	.39	1.09	.41	1.09	.42	1.10	.44	1.10
$\beta = 4, \gamma = 8$	.39	1.08	.56	1.12	.68	1.14	.77	1.16	.83	1.17	.88	1.18	.93	1.19	.96	1.20	.99	1.21
$\beta = 4, \gamma = 10$	.40	1.08	.60	1.12	.72	1.15	.81	1.17	.88	1.18	.94	1.19	.98	1.20	1.02	1.21	1.05	1.21
$\beta = 4, \gamma = 12$	.23	1.05	.47	1.10	.57	1.12	.65	1.13	.71	1.14	.75	1.15	.79	1.16	.82	1.17	.85	1.17
$\beta = 5, \gamma = 4$	-.04	.97	-.06	.95	-.08	.94	-.09	.93	-.10	.93	-.10	.92	-.11	.92	-.11	.91	-.12	.91
$\beta = 5, \gamma = 6$	-.03	.99	-.03	.99	-.03	.99	-.02	.99	-.02	.99	-.02	.99	-.02	.99	-.02	.99	-.02	.99
$\beta = 5, \gamma = 8$	.27	1.06	.42	1.09	.52	1.11	.59	1.12	.64	1.14	.68	1.14	.72	1.15	.74	1.16	.77	1.16
$\beta = 5, \gamma = 10$	.53	1.11	.71	1.14	.86	1.17	.96	1.20	1.04	1.21	1.11	1.22	1.16	1.23	1.20	1.24	1.23	1.25
$\beta = 5, \gamma = 12$	.53	1.11	.73	1.15	.88	1.18	.99	1.20	1.07	1.22	1.13	1.23	1.18	1.24	1.23	1.25	1.26	1.25

EMPIRICAL INVESTIGATION OF THE DISTRIBUTION OF INSTANTANEOUS RATES OF CHANGE

Recall the discussion of Figure 1. The value selected to describe the 21 instantaneous rates of change was the mean. Recall that for illustrative purposes only 21 of the infinitely many rates of change were used for the presumed case of continuous time. At the present time it is not clear that the mean alone is sufficient to fully and adequately represent the set of instantaneous rates of change. Thus, a question that remains is how the change for an individual trajectory from T_1 to T_F should be described. One possibility that emerges is examining the distribution of instantaneous rates of change in order to discern characteristic properties of the instantaneous rates of change.

Apparently the distribution of instantaneous rates of change from an individual trajectory has not previously received much attention. Thus, the shape of such a distribution, its median, and its standard deviation are not well-known. The majority of the present work has been centered around describing growth via a single summary value, namely the mean of all possible instantaneous rates of change for an individual trajectory. Unacknowledged until the present section has been the implicit assumption that the mean of all possible rates of change, rather than some other measure, is the statistic of choice for describing and helping to understand growth for an individual trajectory. Although the mean rate of change is perhaps the most intuitive value for describing a collection of rates of change, questions remain as to whether the mean alone is the optimal value in the context of longitudinal data. Further questions remain regarding the shape and spread of the distribution

of instantaneous rates of change from an individual trajectory. This section explores other descriptive statistics that have the potential to be useful for describing the instantaneous rate of change over time for an individual trajectory.

For 23 functional forms of growth a distribution of instantaneous rates of change was produced by evaluating the derivative of the function at a large number of points (The number of timepoints, $\tau = 10^{-6}$, was 1,000,001 for the first 18 conditions ($T \in [0, 1]$) and 4,000,001 for the remaining 5 conditions ($T \in [1, 5]$), thus time was nearly continuous.). This collection of instantaneous rates of change was used to calculate descriptive statistics in order to document selected distributional properties of the instantaneous rates of change for individual trajectories for a variety of functional forms of growth. The mean, median, standard deviation, skew, kurtosis, and selected percentiles (2.5, 5, 95, 97.5) of the instantaneous rates of change were calculated.²³ The particular functional forms of growth examined and the accompanying descriptive statistics of the distribution of instantaneous rates of change are given in Table 6.

²³Measures of skew and kurtosis were calculated in accord with the recommendations of Stuart and Ord (1994, p. 109). Specifically, skew was defined as $\frac{n^{-1} \sum_{i=1}^n x_i^3}{\left(n^{-1} \sum_{i=1}^n x_i^2\right)^{1.5}}$ whereas kurtosis was defined as $\frac{n^{-1} \sum_{i=1}^n x_i^4}{\left(n^{-1} \sum_{i=1}^n x_i^2\right)^2} - 3$, where x_i is $(X_i - \bar{X})$ and n is the number of X_i values.

Table 6. DESCRIPTIVE VALUES OF THE DISTRIBUTION OF INSTANTANEOUS RATES OF CHANGE FOR THE SELECTED FUNCTIONAL FORMS OF GROWTH

<i>Growth Model</i>	$\overline{RC}$	50%ile	SD	Skew	Kurtosis	2.5%ile	5%ile	95%ile	97.5%ile
$T \in [0, 1]$									
$Y = T^2$	1	1	0.577	0	-1.2	0.05	0.1	1.9	1.95
$Y = T^3$	1	0.75	0.894	0.639	-0.857	0.002	0.008	2.708	2.852
$Y = T^4$	1	0.5	1.134	1.058	-0.092	0	0.001	3.43	3.707
$Y = T^5$	1	0.313	1.333	1.385	0.787	0	0	4.073	4.518
$Y = .5T + .5T^2$	1	1	0.289	0	-1.2	0.525	0.55	1.45	1.475
$Y = 2T - T^2$	1	1	0.577	0	-1.2	0.05	0.1	1.9	1.95
$Y = 2T - .5T^2 - .5T^3$	1	1.125	0.73	-0.403	-1.066	-0.401	-0.304	1.946	1.974
$Y = 3.5T - 1.5T^2 - T^3$	1	1.25	1.746	-0.339	-1.106	-2.277	-2.058	3.343	3.423
$Y = \exp(T)$	1.718	1.649	0.492	0.344	-1.07	1.025	1.051	2.586	2.651
$Y = 5 + 3 \exp(-4T)$	-2.945	-1.624	3.053	-1.269	0.573	-10.858	-9.825	-0.268	-0.243
$Y = 5 + 4 \exp(-8T)$	-3.999	-0.586	6.929	-2.181	4.073	-26.199	-21.45	-0.016	-0.013
$Y = 5 + 5 \exp(-12T)$	-5	-0.149	11.18	-2.862	7.92	-44.449	-32.929	-0.001	0
$Y = 5 \exp(-\exp(3 - 4T))$	3.461	3.587	2.969	0.012	-1.739	0	0	7.321	7.348
$Y = 5 \exp(-\exp(4 - 8T))$	4.909	2.77	5.086	0.709	-0.988	0	0	14.424	14.642
$Y = 5 \exp(-\exp(5 - 12T))$	4.995	0.881	7.074	1.296	0.207	0	0	21.106	21.826
$Y = \frac{5}{1+\exp(3-4T)}$	3.418	3.932	1.381	-0.448	-1.307	0.989	1.081	4.988	4.997
$Y = \frac{5}{1+\exp(4-8T)}$	4.82	4.2	3.168	0.3	-1.379	0.778	0.856	9.901	9.975
$Y = \frac{5}{1+\exp(5-12T)}$	4.962	2.711	5.037	0.784	-0.868	0.074	0.099	14.667	14.916
$T \in [1, 5]$									
$Y = \sqrt{T}$	0.309	0.289	0.071	0.863	-0.234	0.226	0.228	0.456	0.477
$Y = \frac{1}{T}$	-0.2	-0.111	0.207	-1.859	2.914	-0.826	-0.694	-0.043	-0.042
$Y = \frac{1}{\sqrt{T}}$	-0.138	-0.096	0.104	-1.527	1.624	-0.433	-0.38	-0.048	-0.046
$Y = .25\sqrt{T} + .25T + .25T^2 + .25T^3$	9.577	8.572	5.826	0.407	-1.065	1.827	2.044	19.987	20.764
$Y = \log(Time)$	0.402	0.333	0.195	1.191	0.566	0.204	0.208	0.833	0.909

In each of the 23 conditions the obtained mean rate of change ($\overline{RC}$) of the distribution of instantaneous rates of change was equal to the known ARC, as must be the case given the previous discussion of the ARC. It is interesting to compare the $\overline{RC}$ with the 50%ile (median) to see under what conditions the mean and the median are equal. As shown in Table 6, the mean and the median are generally not equal to one another, illustrating that the shape of the distribution of instantaneous rates of change is not often symmetric. In fact, examining the skew of the distributions (skew equals zero when a distribution is symmetric) shows that there are cases where the distribution was positively skewed (12 instances), negatively skewed (8 instances), and few cases where the distribution was symmetric (3 instances).

The three distributions that were not skewed corresponded exactly to a uniform distribution. A distribution where skew = 0 and kurtosis = -1.2 is a uniform (i.e., rectangular) distribution (DeCarlo, 1997, p. 293). These three distributions are cases where β_{SLGM} for the functional form of growth and the ARC are equivalent across time and cases where growth is represented by a quadratic trend or by a combination of linear and quadratic trend. Notice that absent from Table 6 is the functional form for straight-line growth. Information about the distribution of instantaneous rates of change for this particular growth model is not needed, as it is known a priori. The mean instantaneous rates of change is the slope itself over the range of time. Thus, the mean, median and percentiles are equal to the slope, while the standard deviation, skew, and kurtosis are zero.

Table 6 illustrates that understanding a particular trajectory over time, let alone a collection of trajectories, is a potentially daunting task. Because no consistent distributional form emerged for the instantaneous rates of change for the selected functional forms of growth, researchers should be aware that understanding growth trajectories over time potentially involves more than a single descriptive value to

adequately describe the distribution of instantaneous rates of change. Although the ARC is a useful summary value, the same shortcomings of the mean as a descriptive and inferential value, which have been well documented in the literature, also apply to the ARC in the present context. Nevertheless, regardless of the distributional form of the instantaneous rates of change, the ARC describes the mean of all theoretically possible instantaneous rates of change, a value of potential interest to researchers who study change over time. Presumably such a statistic will help researchers describe and potentially better understand the typical rate at which a trajectory is, or is not, changing over time.²⁴

²⁴Although gaining a better understanding of the distribution of the instantaneous rates of change or even the ARC itself may well be helpful for describing and understand the process of change, the importance of estimating and interpreting the parameters of growth should not be neglected. Not only does the ARC provide potentially useful information, so too do the parameters of growth (perhaps even more so by estimating asymptotic values, scaling factors, and linear and/or quadratic components, etc.) and should also be of utmost concern. As previously stated, the proposed usage for the ARC is to supplement not supplant other measures of change. Thus, estimation of the growth parameters should not be neglected because interest is in the ARC.

PRELIMINARY SUGGESTIONS AND CAUTIONS WHEN ESTIMATING THE AVERAGE RATE OF CHANGE

Conspicuously absent from the work thus far have been suggestions and procedures for estimating the ARC. To thoroughly delineate such a topic would require a lengthy work in itself. However, the present section allows for the discussion of potentially useful methods when interest lies in estimating the ARC for a single individual as well as the mean ARC across individuals. This section also provides an opportunity to voice concerns about intuitively appealing, but perhaps not statistically optimal, methods of estimating the ARC.

The Relationship Between the Difference Score and the Average Rate of Change

The observed difference score (also called change score or gain/loss score depending on its sign) for a single individual over the time period T_t to $T_{t+\tau}$ is defined as

$$D_{t+\tau,t} = Y_{t+\tau} - Y_t, \quad (59)$$

which is an estimate of the numerator of the true ARC given in Equation 5c. In the spirit of classical test theory (e.g., Lord & Novick, 1968), an observed score is a composite of a true score and error. An observed Y_t is thus equal to

$$Y_t = \xi_t + \varepsilon_t, \quad (60)$$

where ξ_t is the true score and ε_t is the error at the t th timepoint. Because the expected value of ε_t is zero, the expected value of Y_t is ξ_t . This equality extends

to the difference score, where it is shown that the expected value of the observed difference score is equal to the expected value of the true score difference:

$$E[D_{t+\tau,t}] = E[Y_{t+\tau} - Y_t] \quad (61a)$$

$$= E[Y_{t+\tau}] - E[Y_t] \quad (61b)$$

$$= \xi_{t+\tau} - \xi_t \quad (61c)$$

$$= \Delta_{t+\tau,t}, \quad (61d)$$

where $\Delta_{t+\tau,t}$ is the true difference score over the time period T_t to $T_{t+\tau}$. Thus, the observed difference score is an unbiased estimate of the population difference score, regardless of the amount of measurement error in the observed scores (Rogosa et al., 1982, p. 730).

Even though the difference score is an unbiased estimate, the difference score has been subjected to harsh criticisms (See, for instance, the works contained in Harris, 1963, Lord, 1956 & 1958, Cronbach & Furby, 1970, and Linn & Slinde, 1977, but also Rogosa et al., 1982, Zimmerman & Williams, 1982, and Willett, 1988, who show that many of the previous criticisms of the difference score are unwarranted.). The vast majority of criticisms regarding the difference score have to do with its reliability, and to a lesser extent, the *appearance* that the true initial status and the true difference score are negatively correlated, when in fact the observed correlation is a poor estimate of the true correlation (Rogosa et al., 1982, pp. 734-735; Blomqvist, 1977).²⁵ However, as Willett (1988) argues, “recent methodological research has

²⁵The reason for the “appearance” that true initial status and true change are negatively correlated is because the sample correlation coefficient between the observed initial status and observed change is negatively biased for estimating the population correlation of the true initial status and the true change. The spuriousness of the observed correlation coefficient, however, is “in no way . . . a fundamental problem with the use of the difference score as a measure of individual change” (Rogosa et al., 1982, p. 734). This is actually a long known fact (see Thomson, 1924) that the relationship between true initial status and true change is attenuated because of the errors in the observed scores (Lord, 1958). Because interest lies in the correlation between the true scores, not the observed, the negatively biased correlation between the observed initial status and observed change is not a valid criticism of the analysis of change.

revealed that these deficiencies are *perceived* rather than *actual*, *imaginary* rather than *real*” (p. 367). Modern research on the analysis of change overshadows many of the well-known critiques of yesteryear. Statements such as “the difference between two fallible measures is frequently much more fallible than either” measure (Lord, 1963, p. 32) has been refuted, because it has been realized the reliability of a difference can, in realistic situations, be more reliable than either of the fallible measures.

The definition of reliability in the context of difference scores can be given as

$$\rho(D) = \frac{\sigma_{\xi_{t+\tau}-\xi_t}^2}{\sigma_{\xi_{t+\tau}-\xi_t}^2 + \sigma_{\varepsilon_{t+\tau}-\varepsilon_t}^2}, \quad (62)$$

where $\rho(\cdot)$ represents the population reliability of the quantity in parentheses and σ^2 is the variance of the effect noted in the subscript and can be defined only over a group or population (i.e., not for a single difference; Rogosa et al., 1982, p. 730). When ε_t and $\varepsilon_{t+\tau}$ are independent within and across individuals, Equation 62 can be rewritten as

$$\rho(D) = \frac{\sigma_{Y_t}\rho(Y_t) + \sigma_{Y_{t+\tau}}\rho(Y_{t+\tau}) - 2\sigma_{Y_t}\sigma_{Y_{t+\tau}}\rho_{\xi_t\xi_{t+\tau}}}{\sigma_{Y_t}^2 + \sigma_{Y_{t+\tau}}^2 - 2\sigma_{Y_t}\sigma_{Y_{t+\tau}}\rho_{\xi_t\xi_{t+\tau}}}, \quad (63)$$

where ρ represents the population zero-order correlation coefficient of the two subscripted random variables (Rogosa et al., 1982, pp. 730–731).

Conceptually, the reliability of the difference score represents the accuracy in which individuals can be distinguished from one another on the basis of their difference score. If the variance of the difference scores across a group or population is small, meaning the difference scores are very similar, individuals will not be well differentiated on the basis of their difference score and thus $\rho(D)$ will be low. In such circumstances, even when the error variance ($\sigma_{\varepsilon_{t+\tau}-\varepsilon_t}^2$) is small, the reliability can still be low. In order for the reliability of the difference score to be large, individual

difference in growth must persist, such that individuals can be differentiated from one another on the basis of their difference score. However, as Rogosa et al. (1982) point out, “*low reliability does not necessarily imply lack of precision*” and in fact “the absence of such [individual] differences does not preclude meaningful assessment of individual change” (p. 731).

One of the main reasons why the difference score has so often been criticized is because authors commonly make use of restrictive special cases of Equation 62, which oftentimes may be unrealistic in the context of longitudinal data analysis. As Rogosa and Willett (1983) point out, authors who disparage the difference score usually do so in a context where $\sigma_{\xi_t} = \sigma_{\xi_{t+\tau}}$, $\rho(Y_t) = \rho(Y_{t+\tau})$, and for large positive values of $\rho_{\xi_t \xi_{t+\tau}}$ (p. 336). Thus, low reliability of the difference score comes at no surprise, because when $\rho_{\xi_t \xi_{t+\tau}}$ is large in the presence of constant reliabilities and variances, differentiating among individual difference scores is difficult. Rogosa and Willett (1983) summarize the dubious pre-1980s literature on difference scores by saying that the “difference score cannot detect individual differences in change that do not exist, but it will show good reliability when individual differences in true change are appreciable” (p. 341). Furthermore, the “difference score can be an accurate and useful measure of individual change even in situations where the reliability is low” (Rogosa et al., 1982, p. 730).²⁶

Because an obvious method of estimating the ARC is to divide the difference score by the time interval of interest,

$$\widehat{ARC}_{t+\tau,t} = \frac{D_{t+\tau,t}}{T_{t+\tau} - T_t}, \quad (64)$$

where $\hat{x}$ represents an estimate of the parameter x , the preceding remarks concerning

²⁶In addition to low reliability not markedly affecting the usefulness of the difference score, Overall and Woodward (1975) showed that as the difference score reliability *decreases*, the power of parametric statistical tests to detect group differences *increases*. This “paradox” helps to illustrate the fact that highly unreliable differences can be highly informative.

the difference score were necessary in order to develop the most natural method of estimating the ARC. Specifically, the reliability of the ARC when the ARC is calculated via Equation 64 is closely related to the reliability of the difference score. In fact, for fixed values of time, the reliability of the ARC (calculated via Equation 64) is equivalent to the reliability of the difference score.²⁷ The reason such an equivalency holds is because when the time interval of interest is fixed (i.e., constant), the divisor of the ARC simply scales both $\sigma_{\xi_{t+\tau}-\xi_t}$ and $\sigma_{\varepsilon_{t+\tau}-\varepsilon_t}$ (Equation 62) by $T_{t+\tau} - T_t$. Thus, the constant scaling cancels in the equation for the reliability of the ARC and equals the reliability of the difference score. This relationship is best illustrated by the definitional formula given in Equation 62 applied to the ARC:

$$\rho(\widehat{ARC}_{t+\tau,t}) = \frac{\frac{\sigma_{\xi_{t+\tau}-\xi_t}^2}{(T_{t+\tau}-T_t)^2}}{\frac{\sigma_{\xi_{t+\tau}-\xi_t}^2}{(T_{t+\tau}-T_t)^2} + \frac{\sigma_{\varepsilon_{t+\tau}-\varepsilon_t}^2}{(T_{t+\tau}-T_t)^2}} = \frac{\frac{\sigma_{\xi_{t+\tau}-\xi_t}^2}{(T_{t+\tau}-T_t)^2}}{\frac{\sigma_{\xi_{t+\tau}-\xi_t}^2 + \sigma_{\varepsilon_{t+\tau}-\varepsilon_t}^2}{(T_{t+\tau}-T_t)^2}} = \frac{(\sigma_{\xi_{t+\tau}-\xi_t}^2)(T_{t+\tau}-T_t)^2}{(T_{t+\tau}-T_t)^2(\sigma_{\xi_{t+\tau}-\xi_t}^2 + \sigma_{\varepsilon_{t+\tau}-\varepsilon_t}^2)} = \rho(D). \quad (65)$$

It is important to remember that Equation 65 holds only for fixed values of time. If the time interval of interest is random across replications of the ARC, it is not clear to what extent the reliability of $\widehat{ARC}_{t+\tau,t}$ may change. The reason for such uncertainty is because when the time interval is random across replications, the variance of the true scores and the variance of the errors are themselves divided by a random quantity. Khuri and Casella (2002) provide a discussion on the difficulties in determining $E(1/x)$, where x is some nonnegative continuously distributed random variable, and the sufficient conditions for such an expectation to exist for continuous distributions. Apparently no exact analog of the continuous case exists for $E(1/x)$

²⁷In order to derive the reliability of the ARC, the variance of the ARC must be established. Appendix D shows the derivation of the variance of an individual's difference score as well as the variance of $\widehat{ARC}_{t+\tau,t}$.

when x is discrete.²⁸ Furthermore, when time is fixed and $\widehat{ARC}_{t+\tau,t}$ is used to estimate the ARC, $\widehat{ARC}_{t+\tau,t}$ is an unbiased estimate of the true ARC. This can be seen in an analogous manner as the reliability of the ARC:

$$E\left(\widehat{ARC}_{t+\tau,t}\right) = E\left[\frac{Y_{t+\tau} - Y_t}{T_{t+\tau} - T_t}\right] = \frac{1}{T_{t+\tau} - T_t} E[Y_{t+\tau} - Y_t] = \frac{\xi_{t+\tau} - \xi_t}{T_{t+\tau} - T_t}. \quad (66)$$

Thus, because the expected value is exactly equal to the true $ARC_{t+\tau,t}$, it is an unbiased estimate of the true ARC. Like the reliability of the ARC, it is not clear if the ARC remains unbiased when $(T_{t+\tau} - T_t)$ is a random quantity.

Although Equation 64 yields an unbiased estimate of the true ARC, in cases where $F > 2$ such a method literally ignores all data between T_1 and T_F . Ignoring data is not generally considered advisable. However, as Overall and Tonidandel (2002) show, when the dependent variable follows an order one autoregressive correlational structure with monotonically increasing growth over time, the ordinary difference score is more powerful for detecting group differences than using β_{SLGM} and is essentially as powerful as the generalized least squares approach where the autoregressive correlational structure is explicit in the model. Overall and Tonidandel show this in the context of two stage (i.e., intensive design) growth models (see page 115 and the power comparisons in Table 1 of Overall and Tonidandel, 2002). Thus, if the order one autoregressive correlational structure holds, a potentially complicated generalized least squares analysis can effectively reduce to an analysis of difference scores. The next section illustrates two different methods of estimating the ARC.

²⁸Mood, Graybill, and Boes give an approximation to $E\left[\frac{x_1}{x_2}\right]$, where x_1 and x_2 are random variables (1974, p. 181). Such an approximation would allow for the expected value of Equation 64 to be estimated when the time interval of interest was not considered fixed, as is the case in the present context, but when the time interval is random across the individuals.

Other Suggestions for Estimating the Average Rate of Change

Suppose that one makes the strong, and potentially untenable, assumption that the growth model selected to represent the true functional form of growth is literally the correct model. This supposition, however, is not trivial and the method described under this assumption relies heavily on the model being correct. Nevertheless, if the growth model selected by the researcher is correct, the approach taken for producing the distributions of instantaneous rates of change for the results contained in Table 1 may provide interesting and potentially useful information.

The empirical estimation of the instantaneous rates of change could proceed by first assuming all individuals are governed by the same functional form of growth.²⁹ After the model has been fit and the estimates of the fixed effects obtained, the derivative of the assumed functional form of growth should be computed. The derivative can be for a set of individuals or generalized such that different groups each have their own derivative. Once the derivative (derivatives for multiple group designs) has been computed, the derivative should then be evaluated over a nearly continuous time period in a computer program. The values of time can be thought of as a vector from T_1 to T_F with τ being some small (e.g., 10^{-4}) value. This vector of nearly continuous time values can then be used for evaluating the derivative, such that a large number of instantaneous rates of change are obtained.

²⁹This assumption, while not explicit in applications of HLM type analyses, is generally implicit in the interpretation of results. When individuals (or groups) change following a different functional form of growth, modeling them as though they follow the same functional form leads to potentially misleading results. What does it mean for a significant linear trend to be found across a set of individuals when the individuals' change follows different functional forms of growth? For example, some individuals may follow exponential growth, some Gompertz growth, and still others quadratic growth. Obtaining statistical significance for a particular trend (e.g., linear and/or quadratic) when the individual trajectories follow different forms of growth is not likely to shed much light on the underlying question(s) of interest. The same problems arise when making group comparisons. For example, finding a significant difference for the slope between two groups means little if the functional form of growth is different in the groups (let alone if individuals within the groups have different functional forms of growth). Discussions of this sort do not seem to manifest themselves often in methodological HLM works or in the context of longitudinal data analysis in general. In situations where individuals follow different functional forms of growth, the ARC may be an ideal measure, because even when individuals have different functional forms of growth, their ARC still provides meaningful information about the mean rate of change across time.

Once the set (i.e., vector) of instantaneous rates of change is computed, the mean, median, standard deviation, confidence intervals, et cetera, can be computed as was done in Table 6. Of course, because the obtained fixed effects are literally estimates of the population values they represent, the computed statistics will not literally be equal to their corresponding parameters. At the present time it is not clear how well the confidence interval obtained by finding the value of the instantaneous rate of change for selected lower and upper percentiles corresponds to the percentiles of the population distribution of the instantaneous rate of change.

A possibility for yielding what could probabilistically be thought of as a “worst” and/or “best” case scenario is to calculate the derivative *not* given the obtained point estimates, but rather calculating the derivative using the lower and/or upper confidence limits of the effect. Again, at the present time it is not clear how well such a method may work. However, the basis for such a suggestion is that if the lower and/or upper bound are treated as point estimates when the derivatives are calculated, loosely speaking, those values represent, with the selected probability, likely limits for the population parameter(s). Thus, if the population value(s) are no smaller or no larger (probabilistically speaking) than the bounds of the confidence interval, then the procedure offered here yields bounding distribution(s) for the population instantaneous rates of change. Descriptive statistics can then be calculated for these bounding distributions such that the estimates represent one or both extremes of the probabilistic worst and best case scenarios for the distribution of instantaneous rates of change. As the width of the confidence interval around the population value approaches zero, the upper and/or lower bound(s) of the confidence interval approach the obtained point estimate. Thus, the bounding distributions of the instantaneous rates of change approach the distribution when the point estimate itself is used. Thus, the more precise the obtained point estimate,

the more consistent the statistics from the bounding distributions will be with the distribution of instantaneous rates of change based on the obtained estimate. As a reminder, performing an analysis of this type requires the researcher to make strong assumptions regarding the nature of the underlying growth curve, namely that the selected growth curve is equal to the true growth curve.

Another method of estimating the ARC is based on the strong assumption of straight-line growth. Recall that if the selected growth curve is the straight-line growth model, the instantaneous rate of change is constant across the range of time. Thus, the mean, median, and all percentiles will equal β_{SLGM} . For such straight-line growth models each individual β_{SLGM} provides an estimate of the individuals' ARC. In fact, each of the N β_{SLGM} s represent what can be thought of as estimates for each individual's latent difference score that has been scaled by a factor of $T_{t+\tau} - T_t$ (Rausch & Maxwell, 2003). Such a scaling of the difference score yields what can be thought of as a latent ARC. An individual's latent ARC estimate can also be estimated by making use of the unique slopes calculated via a HLM analysis. Such slopes are empirical Bayes estimates that combine information across individuals in order to have a better estimate of each individual's slopes (See Bryk & Raudenbush, 1987, for empirical Bayes applied within the analysis of change framework and Morris, 1983, for more technical information regarding the general theory of empirical Bayes estimation procedures.). Also of interest in the context of latent estimates of the ARC is the fact that when there is equal spacing between timepoints, growth that follows a quadratic or a combination of linear and quadratic growth fit with the straight-line growth model yields unbiased estimates of each individual's ARC (This fact can be verified via Equations 35 and 36 when time is continuous or via Equations 57 and 58 when time is discrete and equally spaced measurements are obtained.).

One may wonder why a discussion of estimation procedures for the ARC is necessary, given the fact that the ARC calculated via Equation 64 yields an unbiased estimate of the true ARC when time is fixed. While unbiasedness is a desirable property in the context of parameter estimation, the precision of the estimate is also important. For example, when ε_t and $\varepsilon_{t+\tau}$ are independent, which is typically assumed in psychometric contexts, the population variance of the ARC for an individual calculated via Equation 64 is,

$$\sigma^2_{\widehat{ARC}_{t+\tau,t}} = \frac{\sigma_{\varepsilon_t}^2 + \sigma_{\varepsilon_{t+\tau}}^2}{(T_{t+\tau} - T_t)^2}, \quad (67)$$

and equals,

$$\sigma^2_{\widehat{ARC}_{t+\tau,t}} = \frac{2\sigma_\varepsilon^2}{(T_{t+\tau} - T_t)^2}, \quad (68)$$

when there is homogeneity of error variance (Recall that the details leading to the derivation of Equations 67 and 68 are given in Appendix D). The variance for the estimated regression coefficient for an individual when the errors are uncorrelated and have constant variance across time can be written as

$$\sigma^2_{\widehat{\beta}_{SLGM}} = \frac{\sigma_\varepsilon^2}{\sum_{t=1}^F (T_t - \mu_T)^2}. \quad (69)$$

Note that in the special case of two or three equally spaced timepoints, Equation 69 reduces to

$$\sigma^2_{\widehat{\beta}_{SLGM}} = \frac{\sigma_\varepsilon^2}{(T_{t+\tau} - T_t)^2/2} = \frac{2\sigma_\varepsilon^2}{(T_{t+\tau} - T_t)^2}, \quad (70)$$

which is equal to the variance of $\widehat{ARC}_{t+\tau,t}$ (Equation 68).³⁰

The population precision of $\widehat{ARC}_{t+\tau,t}$ can be directly compared to the population precision of $\widehat{\beta}_{SLGM}$. The comparison proceeds in the form of the ratio of variances.

³⁰Note that the estimation of σ_ε^2 necessarily involves more than one individual for the variance of $\widehat{ARC}_{t+\tau,t}$ and $\widehat{\beta}_{SLGM}$ when $F = 2$. As Rogosa et al. (1982, p. 730) have pointed out, with information from only two waves of data, σ_ε^2 cannot be estimated. However, when $F \geq 3$, σ_ε^2 can be estimated for $\widehat{\beta}_{SLGM}$ by $\widehat{\sigma}_Y^2(1 - \widehat{\rho}_{YT}^2)$.

Specifically the variance of $\widehat{ARC}_{t+\tau,t}$ will be compared to the variance of $\widehat{\beta}_{SLGM}$ when $T \in [T_t, T_F]$. The ratio of precisions is given as,

$$\frac{\sigma^2_{\widehat{ARC}_{F,1}}}{\sigma^2_{\widehat{\beta}_{SLGM}}} = \frac{\frac{2\sigma_\varepsilon^2}{(T_F - T_1)^2}}{\frac{\sigma_\varepsilon^2}{\sum_{t=1}^F (T_t - \mu_T)^2}} = \frac{2 \sum_{t=1}^F (T_t - \mu_T)^2}{(T_F - T_1)^2}, \quad (71)$$

and depends only on F . Although Equation 71 may initially appear as though it depends on the particular scaling of time, further investigation reveals that Equation 71 is independent of the scaling of time (i.e., the ratio is invariant to the scaling of time). The irrelevance of the scaling of time can be seen by realizing time can be scaled by a factor of A . In such a case $(T_F - T_1)^2$ in Equation 67 becomes $(AT_F - AT_1)^2$, which reduces to $A^2(T_F - T_1)^2$. Likewise, $\sum_{t=1}^F (T_t - \mu_T)^2$ in Equation 69 becomes $\sum_{t=1}^F (AT_t - A\mu_T)^2$, which reduces to $A^2 \sum_{t=1}^F (T_t - \mu_T)^2$. Thus, in the ratio of precisions, the A^2 s cancel making the scaling of time irrelevant when comparing the precision of $\widehat{ARC}_{t+\tau,t}$ to β_{SLGM} . Note that this is guaranteed to hold only in situations where time is equally spaced (constant τ).

Figure 8 illustrates the ratio of precisions of Equation 71 for three to 25 occasions of measurement. Initial inspection may appear to suggest that there is some constant of proportionality between F and the ratio of precisions. Further investigation by examining the regression line reveals that there is indeed a small amount of curvilinearity. Nevertheless, it is easy to see that the precision of $\sigma^2_{\widehat{\beta}_{SLGM}}$ quickly outperforms the precision of $\sigma^2_{\widehat{ARC}_{F,1}}$. When $F = 2$ or 3, the ratio is 1 and thus the precisions are the same. However, when $F = 5$, $\sigma^2_{\widehat{ARC}_{F,1}}$ has a variance 1.25 times larger than $\sigma^2_{\widehat{\beta}_{SLGM}}$. When $F = 10$ the ratio of $\sigma^2_{\widehat{ARC}_{F,1}}$ to $\sigma^2_{\widehat{\beta}_{SLGM}}$ is slightly larger than 2. When $F = 20$ the ratio is 3.68.

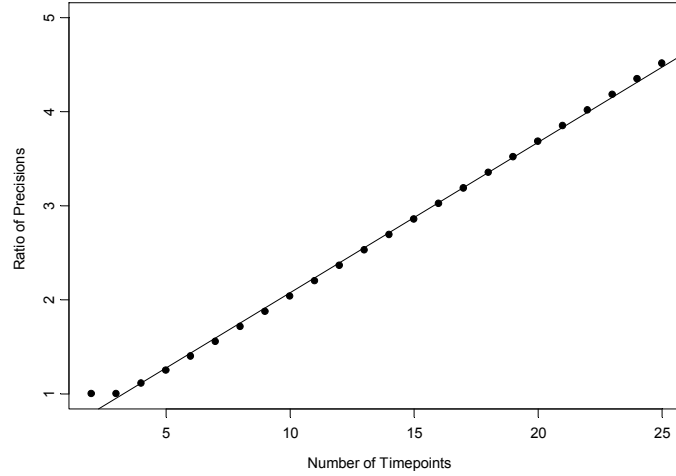


Figure 8. Ratio of the precision of $\widehat{ARC}$ to $\widehat{\beta}_{SLGM}$ for two to 25 equally spaced occasions of measurement when the assumption of straight-line growth is correct

While the most natural estimate of the ARC is dividing the difference score by the time interval (i.e., Equation 64), oftentimes the precision of such an estimate may be less than desired. While the regression coefficient from the straight-line growth model provides a generally more precise and unbiased estimate in the limited case of linear growth as well as combinations of quadratic and linear growth when timepoints are equally spaced, such an estimate of the ARC is not generally unbiased. Recall the previously mentioned estimation procedure that mandates the derivative of the particular growth model be specified and evaluated a large number of times over the time interval of interest in order to produce an empirical distribution of the instantaneous rate of change. While such a method potentially provides meaningful information, the problem is that the true functional form of growth must be known. To the extent that this assumption is false, the results may not be very meaningful. Rogosa et al. (1982) reviews the work of numerous authors in a review of various procedures that yield biased, albeit more precise estimates of the difference score. At the present time it is not clear which method

provides the best overall estimate of the ARC.

Although $\widehat{\beta}_{SLGM}$ is typically more precise than $\widehat{ARC}_{F,1}$, it is also a generally biased estimate of the ARC. However, precision alone does not necessarily lead to an optimal parameter estimate. Bias is also an important property when evaluating parameter estimates. Arguably the most important property of a parameter estimate is accuracy. Conceptually, accuracy is a measure of the discrepancy between an estimate and the true value it represents. Formally the square root of the mean square error is a measure of accuracy (Rozeboom, 1966, p. 500; Hellmann & Fowler, 1999). The accuracy of a measure is a function of the precision and bias and can be expressed as follows:

$$RMSE = \sqrt{E[(\hat{\theta} - \theta)^2]} = \sqrt{E[(\hat{\theta} - E[\hat{\theta}])^2] + (E[\hat{\theta} - \theta])^2}, \quad (72)$$

where RMSE is the square root of the mean square error, $\hat{\theta}$ is an estimate of θ , the true value of the parameter of interest. The first component under the second radical represents the precision while the second component represents the bias. Ideally unbiased estimates are preferred, however, if accuracy can be improved (reducing the RMSE) by adding bias and obtaining a more precise estimate, such a bias-accuracy tradeoff should be considered in order to obtain the estimates that are overall most accurate.

In the context of estimating the ARC, the bias-accuracy tradeoff surfaces in the context of at least two potential estimation techniques. The first situation involves literally using $\widehat{\beta}_{SLGM}$ as an estimate of the ARC. As can be seen from Equation 71 and Figure 8, in many circumstances the precision of $\widehat{\beta}_{SLGM}$ is much better than the precision of $\widehat{ARC}_{F,1}$. The bias, however, can be great for some functional forms of growth when $\widehat{\beta}_{SLGM}$ is used as an estimate of the ARC. Thus, the potential gain in precision is potentially negated by an increase in bias. Further research could

illustrate situations where the overall accuracy is improved by making use of the $\hat{\beta}_{\text{SLGM}}$ rather than Equation 64. Of course, when $F = 2$ or $F = 3$ equally spaced timepoints, both methods yield the same estimate and are equally precise. Another context where the bias-accuracy tradeoff surfaces is making use of biased, albeit more accurate, estimates for the true difference score (for instance, those reviewed in Rogosa et al., 1982). Obtaining better estimates of the difference score translates into better estimates of the ARC if these estimates are used in the numerator of Equation 64.

DISCUSSION

Statistical methods applied within the longitudinal setting should help researchers, and the research community in general, better understand and explain the *what*, *who*, *how*, *when* and *why* of longitudinal research. Research questions involving change over time, either for an individual or for a group, are inherently longitudinal in nature. In the context of experimental design, longitudinal studies are unique, as such designs allow individual trajectories (Rationale 1) to be modeled before identifying differences across individuals (Rationale 2) and correlates of change (Rationale 3). Understanding transitions over time for the individual is a precursor to learning what causes influence individual trajectories over time (Rationale 4). Learning what influences individual trajectories is a precursor to understanding what influences differences among individual trajectories over time (Rationale 5). Implicit in research questions involving change is the individual. Thus, intraindividual change should be the starting point for longitudinal research.

While understanding change for individual trajectories is an important goal for longitudinal data analysis, understanding such a potentially complicated processes is not often straightforward. This is especially true when several polynomial terms are included in the Level 1 growth model. For models more complicated than a single polynomial term, individual parameters cannot be explained without regard to the other parameters in the model, as each parameter affects the trajectory simultaneously. However, it is often the case that complicated polynomial growth models can be avoided by making use of an appropriate nonlinear model. Because nonlinear models generally have parameters whose meaning has a substantive

interpretation, researchers who make use of nonlinear growth models will likely make greater strides in understanding and explaining the process of change over time. It seems likely that as the general research community begins to recognize the value of nonlinear models, more nonlinear models will be employed and exciting questions, both novel and longstanding, will likely be better answered than if polynomial models continue to dominate the landscape of applied growth modeling in the behavioral sciences.

Although the present work contends nonlinear models are underutilized in applied applications of growth models in the behavioral sciences, the overarching theme is that the researchers carefully consider the Level 1 model of individual growth. Making use of a nonlinear model when the true functional form of growth is linear does not help clarify the underlying process of change. Likewise, making use of the straight-line growth model when the true functional form of growth is nonlinear also does not help to clarify the underlying process of change. The importance of the Level 1 model of individual growth (essentially) cannot be overemphasized.

Rather than “always” assuming the straight-line growth is appropriate, researchers should carefully consider other Level 1 models. Although it may seem simpler to consider only an intercept and a slope (i.e., the parameters of the straight-line growth model) than it is to consider the parameters from higher degree polynomial growth models or nonlinear models, such reasoning ignores the accuracy of the growth model itself. The point is that the making use of an incorrect model because the parameters are “straightforward” to interpret, may not shed much light on the underlying process of change. Ideally researchers should carefully consider the appropriateness of the Level 1 model for the study, analysis, and understanding of change. Perhaps the straight-line growth model is deemed the appropriate growth model after carefully considering other alternatives. In such scenarios the use of

the straight-line growth model was not based on a default mind-set, but rather on seriously considered alternatives to the straight-line growth model.³¹

In addition to understanding the importance of the correct Level 1 model, another dimension of the present work was to delineate the meaning and interpretation of the average rate of change (ARC). The ARC is a single value that describes the mean rate at which an individual trend changes over time. The ARC is literally the mean of all theoretically possible instantaneous rates of change across the time interval of interest. The present work contends that the ARC is a measure that can be used by researchers in order for potentially complicated processes of change to be better described and understood, not necessarily the specifics within an interval of time but over the whole interval. The suggested use of the ARC is to supplement, not supplant, descriptors of change that are currently in use. Presumably the ARC will be appreciated by applied researchers who seek to model, understand, and describe overall change parsimoniously.

Unfortunately the ARC has been, both implicitly and explicitly, conceptualized as the regression coefficient from the straight-line growth model in the methodological literature as well as in applications of straight-line growth models in substantive research. However, as Equations 35, 36, 57, and 58 demonstrate, the

³¹Model checking and model diagnostics are important to ensure that the selected model (e.g., the straight-line growth model) is a reasonable choice given the obtained data. Although model checking and diagnostics are beyond the scope of the present work, they should be considered when working within the context of unknown Level 1 growth models. Model checking and diagnostics are especially important when making use of the procedures outlined in the *Empirical Investigation of the Distribution of Instantaneous Rates of Change* section, where the strong assumption is made that the true functional form of growth used in the estimation of the distribution of the instantaneous rates of change is correct. Although model checking and diagnostics are undeniably important, they are more appropriate in the context of exploratory analyses rather than confirmatory ones. For example, if no model is specified a priori and one attempts several models for one that fits the obtained data best, the probability values associated with the statistical significance tests are not accurate, as fitting multiple growth models to find one that “fits better” capitalizes on chance. For confirmatory analyses a model is specified and then fit given the obtained data. Such a confirmatory analysis results in probability values that, provided the assumptions of the statistical procedure are met, do not capitalize on chance.

regression coefficient from the straight-line growth model generally does not equal the ARC for an individual's trajectory. The bias between the two values can be positive or negative, potentially yielding misleading conclusions regarding change over time. One or more of the following five situations must be met in order for the regression coefficient from the straight-line growth model to be an unbiased estimate of the ARC:

1. the true functional form of growth must be completely linear;
2. the true functional form of growth must be completely quadratic with equally spaced measurements;
3. the true functional form of growth must be some combination of linear and quadratic with equally spaced measurements;
4. growth is described by two timepoints;
5. growth is described by three equally spaced timepoints.

In general, the discrepancy between the true β_{SLGM} and the ARC will increase as the number of timepoints increases; the case where time is continuous will thus yield the greatest amount of bias.

The close connection between the ARC and the difference score was delineated. When time is fixed the ARC estimated via Equation 64, like the difference score, is an unbiased estimate (because fixed values of time do not affect the expectation) of the true ARC with reliability equal to that of the difference score. Although the ARC can be estimated in an unbiased fashion (regardless of the extent of measurement error in observed scores), estimation of the ARC was not the purpose of the present work. However, there were several methods proposed that may prove useful in the estimation of the ARC.

Table 6 illustrated the fact that the distribution of instantaneous rates of change for an individual trend, which apparently is not well studied, is often nonsymmetric and can drastically deviate from a normal (even symmetric) distribution. While

it is believed the mean instantaneous rate of change is a useful and important measure for describing and attempting to understand individual trajectories, other measures that describe the distribution of instantaneous rates of change may also provide meaningful information. It would be beneficial to conduct more research on the estimation of the distribution of instantaneous rates of change, in order for the ARC to be supplemented by other measures of the instantaneous rate of change (e.g., various percentiles and measures of dispersion). Estimating derivatives is an area currently receiving attention in the context of dynamical systems (systems where the current state of the system depends, at least in part, on previous states of the system; Boker & Nesselroade, 2002). Perhaps some of the estimation procedures developed within the dynamic systems context will be useful for estimating properties of the distribution of instantaneous rates of change for individual trajectories.

Absent from the present work was the effect of misidentified growth models for individual trends as they relate to group comparisons. Group comparisons of longitudinal data are often carried out in the context of HLMs and HNLMs. The implications that a misspecified Level 1 model can have on group comparisons is potentially troublesome. By misspecifying individual trajectories, group comparisons are effectively comparing sets of erroneous models with other sets of erroneous models. There is no reason to believe the specification errors for individuals will cancel across individuals and then groups. Thus researchers may well be led astray when attempting to make meaningful group comparisons when incorrect models are fit, even in randomized designs.

An example of a misspecification of individual trajectories and its affect on group comparisons is when there is differential attrition across groups. For example, suppose that the null hypothesis is true and the experimental and control group literally do not differ in their mean trajectory over time. Further assume that the

individual trends follow an asymptotic regression growth model. If little or no data are missing, say in the experimental group, but yet the probability of missing data increases over time for the control group, the straight-line growth model would give the impression that the typical trend is a steeper negative slope for those in the control group. Such a “result” would likely be interpreted as the control group’s scores were decreasing quicker when compared to the experimental group. However, because the null hypothesis was true, the discrepant slopes are a function of two events: (a) fitting the wrong model and (b) differential attrition. If the groups had no attrition or even the same type and likelihood of missingness, the model would have been misspecified but, there would have likely been a failure to reject the null hypothesis (i.e., a correct decision would have likely been made). However, because the control group had more participants earlier in the study, the regression slope would have been weighted heavier for the early scores and later scores would have affected the estimated slope less. Such a problem would not have existed if the correct model would have been fit. When the correct model is fit and data are missing at random (or more stringently completely at random), HLMs and HNLMs still yield maximum likelihood estimates of the population parameters (see Raudenbush & Bryk, 2002 but also Dempster, Laird, & Rubin, 1977 for seminal work on the topic).

While there is no doubt that longitudinal data analysis will continue to develop and improve upon existing techniques, it is believed that the present work adds to the rich analysis of change literature by clarifying and extending the current understanding of the ARC. While it is not suggested that the ARC supplant currently used descriptors of change, it is suggested that the ARC supplement them. It is believed that the ARC will help researchers and the research community in the ongoing quest for a better understanding of the dynamic and static relationship that

exist among sets of variables over time.

APPENDIX A

The numerator of Equation 32 can be simplified by realizing the whole quantity can be multiplied by λ_k and then summed over rather than doing the same operations for both components separately:

$$\text{NE32} = \sum_{k=1}^K \left(\lambda_k \left[\frac{(T_F^{\Upsilon_k+2} - T_1^{\Upsilon_k+2})}{\Upsilon_k + 2} - \frac{T_F + T_1}{2} \sum_{k=1}^K \frac{(T_F^{\Upsilon_k+1} - T_1^{\Upsilon_k+1})}{\Upsilon_k + 1} \right] \right), \quad (\text{A1})$$

where NE32 is the numerator of Equation 32.

A common denominator for the two components is obtained by multiplying the first component by $2(\Upsilon_k + 1)$ and the second component by $(\Upsilon_k + 2)$:

$$\text{NE32} = \sum_{k=1}^K \left(\lambda_k \left[\frac{2(\Upsilon_k + 1)(T_F^{\Upsilon_k+2} - T_1^{\Upsilon_k+2}) - (\Upsilon_k + 2)(T_F + T_1)(T_F^{\Upsilon_k+1} - T_1^{\Upsilon_k+1})}{2(\Upsilon_k + 1)(\Upsilon_k + 2)} \right] \right). \quad (\text{A2})$$

After multiplying the appropriate quantities in the numerator of Equation A2, simplifying like-terms, and placing the 2 in the denominator outside of the summation (because it is a constant), the reduced numerator of Equation 32 can be written as:

$$\text{NE32} = \frac{1}{2} \sum_{k=1}^K \left(\frac{\lambda_k \left[\Upsilon_k (T_F^{\Upsilon_k+2} - T_1^{\Upsilon_k+2} + T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1}) + 2 (T_F T_1^{\Upsilon_k+1} - T_1 T_F^{\Upsilon_k+1}) \right]}{(\Upsilon_k + 2)(\Upsilon_k + 1)} \right). \quad (\text{A3})$$

The three components of the denominator of Equation 32 can be multiplied by 4, 6, and 3 respectively, such that a common denominator of 12 allows like terms to

be combined across the three:

$$\text{DE32} = \frac{4(T_F^3 - T_1^3) - 6(T_F + T_1)(T_F^2 - T_1^2) + 3(T_F + T_1)^2(T_F - T_1)}{12}, \quad (\text{A4})$$

where DE32 is the denominator of Equation 32.

After multiplying out the quantities of Equation A4 and then simplifying, the resultant value is:

$$\text{DE32} = \frac{(T_F - T_1)^3}{12}. \quad (\text{A5})$$

Equation 33 in the text results from combining Equations A3 as the numerator and Equation A5 as the denominator.

APPENDIX B

The following Maple (Version 7.00, 2001) syntax allows one to symbolically derive the regression coefficient from the straight-line growth model, ARC, B, and Ψ for (essentially) any functional form of growth when time is considered continuous and Y can be written as a function of time. By setting ‘Y’ equal to some functional form of growth (by replacing ‘f(Time)’ with the functional form of interest), defining the time interval (or leaving it general as is done here), the syntax allows for potentially complicated derivations to be carried out in a straightforward manner. Note that ‘Y’ can be set equal to ‘Asymptotic_Regression,’ ‘Gompertz_Growth,’ or ‘Logistic_Growth’ directly because these functions are defined at the beginning of the syntax.

Maple Syntax

```
Asymptotic_Regression := beta*exp(-gamma*Time) + alpha;

Gompertz_Growth := alpha*exp(-exp(beta - gamma*Time));

Logistic_Growth := alpha/(1 + exp(beta - gamma*Time));

> Y := f(Time);
# Defines ‘Y’ as any functional form of growth, where ‘Time’ is the
# independent variable. Note that ‘Y’ can be any function, a linear
# or nonlinear model for example, with an arbitrary number of
# parameters. For example, ‘Y’ can be defined as some linear model,
# the ‘Asymptotic_Regression’ model, ‘Gompertz_Growth’ model,
# ‘Logistic_Growth’ model, or some other nonlinear model. There
# are (virtually) no restrictions on the complexity of the model.

> Low_Limit := TI;
# Defines the initial value of ‘Time’. ‘Low_Limit’ can be a numerical
```

```

# value or some arbitrary value, such as 'TI'.

> Up_Limit := TF;
# Analogous to 'Low_Limit', 'Up_Limit' defines the final value of
# 'Time'. 'Up_Limit' can be a numerical value or some arbitrary value,
# such as 'TF'.

> Interval_Length := Up_Limit-Low_Limit;
# Subtracts the lower limit from the upper limit to determine the
# total length (i.e., width) of the time interval of interest.

> Limits := Low_Limit..Up_Limit;
# Defines the limits in the language of Maple, such that the limits
# (of the integral) can be used in future calculations.

> mu_Y := int(Y, Time=Limits)/Interval_Length;
# Calculates the mean of 'Y'. Although not explicitly required,
# it may prove useful in certain contexts.

> mu_Time := int(Time, Time=Limits)/Interval_Length;
# Calculates the mean of 'Time'.

> Slope_SLGM := (int((Time-mu_Time)*(Y-mu_Y), Time=Limits))
/(int((Time-mu_Time)^2, Time=Limits));
# Calculates the regression coefficient from the straight-line
# growth model.

> simplify(Slope_SLGM);
# Simplifies the expression for the particular expression, the
# regression coefficient from the straight-line growth model in
# this situation.

> factor(Slope_SLGM);
# Rather than simplifying the particular expression, the regression
# coefficient from the straight-line growth model in this case, it may
# be of interest to factor the expression.

> ARC := (eval(Y, Time=Up_Limit)-eval(Y, Time=Low_Limit))
/(Up_Limit-Low_Limit);
# Defines the average rate of change for the specified function of
# interest.

> simplify(ARC);

```

```

> factor(ARC);

> Bias := Slope_SLGM-ARC;
# Calculates the bias between the regression coefficient from the
# straight-line growth model and the true average rate of change.

> simplify(Bias);

> factor(Bias);

> psi := Slope_SLGM/ARC;
# Calculates the discrepancy factor of the regression coefficient
# from the straight-line growth model as it relates to the true
# average rate of change.

> simplify(psi);

> factor(psi);

```

APPENDIX C

The following three Maple (Version 7.00, 2001) outputs give the regression coefficient from the straight-line growth model and the ARC for the asymptotic regression, Gompertz, and the logistic growth model respectively when time is considered continuous. The derivations are given for the most general case, where the initial and final timepoint are arbitrary, as are the three parameters that define each of the models. Although the mean of Y is not required for the derivations of the slope or the ARC, recall the previous discussion of the first moment about the mean (see page 43), the general mean of Y is given because some researchers may find it useful. The general regression coefficient from the straight-line growth model and the ARC are necessary for the derivations given in the *When Y Conforms to Certain Nonlinear Functions of Time* section. The Maple output below applies the syntax given in Appendix B.

Derivations for the Asymptotic Regression Growth Model:

Making Use of the Maple Syntax From Appendix B

```
> Y := beta*exp(-gamma*Time) + alpha;  
  
Y :=  $\alpha + \beta e^{-\gamma \text{Time}}$   
  
> Low_Limit := T1:  
  
> Up_Limit := TF:  
  
> Interval_Length := Up_Limit-Low_Limit:  
  
> Limits := Low_Limit..Up_Limit:
```

```
> mu_Y := int(Y, Time=Limits)/Interval_Length;
```

$$\text{mu_Y} := -\frac{-\alpha TF \gamma + \beta e^{-\gamma TF} + \alpha T1 \gamma - \beta e^{-\gamma T1}}{\gamma (TF - T1)}$$

```
> mu_Time := int(Time, Time=Limits)/Interval_Length;
```

$$\text{mu_Time} := \frac{1/2 TF^2 - 1/2 T1^2}{TF - T1}$$

```
> Slope_SLGM := simplify(((int((Time-mu_Time)*(Y-mu_Y), Time=Limits))
/(int((Time-mu_Time)^2, Time=Limits))));
```

$$\text{Slope_SLGM} := -6 \frac{(T1 e^{\gamma TF} \gamma + T1 \gamma e^{\gamma T1} - TF \gamma e^{\gamma T1} - TF e^{\gamma TF} \gamma + 2 e^{\gamma TF} - 2 e^{\gamma T1}) \beta e^{-\gamma (TF + T1)}}{\gamma^2 (-TF + T1)^3}$$

```
> ARC := (eval(Y, Time=Up_Limit)-eval(Y, Time=Low_Limit))/(Up_Limit-Low_Limit);
```

$$\text{ARC} := \alpha + \beta e^{-\gamma TF} - \frac{\alpha + \beta e^{-\gamma T1}}{TF - T1}$$

Derivations for the Gompertz Regression Growth Model:

Making Use of the Maple Syntax From Appendix B

```
> Y := alpha*exp(-exp(beta-gamma*Time));
```

$$Y := \alpha e^{-e^{\beta - \gamma \text{Time}}}$$

```
> Low_Limit := T1:
```

```
> Up_Limit := TF:
```

```
> Interval_Length := Up_Limit-Low_Limit:
```

```
> Limits := Low_Limit..Up_Limit:
```

```
> mu_Y := int(Y, Time=Limits)/Interval_Length;
```

$$\text{mu_Y} := \frac{\alpha (Ei(1, e^{\beta - \gamma TF}) - Ei(1, e^{\beta - \gamma T1}))}{\gamma (TF - T1)}$$

```
> mu_Time := int(Time, Time=Limits)/Interval_Length;
```

$$\text{mu_Time} := \frac{1/2 TF^2 - 1/2 T1^2}{TF - T1}$$

```
> Slope_SLGM := simplify((int((Time-mu_Time)*(Y-mu_Y), Time=Limits))
/(int((Time-mu_Time)^2, Time=Limits)));
```

$$\text{Slope_SLGM} := 6 \frac{\alpha \int_{Tl}^{TF} (Tl - 2 \text{Time} + TF) \left(-e^{-e^{\beta-\gamma} \text{Time}} \gamma TF + e^{-e^{\beta-\gamma} \text{Time}} \gamma Tl + Ei(1, e^{\beta-\gamma} TF) - Ei(1, e^{\beta-\gamma} Tl) \right) d\text{Time}}{\gamma (-TF + Tl)^4}$$

```
> ARC := (eval(Y, Time=Up_Limit)-eval(Y, Time=Low_Limit))/(Up_Limit-Low_Limit);
```

$$\text{ARC} := \frac{\alpha e^{-e^{\beta-\gamma} TF} - \alpha e^{-e^{\beta-\gamma} Tl}}{TF - Tl}$$

Derivations for the Logistic Growth Model:

Making Use of the Maple Syntax From Appendix B

```
> Y := alpha/(1 + exp(beta-gamma*Time));
```

$$Y := \frac{\alpha}{1 + e^{\beta-\gamma \text{Time}}}$$

```
> Low_Limit := Tl:
```

```
> Up_Limit := TF:
```

```
> Interval_Length := Up_Limit-Low_Limit:
```

```
> Limits := Low_Limit..Up_Limit:
```

```
> mu_Y := int(Y, Time=Limits)/Interval_Length;
```

$$\text{mu_Y} := -\frac{\alpha \left(-\ln(1 + e^{\beta-\gamma TF}) + \ln(e^{\beta-\gamma TF}) + \ln(1 + e^{\beta-\gamma Tl}) - \ln(e^{\beta-\gamma Tl}) \right)}{\gamma (TF - Tl)}$$

```
> mu_Time := int(Time, Time=Limits)/Interval_Length;
```

$$\text{mu_Time} := \frac{1/2 TF^2 - 1/2 Tl^2}{TF - Tl}$$

```
> Numerator_Slope_SLGM := numer((int((Time-mu_Time)*(Y-mu_Y), Time=Limits))
/(int((Time-mu_Time)^2, Time=Limits)));
```

$$\begin{aligned} \text{Numerator_Slope_SLGM} := & 6\alpha(-TF + Tl)^3(Tl^2\gamma^2 + Tl \ln(e^{\beta-\gamma Tl})\gamma - \\ & Tl \ln(e^{\beta-\gamma TF})\gamma + Tl \ln(1 + e^{\beta-\gamma Tl})\gamma + Tl \ln(1 + e^{\beta-\gamma TF})\gamma - 2\gamma Tl\beta + \\ & 2 \operatorname{dilog}((e^{\gamma TF} + e^{\beta})e^{-\gamma TF}) - TF \ln(1 + e^{\beta-\gamma Tl})\gamma - 2 \ln(e^{\beta-\gamma Tl})\beta - TF^2\gamma^2 - \\ & TF \ln(1 + e^{\beta-\gamma TF})\gamma + 2\gamma\beta TF - 2 \operatorname{dilog}((e^{\gamma Tl} + e^{\beta})e^{-\gamma Tl}) + 2 \ln(e^{\beta-\gamma TF})\beta + \\ & TF \ln(e^{\beta-\gamma Tl})\gamma - TF \ln(e^{\beta-\gamma TF})\gamma) \end{aligned}$$

```
> Denominator_Slope_SLGM := denom((int((Time-mu_Time)*(Y-mu_Y), Time=Limits))
/((int((Time-mu_Time)^2, Time=Limits))));
```

```
Denominator_Slope_SLGM :=  $\gamma^2 (TF^2 - 2 TF TI + TI^2)^3$ 
```

```
> ARC := (eval(Y, Time=Up_Limit)-eval(Y, Time=Low_Limit))/(Up_Limit-Low_Limit);
```

```
ARC :=  $\left( \frac{\alpha}{1+e^{\beta-\gamma TF}} - \frac{\alpha}{1+e^{\beta-\gamma TI}} \right) (TF - TI)^{-1}$ 
```

APPENDIX D

Recall from Equation 60 that an observed score is a composite equal to the true score plus error. Extending this to the context of individual difference scores yields the following:

$$D_{t+\tau,t} = (\xi_{t+\tau} + \varepsilon_{t+\tau}) - (\xi_t + \varepsilon_t). \quad (\text{D1})$$

The variance of $D_{t+\tau,t}$ when the errors are uncorrelated with all other components for an individual is given as the following:

$$\sigma_{D_{t+\tau,t}}^2 = \sigma_{\xi_{t+\tau}}^2 + \sigma_{\xi_t}^2 + \sigma_{\varepsilon_{t+\tau}}^2 + \sigma_{\varepsilon_t}^2 - 2\sigma_{\xi_{t+\tau}-\xi_t}. \quad (\text{D2})$$

For a single individual the true score variance (and thus by implication the covariance) is zero, leading to a reduction of Equation D2:

$$\sigma_{D_{t+\tau,t}}^2 = \sigma_{\varepsilon_{t+\tau}}^2 + \sigma_{\varepsilon_t}^2. \quad (\text{D3})$$

When the error variance at time $t + \tau$ is equal to the error variance at time t (i.e., homoscedasticity), Equation D3 reduces to the following:

$$\sigma_{D_{t+\tau,t}}^2 = 2\sigma_{\varepsilon}^2. \quad (\text{D4})$$

Because $\widehat{ARC}_{t+\tau,t}$ is literally equal to the difference score divided by the (assumed fixed) time interval of interest, the variance of $\widehat{ARC}_{t+\tau,t}$ for an individual is equal to the following:

$$\sigma_{\widehat{ARC}_{t+\tau,t}^2} = \text{Var}\left(\frac{D_{t+\tau,t}}{T_{t+\tau} - T_t}\right) = \frac{1}{(T_{t+\tau} - T_t)^2} \text{Var}(D_{t+\tau,t}) = \frac{2\sigma_{\varepsilon}^2}{(T_{t+\tau} - T_t)^2}, \quad (\text{D5})$$

where $\text{Var}(\cdot)$ is the variance of the quantity. It is important to realize the

assumptions both implicit and explicit in Equation D5. These assumptions are that the variance pertains to one individual, errors at time T_t and $T_{t+\tau}$ are uncorrelated with one another and each true score, and that the error variance is equal for the two occasions of measurement.

REFERENCES

- Abramowitz, M., & Stegun, I. (1965). *Handbook of mathematical functions*. New York, NY: Dover Publications.
- Baltes, P. B., & Nesselroade, J. R. (1979). History and rationale of longitudinal research. In J. R. Nesselroade & P. B. Baltes (Eds.), *Longitudinal research in the study of behavior and development* (pp. 1–39). New York, NY: Academic Press.
- Billings, L., & Shepard, J. (1910). The change of heart rate with attention. *Psychological Review*, 17(3), 217–228.
- Blomqvist, N. (1977). On the relation between change and initial value. *Journal of the American Statistical Association*, 72(360), 746–749.
- Boker, S. M., & Nesselroade, J. R. (2002). A method for modeling the intrinsic dynamics of intraindividual variability: Recovering the parameters of simulated oscillators in multi-wave panel data. *Multivariate Behavioral Research*, 37(1), 127–160.
- Box, G. (1979). Robustness in the strategy of scientific model building. In R. L. Lauder & G. N. Williamson (Eds.), *Robustness in statistics* (pp. 201–236). New York, NY: Academic Press.
- Box, G. (1984). Science and statistics. *Journal of the American Statistical Association*, 71(356), 791–799.
- Browne, M., & Du Toit, S. (1991). Models for learning data. In L. M. Collins & J. L. Horn (Eds.), *Best methods for the analysis of change: Recent advances, unanswered questions, future directions*. Washington, DC: American Psychological Association.
- Bryk, A. S., & Raudenbush, S. W. (1987). Application of hierarchical linear models to assessing change. *Psychological Bulletin*, 101(1), 147–158.
- Bryk, A. S., & Raudenbush, S. W. (1992). *Hierarchical linear models: Applications and data analysis methods*. Newbury Park, CA: Sage Publications.
- Collins, L. M. (1996). Measurement of change in research on aging: Old and new issues from an individual growth perspective. In J. E. Birren & K. Schaie (Eds.), *Handbook of the psychology of the aging* (4th ed., pp. 38–58). San Diego, CA: Academic Press.

- Coren, S., Ward, L. M., & Enns, J. T. (1994). *Sensation and perception* (4th ed.). New York, NY: Harcourt Brace & Company.
- Cronbach, L. J., & Furby, L. (1970). How we should measure “change”: Or should we? *Psychological Bulletin*, 74, 68–80.
- Cudeck, R. (1996). Mixed-effects models in the study of individual differences with repeated measures data. *Multivariate Behavioral Research*, 31(3), 371–403.
- Davidian, M., & Giltinan, D. M. (1995). *Nonlinear models for repeated measurement data*. New York, NY: Chapman & Hall.
- Davies, A. E. (1900). The concept of change. *The Philosophical Review*, 9(5), 502–517.
- DeCarlo, L. (1997). On the meaning and use of kurtosis. *Psychological Methods*, 2(3), 292–307.
- Dempster, A., Laird, N., & Rubin, D. (1977). Maximum likelihood from incomplete data via the EM algorithm. *Journal of the Royal Statistical Society, Series B*, 39, 1–8.
- Finney, R. L., Weir, M. D., & Giordano, F. R. (2001). *Thomas’ calculus* (10th ed.). New York, NY: Addison Wesley.
- Francis, D. J., Fletcher, J. M., Stuebing, K. K., Davidson, K. C., & Thompson, N. M. (1991). Analysis of change: Modeling individual growth. *Journal of Consulting and Clinical Psychology*, 59(1), 27–37.
- Francis, J., David, Schatschneider, C., & Carlson, C. D. (2000). Introduction to individual growth curve analysis. In D. Drotar (Ed.), *Handbook of research in pediatric and clinical child psychology: Practical strategies and methods* (pp. 51–73). New York, NY: Kluwer Academic/Plenum Publishers.
- Goldstein, H. (1995). *Multilevel statistical models* (Vol. 3, 2nd ed.). New York, NY: Halsted Press.
- Harris, C. W. (Ed.). (1963). *Problems in measuring change*. Madison, WI: University of Wisconsin Press.
- Hartz, S., Ben-Shahar, Y., & Tyler, M. (2001). Logistic growth curve analysis in associative learning data. *Animal Cognition*, 4, 185–189.
- Hellmann, J. J., & Fowler, G. W. (1999). Bias, precision, and accuracy of four measures of species richness. *Ecological Applications*, 9(3), 824–834.
- Karney, B. R., & Bradbury, T. N. (1995). Assessing longitudinal change in marriage: An introduction to the analysis of growth curves. *Journal of Marriage and the Family*, 57, 1091–1108.
- Khuri, A., & Casella, G. (2002). The existence of the first negative moment revisited. *The American Statistician*, 56(1), 44–47.

- Kline, M. (1977). *Calculus: An intuitive and physical approach*. New York, NY: John Wiley and Sons.
- Kraemer, H. C., & Thiemann, S. (1989). A strategy to use soft data effectively in randomized controlled clinical trials. *Journal of Consulting and Clinical Psychology*, 57(1), 148–154.
- Laird, N. M., & Wang, F. (1990). Estimating rates of change in randomized clinical trials. *Controlled Clinical Trials*, 11, 405–419.
- Laird, N. M., & Ware, H. (1982). Random-effects models for longitudinal data. *Biometrics*, 38, 963–974.
- Lewin, L. (1981). *Polylogarithms and associated functions*. New York, NY: North-Holland.
- Linn, R. L., & Slinde, J. A. (1977). The determination of the significance of change between pre- and posttesting periods. *Review of Educational Research*, 47(1), 121–150.
- Lord, F. M. (1956). The measurement of growth. *Educational and Psychological Measurement*, 16, 421–437.
- Lord, F. M. (1958). Further problems in the measurement of growth. *Educational and Psychological Measurement*, 18(3), 437–451.
- Lord, F. M. (1963). Elementary models for measuring change. In C. W. Harris (Ed.), *Problems in measuring change* (pp. 21–38). Madison, WI: University of Wisconsin Press.
- Lord, F. M., & Novick, M. R. (1968). *Statistical theories of mental test scores*. Reading, MA: Addison-Wesley.
- Maple (Version 7.00). (2001). [Computer software] Waterloo Maple Inc.
- Mehta, P. D., & West, S. G. (2000). Putting the individual back into individual growth curves. *Psychological Methods*, 5(1), 23–43.
- Mood, A. M., Graybill, F. A., & Boes, D. C. (1974). *Introduction to the theory of statistics* (3rd ed.). New York, NY: McGraw-Hill.
- Morris, H. (1983). Parametric empirical Bayes inference: Theory and applications. *Journal of the American Statistical Association*, 78, 47–65.
- Overall, J. E., & Tonidandel, S. (2002). Measuring change in controlled longitudinal studies. *British Journal of Mathematical and Statistical Psychology*, 55, 109–124.
- Overall, J. E., & Woodward, J. A. (1975). Unreliability of difference scores: A paradox for measurement of change. *Psychological Bulletin*, 82(1), 85–86.
- Pearson, E., & Hartley, H. (Eds.). (1970). *Biometrika tables for statisticians* (Vol. 1, 3rd ed.). New York, NY: Cambridge University Press.

- Pinheiro, J., & Bates, D. (2000). *Mixed-effects models in S and S-Plus*. New York, NY: Springer.
- Ratkowsky, D. A. (1983). *Nonlinear regression modeling: A unified practical approach*. New York, NY: Marcel Dekker, Inc.
- Raudenbush, S., & Xiao-Feng, L. (2001). Effects of study duration, frequency of observation, and sample size on power in studies of group differences in polynomial change. *Psychological Methods*, 6(4), 387–401.
- Raudenbush, S. W. (2001). Comparing personal trajectories and drawing causal inferences from longitudinal data. *Annual Review of Psychology*, 52, 501–525.
- Raudenbush, S. W., & Bryk, A. S. (2002). *Hierarchical linear models: Applications and data analysis methods* (2nd ed.). Thousand Oaks, CA: Sage.
- Rausch, J. R., & Maxwell, S. E. (2003). Longitudinal designs in randomized group comparisons: Optimizing power when the latent individual growth trajectories follow straight-lines. Unpublished Manuscript.
- Rogosa, D. (1995). Myths and methods: “Myths about Longitudinal Research” plus supplemental questions. In J. M. Gottman (Ed.), *The analysis of change* (pp. 3–66). Mahwah, NJ: Lawrence Earlbaum Associates.
- Rogosa, D., Brandt, D., & Zimowski, M. (1982). A growth curve approach to the measurement of change. *Psychological Bulletin*, 92(3), 726–748.
- Rogosa, D., & Willett, J. B. (1985). Understanding correlates of change by modeling individual differences in growth. *Psychometrika*, 50(2), 203–228.
- Rogosa, D. R., & Willett, J. B. (1983). Demonstrating the reliability of the difference score in the measurement of change. *Journal of Educational Measurement*, 20(4), 335–343.
- Rozeboom, W. W. (1966). *Foundations of the theory of prediction*. Homewood, IL: The Dorsey Press.
- Seigel, D. G. (1975). Several approaches for measuring average rates of change for a second degree polynomial. *The American Statistician*, 29(1), 36–37.
- Shadish, W. R. (2002). Revisiting field experimentation: Field notes for the future. *Psychological Methods*, 7(1), 3–18.
- Stevens, W. (1951). Asymptotic regression. *Biometrics*, 7(3), 247–267.
- Stewart, J. (1998). *Calculus: Concepts and contexts*. Cincinnati, OH: Brooks/Cole Publishing Company.
- Stuart, A., & Ord, J. K. (1994). *Kendall’s advanced theory of statistics: Distribution theory* (Vol. 1, 6th ed.). New York, NY: John Wiley & Sons.
- Svartberg, M. (1999). Therapist competence: Its temporal course, temporal stability, and determinants in short-term anxiety-provoking psychotherapy. *Journal of Clinical Psychology*, 55(10), 1313–1319.

- Thomson, G. H. (1924). A formula to correct for the effect of errors of measurement on the correlation of initial values with gains. *Journal of Experimental Psychology*, 7, 321–324.
- Van Gert, P. (1991). A dynamic systems model of cognitive and language growth. *Psychological Review*, 98(1), 3–52.
- Vonesh, E. F., & Chinchilli, V. M. (1997). *Linear and nonlinear models for the analysis of repeated measurements*. New York, NY: Marcel Dekker.
- Willet, J. B. (1988). Questions and answers in the measurement of change. In E. Z. Rothkopf (Ed.), *Review of research in education* (Vol. 15, pp. 345–422). Washington, DC: American Educational Research Association.
- Willet, J. B. (1989). Some results on the reliability for the longitudinal measurement of change: Implications for the design of studies of individual change. *Educational and Psychological Measurement*, 49, 587–601.
- Winsor, C. P. (1932). The Gompertz curve as a growth curve. *The Proceedings of the National Academy of Sciences of the United States of America*, 18(1), 1–8.
- Zimmerman, D., & Williams, R. (1982). Gain scores in research can be highly reliable. *Journal of Educational Measurement*, 19, 149–154.