Name: ________________________________

Math 20610: Linear Algebra
Spring **Semester 2025**-26
Practice **Exam 1**

No calculators or notes allowed.

Scores

Question	Possible	Actual
1	12	
2	10	
3 & 4	10	
5	10	
6	10	
7	15	
Total	67	

GOOD LUCK

Note: our exam will be out of 100 points. This practice exam just collects sample problems and is not intended to be an exact parallel to the real exam.

1. (10 points) Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ or explain why none exists.

Row reduce:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

So $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

Check:

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \checkmark$$

(The problem didn't ask you to check, but it's a good habit.)

2. (5 points) For which numbers $t \in \mathbf{R}$ (if any) is $\begin{pmatrix} t \\ t+1 \\ t \end{pmatrix}$ a linear combination of the vectors $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$?

$$x\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} t \\ t+1 \\ t \end{bmatrix} \rightarrow \begin{aligned} x - y &= t \\ 2x &= t+1 \\ y &= t \end{aligned}$$

If $y=t$ (3^{rd} eqn) then $x - t = t$ (1^{st} eqn) so $x = 2t$. Then 2^{nd} eqn gives $4t = t+1$ so $t = 1/3$

3. (5 points) Find the reflection of the vector $\vec{x} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$ about the line L through $\mathbf{0}$ and $\vec{v} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$. (Half-credit for finding the *projection* onto L) Remember $ref_\ell(\vec{x}) = 2\,proj_\ell(\vec{x}) - \vec{x}$.

(This formula will be provided in the exam.)

$$\left\| \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \right\| = \sqrt{4+4+1} = 3$$

unit vector: $\vec{u} = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$$proj_\ell(\vec{x}) = (\vec{u}\cdot\vec{x})\,\vec{u}$$

$$= \left(\frac{-2}{3}\cdot(-2) + \frac{2}{3}(3) + \frac{1}{3}(-1)\right)\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= 3\begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$ref_\ell(\vec{x}) = 2\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

↖ answer

4. Suppose that A is a 3×6 matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_6 \in \mathbf{R}^3$ and that one can perform row operations on A to arrive at the matrix

$$\begin{bmatrix} 1 & 0 & 2 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (5 points) What are the solutions of $A\mathbf{x} = \mathbf{0}$?

Being homogeneous, the original equations (whatever they were) row reduce to

$x_1 + 2x_3 - 4x_5 = 0$

$x_4 + 2x_5 = 0$

$x_6 = 0$

So if we let u, s, t be arbitrary then

$x_1 = -2s + 4t$

$x_2 = u$

$x_3 = s$

$x_4 = -2t$

$x_5 = t$

$x_6 = 0$

(infinitely many solutions)

(b) (3 points) What can you say about the number of solutions of the linear system $A\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$? Explain briefly.

There are infinitely many, because this matrix has rank 3 so we can never get $0 = 1$ in the bottom row of the augmented matrix (i.e. it is consistent) and there are more variables than equations.

5. Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^3$ is a linear transformation such that $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

(a) (4 points) Find $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 3 \end{bmatrix}$$

(b) (4 points) Find the matrix for T.

$$A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \right] = \left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \mid T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right] = \left[-T\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right) \mid T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right]$$

$$= \begin{bmatrix} -2 & -2 \\ -2 & 3 \\ -3 & 3 \end{bmatrix}$$

(c) (2 points) Compute $T\begin{pmatrix} e \\ \pi \end{pmatrix}$.

$$\begin{bmatrix} -2 & -2 \\ -2 & 3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} e \\ \pi \end{bmatrix} = \begin{bmatrix} -2e - 2\pi \\ -2e + 3\pi \\ -3e + 3\pi \end{bmatrix}$$

$$(3 \times 2) \quad (2 \times 1) \quad = \quad (3 \times 1)$$

6. Two of the following six assertions are false (meaning 'not always true'). Identify them and give **specific** counterexamples at the bottom (and, if needed, back) of this page (go to the back if necessary). Note that you do not have to justify your counterexamples—only present them. (5 points each)

True **(a)** If A is a 3×4 matrix with rank 3 and $\mathbf{b} \in \mathbf{R}^3$ is a vector, then the linear system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

True **(b)** If A is an invertible $n \times n$ matrix, then A is row equivalent to the identity matrix.

False **(c)** If A and B are 2×2 matrices, then $AB = BA$.

False **(d)** If A is an 3×2 matrix with linearly independent columns and $\mathbf{b} \in \mathbf{R}^m$ is a vector, then $A\mathbf{x} = \mathbf{b}$ is consistent. Two columns are linearly independent if neither is a multiple of the other.

True **(e)** If $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

True **(g)** If $\mathbf{v} \in \mathbf{R}^3$ is a non-zero vector and L is the line joining $\mathbf{0}$ to $\mathbf{v}$, then $\text{proj}_L(\mathbf{v}) = \mathbf{v}$.

Counterexamples

(c)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 8 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 11 & 16 \end{bmatrix}$$

(d) This only makes sense if $m = 3$.

But even for $m = 3$ it is not true.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{aligned} x + 0y &= 1 \\ 0x + y &= 1 \\ 0 &= 1 \end{aligned}$$

not consistent