

# Show all work!

The Honor Code is in effect for this exam, including keeping your own exam under cover.

No calculators are allowed to be used on this exam.

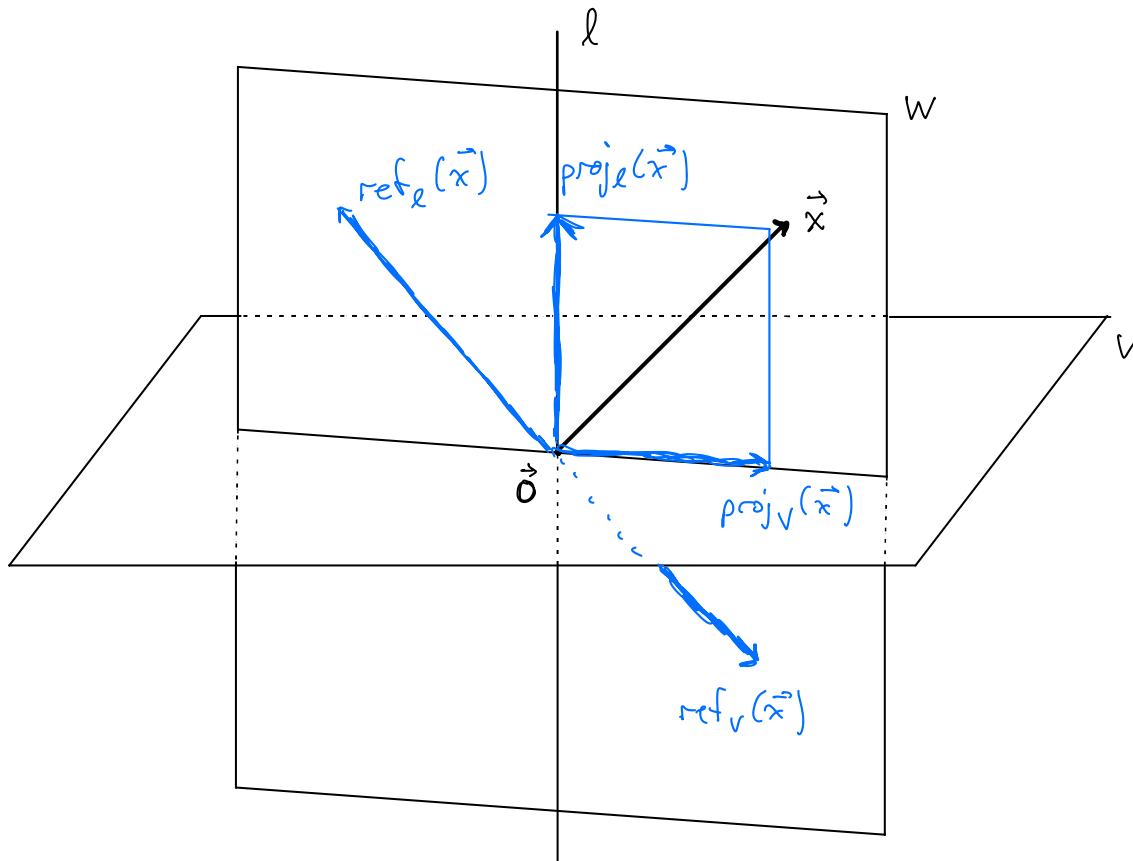
Good luck!!

Several formulas that may be helpful for this exam can be found on the last page.

1. (12 points) In the picture below, note that (i)  $\ell$  is a line through the origin  $\vec{0}$  and  $V$  is a plane through the origin, (ii)  $\ell$  is perpendicular to  $V$ , and (iii)  $W$  is the plane spanned by  $\ell$  and the vector  $\vec{x}$ .

Draw and clearly label the following vectors. No formulas are needed for this problem, but make sure your drawn vectors start at  $\vec{0}$  (for example, as  $\vec{x}$  does in the picture).

- $\text{proj}_\ell(\vec{x})$
- $\text{ref}_\ell(\vec{x})$
- $\text{proj}_V(\vec{x})$
- $\text{ref}_V(\vec{x})$



2. (points) For all parts of this problem, let  $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^4$ . Each part of this problem is helpful for the subsequent part.

- (a) Find  $\|\vec{v}\|$ .

$$\|\vec{v}\| = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

Answer:

$$\|\vec{v}\| = \sqrt{30}$$

- (b) Find a unit vector,  $\vec{u}$ , on the line defined by  $\vec{v}$ . **Your answer should be written in the usual way for a vector in  $\mathbb{R}^4$ , namely it should be represented by a  $4 \times 1$  matrix of real numbers.**

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$$

Answer:

$$\vec{u} = \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 3/\sqrt{30} \\ 4/\sqrt{30} \end{bmatrix}$$

- (c) Let  $\ell$  be the line spanned by  $\vec{v}$ . If  $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ , find  $\text{proj}_\ell(\vec{x})$ . (Hint: the formula for the projection to a line in  $\mathbb{R}^3$  is exactly the same as the one in  $\mathbb{R}^4$ .)

$$\text{proj}_\ell(\vec{x}) = (\vec{x} \cdot \vec{u}) \vec{u}$$

$$(\vec{x} \cdot \vec{u}) = \frac{1}{\sqrt{30}} + \frac{2}{\sqrt{30}} + \frac{3}{\sqrt{30}} + \frac{4}{\sqrt{30}} = \frac{10}{\sqrt{30}}$$

$$\text{proj}_\ell(\vec{x}) = \frac{10}{\sqrt{30}} \begin{bmatrix} 1/\sqrt{30} \\ 2/\sqrt{30} \\ 3/\sqrt{30} \\ 4/\sqrt{30} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \\ 4/3 \end{bmatrix}$$

Answer:

$$\text{proj}_\ell(\vec{x}) = \begin{bmatrix} 1/3 \\ 2/3 \\ 1 \\ 4/3 \end{bmatrix}$$

3. Let  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a linear transformation and let

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \vec{v} = 2\vec{v}_2 - \vec{v}_1 = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \quad T(\vec{v}_1) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad T(\vec{v}_2) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Find  $T(\vec{v})$ . (Hint: pay attention to the fact that  $\vec{v} = 2\vec{v}_2 - \vec{v}_1$ .)

$$\begin{aligned} T(\vec{v}) &= 2T(\vec{v}_2) - T(\vec{v}_1) \\ &= \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{aligned}$$

**Answer:**

$$T(\vec{v}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

4. (points) Let  $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ . Let  $V$  be the plane defined by the equation  $2x + y - 2z = 0$ . Let

$$\vec{w} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}. \text{ Let } \ell \text{ be the line spanned by } \vec{w}.$$

(a) Show that  $\vec{w}$  lies on  $V$ .

$$2(3) + (-2) - 2(2) = 0 \text{ so } \vec{w} \text{ satisfies the equation of } V.$$

(b) Find  $(\vec{v} \cdot \vec{w})$ .

$$\begin{aligned} (\vec{v} \cdot \vec{w}) &= (2)(3) + (1)(-2) + (-2)(2) \\ &= 0 \end{aligned}$$

**Answer:**

$$(\vec{v} \cdot \vec{w}) = 0$$

(c) Find the angle,  $\theta$ , between  $\vec{v}$  and  $\vec{w}$ .

$$\begin{aligned} \text{since } (\vec{v} \cdot \vec{w}) &= \|\vec{v}\| \|\vec{w}\| \cos \theta \text{ and } \vec{v} \text{ and } \vec{w} \text{ are} \\ \text{not zero vectors, } \cos \theta &= 0 \text{ so } \theta = \frac{\pi}{2} (= 90^\circ) \end{aligned}$$

**Answer:**

$$\theta = 90^\circ$$

5. Consider the linear system

$$\begin{aligned}x + 2y + 3z &= 4 \\2x + 5y + 8z &= 5 \\x + 3y + 5z &= 1\end{aligned}$$

(a) Find the reduced row-echelon form of the *augmented* matrix. Please circle the leading 1's.

$$\left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 5 \\ 1 & 3 & 5 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 1 & 2 & -3 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|c} 1 & 0 & -1 & 10 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

(b) Using your answer to (a), solve the linear system given at the start of this problem. In particular,

- If there is no solution, explain why not.
- If there is a unique solution, give it.
- If there are infinitely many solutions, say so and give the general solution.

Make sure that your answer to (b) is consistent with your answer to (a).

There are infinitely many solutions since it is consistent (bottom row is all 0) and the RREF has more rows than variables.

The solution is 
$$\begin{aligned}x &= 10 + t \\ y &= -3 - 2t \\ z &= t\end{aligned}$$
 where  $t$  is an arbitrary real number.

(c) In the linear system at the start of this problem, is the *coefficient* matrix invertible? Briefly explain why or why not. (Either way, you are not asked to **find** the inverse in this problem.)

No — when you row reduce just the coefficient matrix (ignore the 4<sup>th</sup> column) you don't get the identity  $I_3$ , you get 
$$\left[ \begin{array}{ccc} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right].$$

6. Suppose that  $A$  is some (unknown)  $3 \times 5$  matrix and that the reduced row-echelon form of  $A$  is

$$RREF(A) = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Consider the linear system

$$A\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \quad (*)$$

(Note that the matrix on line 2 of this problem is not  $A$ , it's  $RREF(A)$ .) Since we don't know the series of row operations that took us from  $A$  to its reduced row-echelon form, we can't solve the linear system (\*) (you don't have to prove this). However, explain why we do know that this linear system is consistent and has infinitely many solutions.

whatever the sequence of row operations does to  $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$  we know we'll never get  $\mathbf{0}$  since  $\text{rank}(A) = 3$ . Since there are 5 variables, it has infinitely many solutions. (There are 2 "free" variables.)

(b) Find the general solution of the linear system

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that this linear system involves  $A$ , not its reduced row-echelon form  $RREF(A)$ .

when you row-reduce  $\left[ A \mid \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right]$  you get  $\left[ \begin{array}{ccccc|c} 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$

so the solution is  $x_1 = -2s - 3t$

$$x_2 = s$$

$$x_3 = t$$

$$x_4 = 0$$

$$x_5 = 0$$

where  $s, t$  are arbitrary real numbers

(c) Briefly explain why having  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$  on the right in (b) rather than  $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$  makes the linear system in (b) possible to solve even though the linear system (\*) in (a) is not possible with the information we have.

we don't know what the unknown row operations do to  $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$  but

whatever they are, they don't affect  $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ .

**Formulas**

Here are some formulas that may help in this exam. Remember that you are responsible for the formula for  $\text{proj}_\ell(\vec{x})$ , so it is omitted from the list.

We always assume that  $\ell$  is a line through the origin in  $\mathbb{R}^3$ ,  $V$  is a plane through the origin in  $\mathbb{R}^3$ ,  $\ell$  is perpendicular to  $V$ , and  $\vec{u}$  is a unit vector on  $\ell$ .

1.  $\text{proj}_V(\vec{x}) = \vec{x} - \text{proj}_\ell(\vec{x})$
2.  $\text{ref}_\ell(\vec{x}) = \text{proj}_\ell(\vec{x}) - \text{proj}_V(\vec{x})$
3.  $\text{ref}_\ell(\vec{x}) = 2\text{proj}_\ell(\vec{x}) - \vec{x}$
4.  $\text{proj}_V(\vec{x}) = \text{ref}_V(\vec{x}) + \text{proj}_\ell(\vec{x})$
5.  $\text{ref}_V(\vec{x}) + \text{ref}_\ell(\vec{x}) = \vec{0}$