Math 20610
Exam 1
February 25, 2026 **Name** ______________________________

Show all work!

The Honor Code is in effect for this exam, including keeping your own exam under cover.

No calculators are allowed for this exam. A blank page is provided at the end if you need more space.

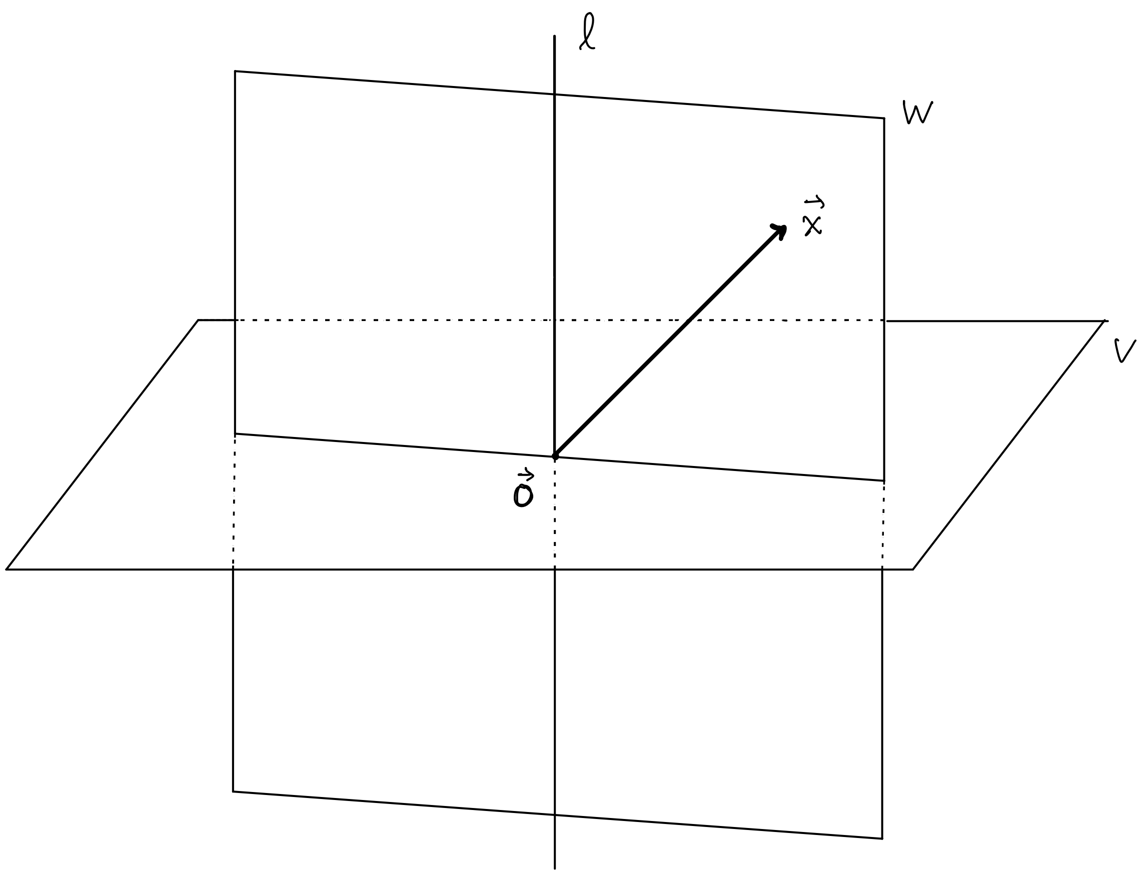
Good luck!!

Several formulas that may be helpful for this exam can be found on the last page.

1. (*12 points*) In the picture below, note that (i) ℓ is a line through the origin $\overrightarrow{0}$ and V is a plane through the origin, (ii) ℓ is perpendicular to V, and (iii) W is the plane spanned by ℓ and the vector $\overrightarrow{x}$.

   Draw and clearly label the following vectors. No formulas are needed for this problem, but make sure your drawn vectors start at $\overrightarrow{0}$ (for example, as $\overrightarrow{x}$ does in the picture).

   - $\text{proj}_\ell(\overrightarrow{x})$
   - $\text{ref}_\ell(\overrightarrow{x})$
   - $\text{proj}_V(\overrightarrow{x})$
   - $\text{ref}_V(\overrightarrow{x})$



2. For all parts of this problem, let $\overrightarrow{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathbb{R}^4$. Each part of this problem is helpful for the subsequent part.

   (a) (*7 points*) Find $\|\overrightarrow{v}\|$.

   Answer:

   $\|\overrightarrow{v}\| =$

   (b) (*7 points*) Find a unit vector, $\overrightarrow{u}$, on the line defined by $\overrightarrow{v}$. **Your answer should be written in the usual way for a vector in $\mathbb{R}^4$, namely it should be represented by a 4×1 matrix of real numbers.**

   Answer:

   $\overrightarrow{u} =$

   (c) (*7 points*) Let ℓ be the line spanned by $\overrightarrow{v}$. If $\overrightarrow{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, find $\text{proj}_\ell(\overrightarrow{x})$. (Hint: the formula for the projection to a line in $\mathbb{R}^3$ is exactly the same as the one in $\mathbb{R}^4$.)

   Answer:

   $\text{proj}_\ell(\overrightarrow{x}) =$

3. (*7 points*) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear tranformation and let

$$\overrightarrow{v_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \overrightarrow{v_2} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \quad \overrightarrow{v} = 2\overrightarrow{v_2} - \overrightarrow{v_1} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}, \quad T(\overrightarrow{v_1}) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad T(\overrightarrow{v_2}) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}.$$

Find $T(\overrightarrow{v})$. (Hint: pay attention to the fact that $\overrightarrow{v} = 2\overrightarrow{v_2} - \overrightarrow{v_1}$.)

Answer:

$T(\overrightarrow{v}) =$

4. Let $\overrightarrow{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$. Let V be the plane defined by the equation $2x + y - 2z = 0$. Let $\overrightarrow{w} = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$.
Let ℓ be the line spanned by $\overrightarrow{w}$.

(a) (*7 points*) Show that $\overrightarrow{w}$ lies on V.

(b) (*6 points*) Find $(\overrightarrow{v} \cdot \overrightarrow{w})$.

Answer:

$(\overrightarrow{v} \cdot \overrightarrow{w}) =$

(c) (*6 points*) Find the angle, θ, between $\overrightarrow{v}$ and $\overrightarrow{w}$.

Answer:

$\theta =$

5. Consider the linear system

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 2x & + & 5y & + & 8z & = & 5 \\ x & + & 3y & + & 5z & = & 1 \end{array}$$

(a) (*7 points*) Find the reduced row-echelon form of the *augmented* matrix. Please circle the leading 1's.

(b) (*7 points*) Using your answer to (a), solve the linear system given at the start of this problem. In particular,

- If there is no solution, explain why not.
- If there is a unique solution, give it.
- If there are infinitely many solutions, say so and give the general solution.

Make sure that your answer to (b) is consistent with your answer to (a).

(c) (*7 points*) In the linear system at the start of this problem, is the *coefficient* matrix invertible? Briefly explain why or why not. (Either way, you are not asked to **find** the inverse in this problem.)

6. Suppose that A is some (unknown) 3×5 matrix and that the reduced row-echelon form of A is

$$RREF(A) = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) (*7 points*) Consider the linear system

$$A\vec{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}. \qquad (*)$$

(Note that the matrix on line 2 of this problem is not A, it's $RREF(A)$.) Since we don't know the series of row operations that took us from A to its reduced row-echelon form, we can't solve the linear system $(*)$ (you don't have to prove this). However, explain why we do know that this linear system is consistent and has infinitely many solutions.

(b) (*7 points*) Find the general solution of the linear system

$$A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Note that this linear system involves A, not its reduced row-echelon form $RREF(A)$.

(c) (*6 points*) Briefly explain why having $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ on the right in (b) rather than $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ makes the linear system in (b) possible to solve even though the linear system $(*)$ in (a) is not possible with the information we have.

Formulas

Here are some formulas that may help in this exam. Remember that you are responsible for the formula for $\text{proj}_\ell(\overrightarrow{x})$, so it is omitted from the list.

We always assume that ℓ is a line through the origin in $\mathbb{R}^3$, V is a plane through the origin in $\mathbb{R}^3$, ℓ is perpendicular to V, and $\overrightarrow{u}$ is a unit vector on ℓ.

1. $\text{proj}_V(\overrightarrow{x}) = \overrightarrow{x} - \text{proj}_\ell(\overrightarrow{x})$
2. $\text{ref}_\ell(\overrightarrow{x}) = \text{proj}_\ell(\overrightarrow{x}) - \text{proj}_V(\overrightarrow{x})$
3. $\text{ref}_\ell(\overrightarrow{x}) = 2\text{proj}_\ell(\overrightarrow{x}) - \overrightarrow{x}$
4. $\text{proj}_V(\overrightarrow{x}) = \text{ref}_V(\overrightarrow{x}) + \text{proj}_\ell(\overrightarrow{x})$
5. $\text{ref}_V(\overrightarrow{x}) + \text{ref}_\ell(\overrightarrow{x}) = \overrightarrow{O}$

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