The Monotone Secant Conjecture in the Real Schubert Calculus

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Métodos Efectivos en Geometría Algebraica
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Joint work with:
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How many lines meet four given general lines in 3-space?

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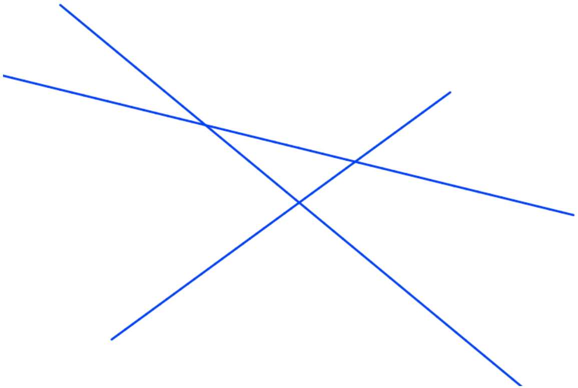
Three of the lines



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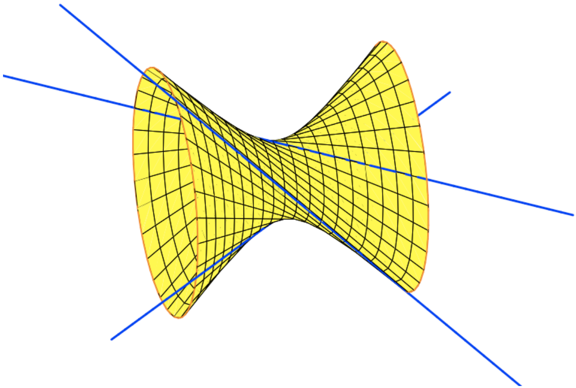
Quadric surface



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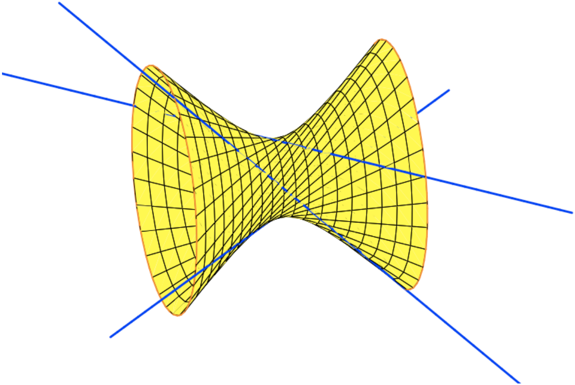
Second ruling



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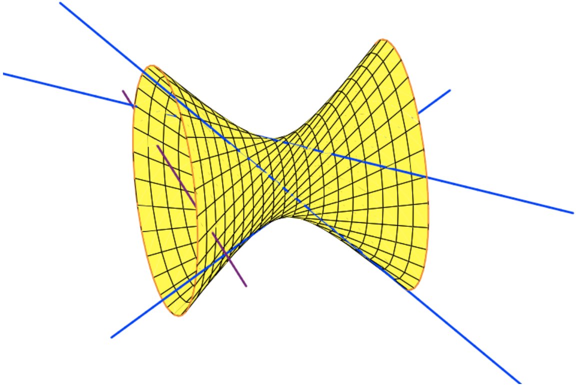
Fourth line



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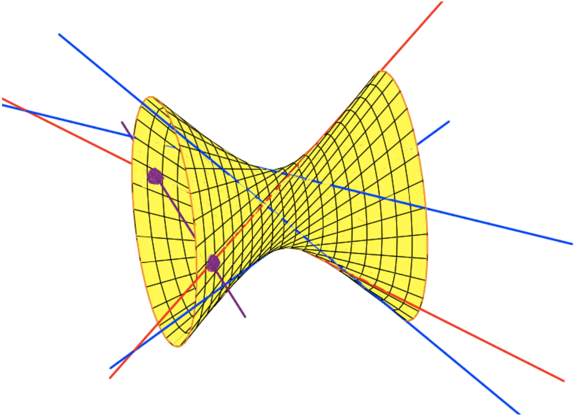
Two solutions



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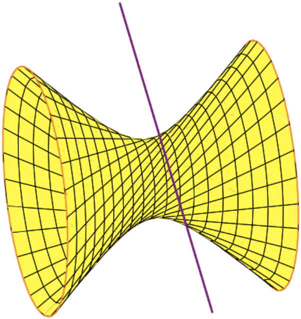
The solutions might not be real—they might be complex conjugates.



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Flags

Given $\alpha = (a_1 < a_2 < \cdots < a_k)$ and n,

$$E_\bullet : \{0\} \subset E_{a_1} \subset E_{a_2} \subset \cdots \subset E_{a_k} \subset \mathbb{C}^n, \quad \text{where } \dim E_{a_i} = a_i.$$

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Example

If $\alpha = (1, 2)$, then $E_\bullet = E_1 \subset E_2$ in $\mathbb{C}^n$

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Example:

The set of lines in $\mathbb{P}^3$ is $\mathrm{Gr}(2, 4)$

Definition

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Definition

A *Schubert problem* in $\mathbb{F}\ell(\alpha; n)$ is a list of conditions $(\sigma_1, \ldots, \sigma_m)$ such that

$$X_{\sigma_1} F^1_\bullet \ \cap \ X_{\sigma_2} F^2_\bullet \ \cap \cdots \cap X_{\sigma_m} F^m_\bullet$$

is finite (when the flags $F^i_\bullet$ are in general position.)

In the last $\sim$15 years, a series of conjectures, experiments, and theorems has explored the reality of Schubert calculus:

What conditions on real reference flags ensure that a Schubert problem has all its solutions real?

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Shapiro Conjecture

Let γ *be a real rational normal curve.*

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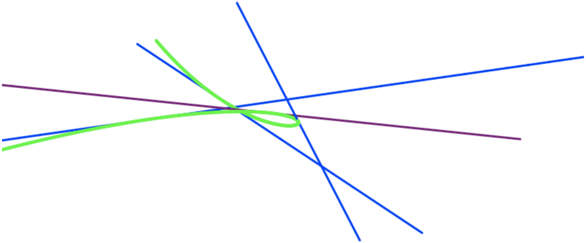
Four tangent lines



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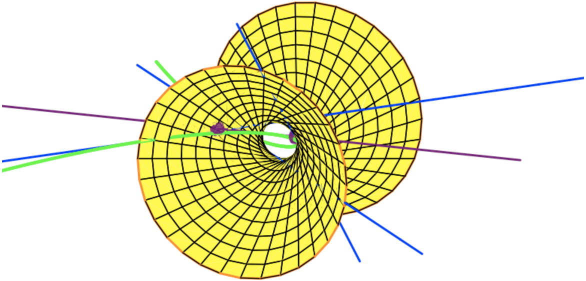
Always real solutions



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[**Ruffo-Sivan-Soprunova-Sottile**, 2005]: Tested 520,420,135 instances of 1,126 Schubert problems, taking 15.76 GHz-years.

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Shapiro Conjecture

Let γ be a real rational normal curve. If $F_\bullet^1, \ldots, F_\bullet^m$ are real flags tangent to γ, then $X_{\sigma_1}F_\bullet^1 \cap X_{\sigma_2}F_\bullet^2 \cap \cdots \cap X_{\sigma_m}F_\bullet^m$ is transverse and all points are real.

The Monotone-Secant Conjecture

Is a generalization of the Shapiro conjecture in two directions:

1. For Schubert problems in $\mathbb{F}\ell(\alpha; n)$.
2. When the flags $F_\bullet^1, \ldots, F_\bullet^m$ are secant to γ.

Consider the problem in $\mathbb{F}\ell(2,3;4)$ of finding $\ell \subset \Pi$ such that ℓ meets three lines secant to γ and Π meets two points of γ

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Begin with a rational normal curve in 3-space:

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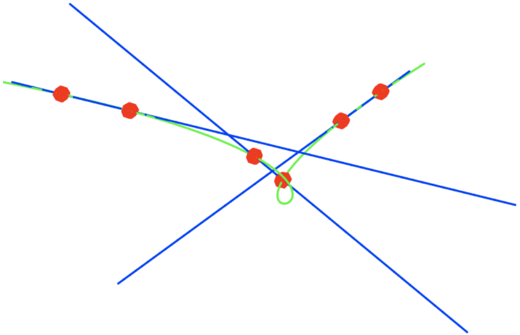
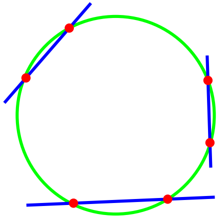
Consider the problem in $\mathbb{F}\ell(2,3;4)$ of finding $\ell \subset \Pi$ such that ℓ meets three lines secant to γ and Π meets two points of γ

Select three secant lines:





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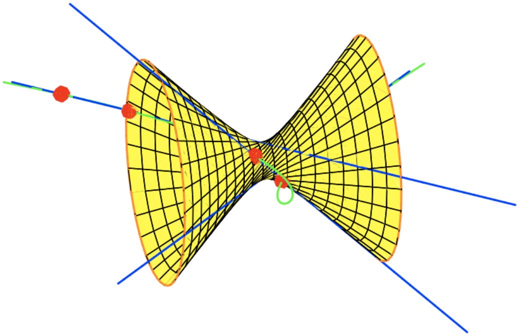
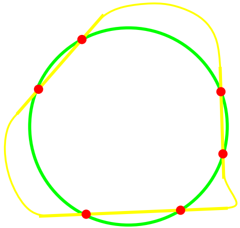
Introduce the hyperboloid defined by the three secant lines:





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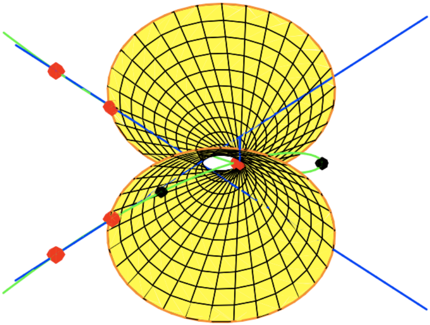
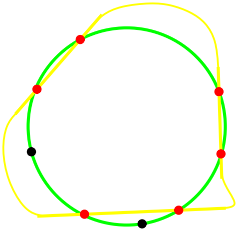
Consider an extra pair of points in the rational normal curve:





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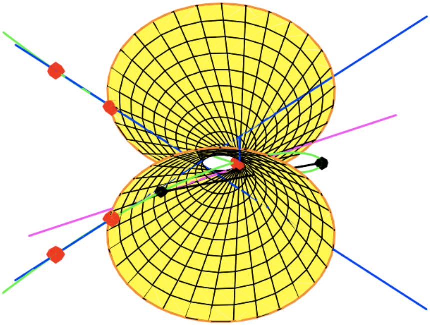
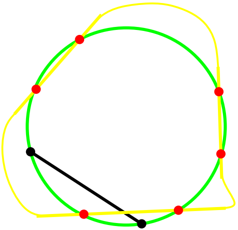
There is one (hence two) real solution(s):





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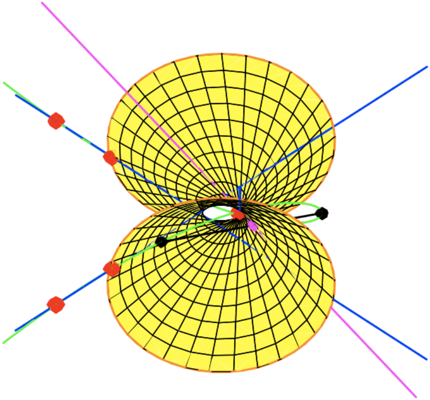
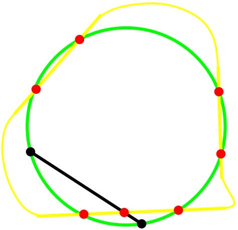
Here is the other solution:





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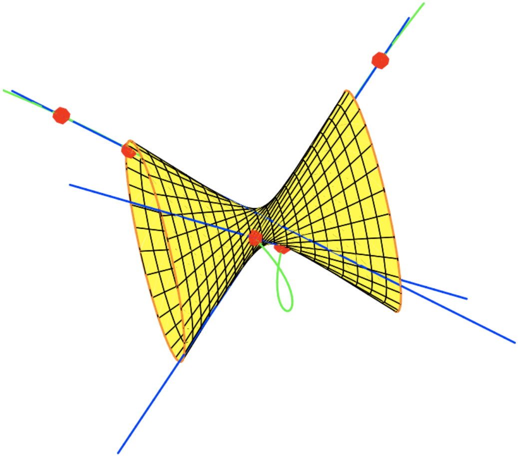
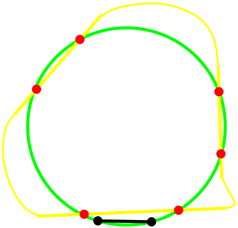
Two points in the curve may lead to no real solutions:





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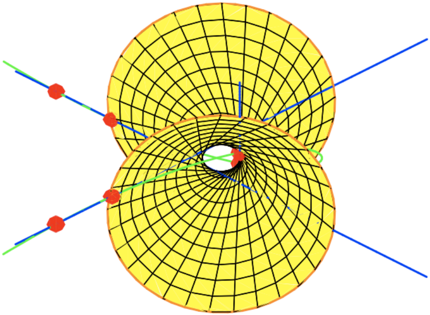
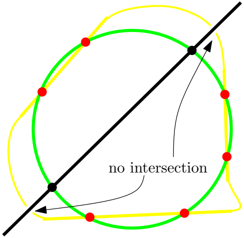
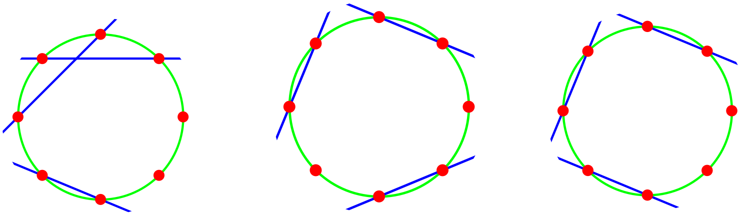
This view shows the same behavior "inside" the hyperboloid:





Monotone Secant conjecture

If $F_\bullet^1, \ldots, F_\bullet^m$ are real secant that are separated and monotone, then $X_{\sigma_1}F_\bullet^1 \cap X_{\sigma_2}F_\bullet^2 \cap \cdots \cap X_{\sigma_m}F_\bullet^m$ is transverse and all points are real.



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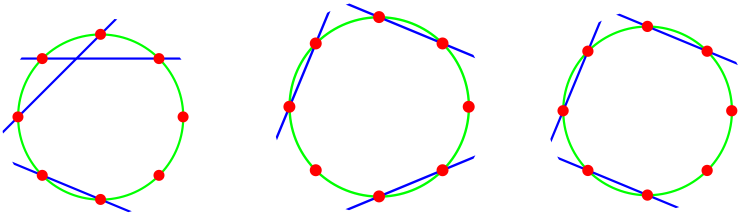

Not separated — Separated but not Monotone — Separated and Monotone

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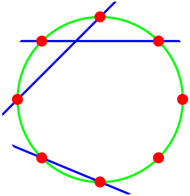

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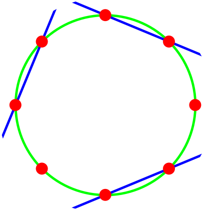

Separated but not Monotone

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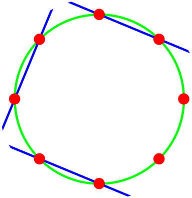

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You can see our data here:

`http://www.math.tamu.edu/~secant/`

Thank you!

Abraham Martín del Campo

www.math.tamu.edu/~asanchez/

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