Witness sets of projections

Jonathan D. Hauenstein* Andrew J. Sommese †

August 26, 2010

Abstract

Elimination is a basic algebraic operation which geometrically corresponds to projections. This article describes using the numerical algebraic geometric concept of witness sets to compute the projection of an algebraic set. The ideas described in this article apply to computing the image of an algebraic set under any linear map.
Keywords. Numerical algebraic geometry, polynomial system, algebraic sets, witness sets, projections
AMS Subject Classification. 65H10, 68W30

Introduction

Given a polynomial system $f : \mathbb{C}^N \to \mathbb{C}^n$; an ℓ-dimensional irreducible algebraic set $V \subset f^{-1}(0)$; and a linear map $\pi : \mathbb{C}^N \to \mathbb{C}^K$, this article defines and describes how to compute a "witness set" for $\overline{\pi(V)}$ given a witness set for V, which allows us to efficiently reduce computations on $\overline{\pi(V)}$ to computations on V.

If we are working with an ℓ-dimensional component V of $\mathrm{Var}(f)$, we may *without loss of generality* assume that $n = N - \ell$. If $n \neq N - \ell$, we may achieve $n = N - \ell$ by the randomization procedure of replacing f by $\begin{bmatrix} I & A \end{bmatrix} \cdot f$ where A is a random $(N - \ell) \times n$ matrix (see [17, §13.5] and

*Department of Mathematics, Mailstop 3368, Texas A&M University, College Station, TX 77843 (jhauenst@math.tamu.edu, `www.math.tamu.edu/~jhauenst`). This author was supported by Texas A&M University, the Duncan Chair of the University of Notre Dame, and NSF grant DMS-0915211.

†Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN 46556 (sommese@nd.edu, `www.nd.edu/~sommese`). This author was supported by the Duncan Chair of the University of Notre Dame and NSF grant DMS-0712910.

in particular [17, Theorem 13.5.1]). This procedure, which plays a key role in the computation of the irreducible components of $f^{-1}(0)$ stabilizes the numerics of computing the solution component V of the polynomial system.

Without loss of generality, the irreducible dimension ℓ algebraic set V may be assumed to be generically reduced, since if it is not, we may reduce to this situation using deflation: this is discussed in §1.3. We denote the reduced algebraic set of solutions of $f = 0$ by $\mathrm{Var}(f)$, i.e., $\mathrm{Var}(f)$ is the reduction of $f^{-1}(0)$.

In this case a witness set is a triple $\{f, \mathcal{L}, W\}$, where $\mathcal{L}$ is a system of ℓ generic linear equations on $\mathbb{C}^N$ and where $W := V \cap \mathrm{Var}(\mathcal{L})$. This is discussed in §1.2.

We know that $\pi(V)$ is constructible and thus contains a nonempty Zariski open set $\mathcal{V}$ which satisfies $\overline{\mathcal{V}} = \overline{\pi(V)}$. Assume that $\ell' := \dim \overline{\pi(V)}$: for how to compute this dimension see Lemma 3. If we had a polynomial system $g : \mathbb{C}^K \to \mathbb{C}^{K-\ell'}$ with $\overline{\pi(V)}$ an irreducible component of $\mathrm{Var}(g)$, then setting $W_\pi = \overline{\pi(V)} \cap \mathrm{Var}(\mathcal{K})$ for a system of ℓ' generic linear equations on $\mathbb{C}^K$, $\{g, \mathcal{K}, W'\}$ will be a witness set for $\overline{\pi(V)}$. There are three difficulties with this:

1. computation of polynomial equations g on $\mathbb{C}^K$ with $\overline{\pi(V)}$ an irreducible component of $\mathrm{Var}(g)$ is computationally a very expensive interpolation problem;

2. without using special case-by-case information, the interpolation methods to compute the equations degree lead to $\deg \overline{\pi(V)}$ polynomials, which usually having higher degree than those of f and are less accurately known; and

3. even if we did compute such a system g, decomposing $\mathrm{Var}(g)$ to find $\overline{\pi(V)}$ can be computationally expensive.

Classically finding equations g vanishing on $\overline{\pi(V)}$ is a prime (and computational expensive) goal of elimination theory.

Our approach is to find a system $\mathcal{L}'$ of ℓ appropriate general linear equations on $\mathbb{C}^N$ such that $\mathcal{W} = V \cap \mathrm{Var}(\mathcal{L}')$ is a finite set and such that $\pi(\mathcal{W}) = \overline{\pi(V)} \cap \mathrm{Var}(\mathcal{K})$ for a system of $\ell' := \dim \overline{\pi(V)}$ linear equations on $\mathbb{C}^K$. The triple $\{f, \mathcal{L}', \mathcal{W}\}$ may be used to carry out computations that a witness set for $\overline{\pi(V)}$ would be used for. We make precise the class of $\mathcal{L}'$ we use and call the quadruple $\{f, \pi, \mathcal{L}', \mathcal{W}\}$ a witness set for $\overline{\pi(V)}$. Note that moving computations from an algebraic set Y, which it is hard to work with directly, to a well-behaved algebraic set mapping X onto Y has occurred in several situations, e.g., deflation (discussed in §1.3) and rank deficiency sets (see in particular [1, §2.1.3]).

For the system $\mathcal{L}'$ we take $\ell - \ell'$, i.e., $\dim V - \dim \overline{\pi(V)}$, general linear equations on $\mathbb{C}^N$ plus a system of ℓ' linear equations on $\mathbb{C}^N$ of the form $\mathcal{K}' \circ \pi$, where $\mathcal{K}'$ is a system of ℓ' general linear equations on $\mathbb{C}^K$. Using a standard homotopy presented in Equation 1 with start points W, the finite endpoints form $\mathcal{W}$.

In §1 we present background on Bertini's Theorem (see §1.1), witness sets (see §1.2), deflation (see 1.3), and moving linear spaces (see 1.4).

In §2 we discuss linear maps and projections and give the algorithm for finding a witness set for $\overline{\pi(V)}$. We also show that the algorithm works equally well with projective space replacing Euclidean space.

Examples are presented in §3.

1 Some Background Material

In this section we collect some background material.

1.1 Bertini's Theorem

Bertini's Theorem underlies many of the probability-one algorithms of numerical algebraic geometry, e.g., see [17, §A.9]. We use the following version of Bertini's Theorem, which is weaker than the version in [17, §A.9].

Theorem 1 (Bertini's Theorem)**.** *Let V be a reduced pure N-dimensional algebraic set. Let*

$$R := \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

be a collection of n algebraic functions $R_1, \ldots, R_n$ such that at each point $x \in V$ there is at least one R_i that is not zero. Then for each $k \leq N+1$ there is a nonempty Zariski open set $\mathcal{A} \subset \mathbb{C}^{k \times n}$ of the $k \times n$ matrices such that for $A \in \mathcal{A}$, it follows that $\mathrm{Var}(A \cdot R)$ *is either empty or of pure dimension $N-k$. Here a set of dimension -1 is empty. Moreover,* $\mathrm{Var}(A \cdot R) \cap (V \setminus \mathrm{Sing}(V))$ *is either empty or a smooth submanifold of U of dimension $N-k$. The rank of the Jacobian matrix of $A \cdot R$ at any point of* $\mathrm{Var}(A \cdot R) \cap (V \setminus \mathrm{Sing}(V))$ *is equal to $N-k$.*

Since the intersection of any finite number of Zariski open sets is a nonempty Zariski open set, we may assume that $A \cdot R$ has the same genericity properties for any finite number of previously given algebraic subsets of V.

We have the following useful corollary of Theorem 1.

Corollary 2. *Let V be a reduced pure N-dimensional algebraic set. Let*

$$R := \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}$$

be a collection of n algebraic functions $R_1, \ldots, R_n$ such that at each point $x \in V$ there is at least one R_i that is not zero. Given a Zariski open and dense subset $U \subset V$, there is a nonempty Zariski open set $\mathcal{A} \subset \mathbb{C}^{N\times n}$ of the $N \times n$ matrices such that for $A \in \mathcal{A}$, it follows that $\mathrm{Var}(A \cdot R) \subset U$.

The above results are true also for homogeneous polynomials on projective algebraic sets, when interpreted in terms of line bundles.

1.2 Witness sets

A *witness set* for an irreducible algebraic set $V \subset f^{-1}(0) \subset \mathbb{C}^N$ of dimension ℓ and degree d is the triple $\{f, \mathcal{L}, W\}$ where $\mathcal{L}$ consists of ℓ general linear polynomials on $\mathbb{C}^N$ and $W = V \cap \mathrm{Var}(\mathcal{L})$. The set W, which consists of d points, is called a *witness point set* for V.

The *multiplicity* of V with respect to f is equal to the multiplicity of any $w \in W$ as a solution of $\begin{bmatrix} f \\ \mathcal{L} \end{bmatrix} = 0$. If the multiplicity of V is 1, then V is said to be generically reduced. Otherwise, V is said to be generically nonreduced and, in this case, additional items can be added to the witness set to facilitate numerical computations on V. See [17] for more details regarding witness sets.

1.3 Deflation

Deflation is a regularization procedure to facilitate numerical computations for generically nonreduced irreducible components. Started by Ojika [10, 11], the first variant guaranteed to terminate was done for isolated points in [8] and for components in [17, §13.3.2, §15.2.2]. Improvements were made in [4], [8], and [6, §4.1].

Let $f : \mathbb{C}^N \to \mathbb{C}^{N-\ell}$ be a polynomial system with $x = (x_1, \ldots, x_N) \in \mathbb{C}^N$. Let V denote an irreducible component of dimension ℓ of $\mathrm{Var}(f)$. Deflation produces a polynomial system $\widehat{f} : \mathbb{C}^N \times \mathbb{C}^{N'} \to \mathbb{C}^{N+N'-\ell}$ in the variables $(x, \xi) = (x_1, \ldots, x_N, \xi_1, \ldots, \xi_{N'})$ for some $N' \geq 0$ such that there is a dimension ℓ generically reduced irreducible component $\widehat{V} \subset \mathrm{Var}(\widehat{f})$ which maps generically one-to-one onto a Zariski dense set of V under the map $(x, \xi) \to x$.

We assume from this point on that V is irreducible and generically reduced.

1.4 Moving linear spaces

The key operation used to compute a witness set for $\overline{\pi(V)}$ given a witness set for V is the moving of linear spaces. Let $\{f, \mathcal{L}, W\}$ be a witness set for an irreducible and generically reduced $V \subset f^{-1}(0)$ of dimension ℓ where $f : \mathbb{C}^N \to \mathbb{C}^n$ and $\mathcal{L} : \mathbb{C}^N \to \mathbb{C}^\ell$. If $\widehat{\mathcal{L}} : \mathbb{C}^N \to \mathbb{C}^\ell$ consists of ℓ linear polynomials with $\dim \mathrm{Var}(\widehat{\mathcal{L}}) = N - \ell$, we want to compute $\widehat{W} = V \cap \mathrm{Var}(\widehat{\mathcal{L}})$ given $W = V \cap \mathrm{Var}(\mathcal{L})$. Since $\mathcal{L}$ consists of ℓ general linear polynomials, we know, counting multiplicity in $\widehat{W}$, $|\widehat{W}| \leq |W|$. Define $n_\infty = |W| - |\widehat{W}|$.

Let $A \in \mathbb{C}^{(N-l) \times n}$ be generic and consider a general randomization of f, $\mathcal{R}(f) : \mathbb{C}^N \to \mathbb{C}^{N-l}$, defined by $\mathcal{R}(f) = A \cdot f$. Since V is ℓ dimensional, V is an irreducible component of $\mathcal{R}(f)^{-1}(0)$. The randomization $\mathcal{R}(f)$ allows us to construct the "square" homotopy $H : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N$ defined by

$$H(x, t) = \begin{bmatrix} \mathcal{R}(f)(x) \\ (1-t)\widehat{\mathcal{L}}(x) + t\mathcal{L}(x) \end{bmatrix}. \tag{1}$$

Starting at $t = 1$ with the points in W, continuation allows one to track the path defined by $H(x, t) \equiv 0$ as t goes from 1 to 0. See [17] for more information about path tracking and continuation.

Of the $|W|$ paths tracked using the homotopy H, n_∞ of them will diverge as t approaches 0. The set $\widehat{W} = V \cap \mathrm{Var}(\widehat{\mathcal{L}})$ is the set of endpoints of the paths that converge to a point in $\mathbb{C}^N$ as t approaches 0.

2 Linear maps

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system having an irreducible component $V \subset f^{-1}(0)$ of dimension ℓ and $\pi : \mathbb{C}^N \to \mathbb{C}^K$ be a linear map. In particular, there exists $B \in \mathbb{C}^{K \times N}$ such that $\pi(x) = Bx$. By deflating and updating the objects as needed, we may assume that V is generically reduced with witness set $\{f, \mathcal{L}, W\}$.

Since V is irreducible and π is a linear map, $\overline{\pi(V)}$ is irreducible. Since $\pi(V)$ is constructible, it contains a nonempty Zariski open set $\mathcal{V}$ with $\overline{\mathcal{V}} =$

$\overline{\pi(V)}$ [17, §12.5]. Thus $\dim \pi(V)$ is well defined, i.e., $\dim \mathcal{V} = \dim \overline{\pi(V)}$. Moreover in light of Corollary 2, $\deg \pi(V)$ is also well defined. Thus we will abuse notation and deal with $\pi(V)$ as if it was an algebraic set, while in fact we are talking of properties of $\mathcal{V}$ or $\overline{\pi(V)}$.

There are four important invariants related to $\pi(V)$. The first two are the dimension and degree of $\pi(V)$, denoted $\dim \pi(V)$ and $\deg \pi(V)$, respectively. For general $y \in \pi(V)$, the fiber over y with respect to V is the algebraic set $\{v \in V \mid \pi(v) = y\}$. The other two invariants are the dimension and degree of the fiber over a generic point in $\pi(V)$ with respect to V, denoted $\dim_{gf}(V, \pi)$ and $\deg_{gf}(V, \pi)$, respectively.

An important part of constructing a witness set for $\pi(V)$ is to compute the dimension of $\pi(V)$, which is described in the following lemma.

Lemma 3. *Let $x^* \in V$ be generic and $J(x^*)$ be the Jacobian matrix of f at x^*. The dimension of the null space of $\begin{bmatrix} J(x^*) \\ B \end{bmatrix}$, say p, is the dimension of the fiber over x^*. In particular, $\dim_{gf}(V, \pi) = p$ and $\dim \pi(V) = \ell - p$.*

Proof. This is a special case of [17, Theorem A.6.1]. □

The following definition describes a witness set for $\pi(V)$.

Definition 4. *Using the setup described above, a witness set for $\pi(V)$ is the quadruple $\{f, \pi, \widehat{\mathcal{L}}, \widehat{W}\}$ where $\widehat{W} = V \cap \mathrm{Var}(\widehat{\mathcal{L}})$ for a system $\widehat{\mathcal{L}} : \mathbb{C}^N \to \mathbb{C}^\ell$ of ℓ linear equations $\widehat{L}_1, \ldots, \widehat{L}_\ell$ of the form $\widehat{L}_i(x) = b_i \cdot x - 1$ where for $i = 1, \ldots, \ell' := \dim \pi(V)$, $b_i \in \mathbb{C}^N$, b_i is a general element in the row span of B and, for $i = \ell' + 1, \ldots, \ell$, the b_i are general elements of $\mathbb{C}^N$.*

Remark 5. *Computations performed on $\pi(V)$ using this definition of a witness set are actually performed on V.*

Upon computing $\dim \pi(V)$ using Lemma 3, we can construct $b_1, \ldots, b_\ell$, $\widehat{L}_1, \ldots, \widehat{L}_\ell$, and $\widehat{\mathcal{L}}$ as in Definition 4. Since $\dim \pi(V) \leq \operatorname{rank} B$ and

$$\dim_{gf}(V, \pi) = l - \dim \pi(V) \leq \dim \operatorname{null} B,$$

$b_1, \ldots, b_\ell$ are linearly independent yielding that $\dim \mathrm{Var}(\widehat{\mathcal{L}}) = N - \ell$. In particular, $\widehat{W} = V \cap \mathrm{Var}(\widehat{\mathcal{L}})$ consists of finitely many points.

The set $\widehat{W}$ is computed using the homotopy H defined by Equation 1. Starting from the points in the witness point set W, the set $\widehat{W}$ is the set of endpoints that converge in $\mathbb{C}^N$ and let n_∞ be the number of paths that diverge. Due to the genericity of the b_i's, each $\widehat{w} \in \widehat{W}$ is the endpoint of a unique path. The following lemma uses $\widehat{W}$ to compute $\deg \pi(V)$ and $\deg_{gf}(V, \pi)$.

Theorem 6. *With the setup described above,* $\deg \pi(V)$ *is the number of distinct elements in* $\pi(\widehat{W}) = \{\pi(\widehat{w}) \mid \widehat{w} \in \widehat{W}\}$*. Given any* $y \in \pi(\widehat{W})$*,* $\deg_{gf}(V, \pi) = \left|\{\widehat{w} \in \widehat{W} \mid \pi(\widehat{w}) = y\}\right|$*. In particular,* $\deg \pi(V) \cdot \deg_{gf}(V, \pi) = \left|\widehat{W}\right|$ *and*

$$\deg \pi(V) \cdot \deg_{gf}(V, \pi) + n_\infty = \deg V.$$

Proof. This is a variant of a classical result [9, Theorem 5.11]. We assume without loss of generality that $\ell' := \dim \pi(V)$ has dimension at least one.

Let $\overline{L}_{\ell'+1}, \ldots, \overline{L}_\ell$ be general linear equations on $\mathbb{C}^N$. By Seidenberg's Theorem [3, Theorem 1.7.1], $V' = V \cap \mathrm{Var}\left(\overline{L}_{\ell'+1}, \ldots, \overline{L}_\ell\right)$ is irreducible of dimension $\ell - \ell'$. Since it meets a general fiber in a finite set of points (by Theorem 1), $\pi(V')$ is dense in $\overline{\pi(V)}$. Note (e.g., by [3, Theorem A.4.20]) that $\pi_{V'}$ is a covering when restricted to $V' \cap \pi^{-1}(U')$ for a nonempty Zariski open U' of $\overline{\pi(V')}$. Note that the number of points in a general fiber is $\deg_{gf}(V, \pi)$.

Let $L_1, \ldots, L_{\ell'}$ be general linear equations on $\mathbb{C}^K$. We have

$$\deg \overline{\pi(V)} = |\overline{\pi(V)} \cap \mathrm{Var}(L_1, \ldots, L_{\ell'})|.$$

Note that by multiplying each L_i by a nonzero number and composing with π, we get $\widehat{L}_1, \ldots, \widehat{L}_{\ell'}$ as in Definition 4.

Using Corollary 2, we see that $V' \cap \mathrm{Var}\left(\widehat{L}_1, \ldots, \widehat{L}_{\ell'}\right)$ consists $\deg \overline{\pi(V)}$ general fibers of $\pi : V' \to \overline{\pi(V')}$. This proves the theorem. ∎

Remark 7. *It is helpful to consider the simple case of* $\mathrm{Var}(y - x^2) \subset \mathbb{C}^2$*. Let* $\pi : \mathbb{C}^2 \to \mathbb{C}$ *denote the projection* $(x, y) \to x$*. The fiber over any point* x *is the unique point* (x, x^2)*. The degree of* $\mathrm{Var}(y - x^2)$ *is two, so* $n_\infty = 1$*.*

Lemma 3 and Theorem 6 justify the following algorithm which summarizes the computation of a witness set for $\pi(V)$.

Procedure $\{f, \pi, \widehat{\mathcal{L}}, \widehat{W}\} = \mathrm{ProjectionWitnessSet}(\pi, \{f, \mathcal{L}, W\})$

Input A linear map $\pi : \mathbb{C}^N \to \mathbb{C}^k$ and a witness set for an irreducible and generically reduced $V \subset f^{-1}(0)$ where $f : \mathbb{C}^N \to \mathbb{C}^n$ and $\mathcal{L} : \mathbb{C}^N \to \mathbb{C}^\ell$.

Output A witness set for $\pi(V)$.

Begin
1. Construct $B \in \mathbb{C}^{k \times N}$ such that $\pi(x) = Bx$.
2. Pick $w \in W$ and compute $r := l - \dim \mathrm{null} \begin{bmatrix} J(w) \\ B \end{bmatrix}$, where $J(w)$ is the Jacobian of f at w.

3. Let $a_1, \ldots, a_r \in \mathbb{C}^k$ be random and, for $i = 1, \ldots, r$, compute $b_i := B^T a_i$.
4. Compute $j := \dim \operatorname{null} B$ and a matrix $N \in \mathbb{C}^{N \times j}$ whose columns form a basis for $\operatorname{null} B$.
5. Let $a_{r+1}, \ldots, a_l \in \mathbb{C}^j$ be random and, for $i = r+1, \ldots, \ell$, compute $b_i := N a_i$.
6. Set $\widehat{L}_i(x) := b_i \cdot x - 1$ and $\widehat{\mathcal{L}} := \begin{bmatrix} \widehat{L}_1 \\ \vdots \\ \widehat{L}_l \end{bmatrix}$.
7. Let $A \in \mathbb{C}^{(N-l)\times n}$ be random. Construct $\mathcal{R}(f) := A \cdot f$ and $H(x,t) := \begin{bmatrix} \mathcal{R}(f)(x) \\ (1-t)\widehat{\mathcal{L}}(x) + t\mathcal{L}(x) \end{bmatrix}$.
8. Initialize $\widehat{W} := \{\}$.
9. For each $w \in W$
   (a) Track the homotopy path defined by $H(x,t) \equiv 0$ starting at w at $t = 1$.
   (b) If the path converges to, say $\widehat{w} \in \mathbb{C}^N$, as $t \to 0$, update $\widehat{W} := \widehat{W} \cup \{\widehat{w}\}$.

Return $\{f, \pi, \widehat{\mathcal{L}}, \widehat{W}\}$**.**

Remark 8. *If we work over projective space and use linear projections, and consider homogeneous coordinates as sections of the hyperplane section bundle, the algorithm is still true. The key fact is that linear projection* $\pi : \mathbb{P}^N \to \mathbb{P}^K$ *are not everywhere well defined. The center* $\mathcal{C}$ *of the projection* π *(see [17, §A.8.2]) is the set of indeterminacy of* π*. It is of dimension* $N - K - 1$ *and contained in each fiber of* π*.*

3 Examples

We first restate part of the construction of a witness set for a projection when the projection takes a simplified form. We then deal with two examples.

3.1 Projections

A *projection map* is a linear map $\pi : \mathbb{C}^N \to \mathbb{C}^K$ such that, upon possibly renaming variables, $\pi(x_1, \ldots, x_N) = (x_1, \ldots, x_K)$. In particular, $\pi(x) = Bx$

where $B = \begin{bmatrix} I_K & 0 \end{bmatrix} \in \mathbb{C}^{K\times N}$ and I_K is the $K \times K$ identity matrix. Due to the structure of B, computing a witness set for the projection of an irreducible algebraic set is simplified.

Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system and $\{f, \mathcal{L}, W\}$ be a witness set for an irreducible, generically reduced component $V \subset f^{-1}(0)$ of dimension ℓ. Let $J(x)$ denote the Jacobian matrix of f at x and $J(x)_{[\cdots,K+1:N]}$ denote the matrix consisting of the last $N - K$ columns of $J(x)$. By Lemma 3, for any $w \in W$, $\dim_{gf}(V, \pi) = \dim \operatorname{null} J(w)_{[\cdots,K+1:N]}$ and $\dim \pi(V) = \ell - \dim_{gf}(V, \pi)$.

If $\ell' = \dim \pi(V)$, let $C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix} \in \mathbb{C}^{\ell\times N}$ where $C_1 \in \mathbb{C}^{\ell'\times K}$ and $C_2 \in \mathbb{C}^{(\ell-r)\times(N-K)}$ are general. Let b_i denote the i^{th} row of C, $\widehat{L}_i(x) = b_i \cdot x - 1$, and $\widehat{\mathcal{L}} = \begin{bmatrix} \widehat{L}_1 \\ \vdots \\ \widehat{L}_\ell \end{bmatrix}$. In particular, C_1 defines r general linear polynomials in the "projection variables" $x_1, \ldots, x_K$ and C_2 defines $\ell - \ell'$ general linear polynomials in the "fiber variables" $x_{K+1}, \ldots, x_N$.

3.2 A basic example

Consider $f(a, b, c, x) = \begin{bmatrix} ax^2 + bx + c \\ 2ax + b \end{bmatrix}$ and $V = \operatorname{Var}(f)$ with $\dim V = 2$, $\deg V = 3$, and witness set $\{f, \mathcal{L}, W\}$. Let $\pi(a, b, c, x) = (a, b, c)$. The set $\pi(V)$ is the discriminant locus of quadratic univariate polynomials. Since π is a projection onto the first three coordinates and

$$J(a, b, c, x)_{[\cdots,4:4]} = \begin{bmatrix} 2ax + b \\ 2a \end{bmatrix}$$

is nonzero for generic $(a, b, c, x) \in \operatorname{Var}(f)$,

$$\dim \pi(V) = 2 \quad \text{and} \quad \dim_{gf}(V, \pi) = 0.$$

Let $\alpha_{i,j} \in \mathbb{C}$ be random for $i = 1, 2$ and $j = 1, 2, 3$. For $i = 1, 2$, define $\widehat{L}_i(a, b, c, x) = \alpha_{i,1}a + \alpha_{i,2}b + \alpha_{i,3}c - 1$ and $\widehat{\mathcal{L}} = \begin{bmatrix} \widehat{L}_1 \\ \widehat{L}_2 \end{bmatrix}$. When using the homotopy H defined by Equation 1, one of the three paths diverges as t approaches 0. Since the endpoints of the two convergent paths map to distinct points under π, $\deg \pi(V) = 2$ and $\deg_{gf}(V, \pi) = 1$.

3.3 Dual variety

Consider computing the degree of the dual variety of the smooth sextic curve $C = \mathrm{Var}(f) \subset \mathbb{P}^3$ where

$$f(x,y,z,w) = \left[\begin{array}{c} f_1(x,y,z,w) \\ f_2(x,y,z,w) \end{array}\right] = \left[\begin{array}{c} x^2+y^2+z^2+w^2 \\ xyz - w^3 \end{array}\right]$$

which is discussed in [12, §4.4]. The dual variety $\mathcal{D}$ of C is the union of all hyperplanes in $\mathbb{P}^3$ which are tangent to C. See [5] for more details regarding dual varieties.

We computed the degree for the dual variety $\mathcal{D}$ of C as follows. Let $L_1, L_2 : \mathbb{C}^4 \to \mathbb{C}$ be general linear polynomials. Then $L_1(x,y,z,w) = L_2(X,Y,Z,W) = 0$ defines an affine set, which under the natural projection $\left[\mathbb{C}^4 \setminus \{(0,0,0,0)\right] \times \left[\mathbb{C}^4 \setminus \{(0,0,0,0)\right] \to \mathbb{P}^3 \times \mathbb{P}^3$ goes one-to-one and onto a general coordinate patch $\mathbb{C}^3 \times \mathbb{C}^3 \subset \mathbb{P}^3 \times \mathbb{P}^3$.

We work with the set of pairs consisting of a point in $\mathbb{P}^3$ and a hyperplane containing it. The condition that an hyperplane $[X,Y,Z,W]$ of $\mathbb{P}^3$ vanish at a point $[x,y,z,w]$ gives the polynomial

$$xX + yY + zZ + wW = 0.$$

Thus we have five equations so far

$$\left[\begin{array}{c} f_1(x,y,z,w) \\ f_2(x,y,z,w) \\ L_1(x,y,z,w) \\ L_2(X,Y,Z,W) \\ xX+yY+zZ+wW \end{array}\right] = 0.$$

Finally we need that the hyperplane $[X,Y,Z,W]$ is tangent to C at $[x,y,z,w]$. This condition translates into

$$\mathrm{rank}\left[\begin{array}{c} \nabla f_1(x,y,z,w) \\ \nabla f_2(x,y,z,w) \\ (X,Y,Z,W) \end{array}\right] = 2.$$

As explained in [1], a dense Zariski open set of this set may be identified with the solution set of

$$\left[\begin{array}{c} v_1 \nabla f_1(x,y,z,w)^T + v_2 \nabla f_2(x,y,z,w)^T + v_3\,(X,Y,Z,W)^T \\ L_3(v_1,v_2,v_3) \end{array}\right] = 0$$

where $L_3 : \mathbb{C}^3 \to \mathbb{C}$ is a general linear polynomial.

If π_1 and π_2 are the projections of $\mathbb{C}^{11}$ onto the first and second four coordinates, respectively, then the images $\pi_1(S)$ and $\pi_2(S)$ of the algebraic set S consisting of points $(x, y, z, w, X, Y, Z, W, v_1, v_2, v_3) \in \mathbb{C}^{11}$ such that

$$\begin{bmatrix} f_1(x,y,z,w) \\ f_2(x,y,z,w) \\ xX + yY + zZ + wW \\ v_1 \nabla f_1(x,y,z,w)^T + v_2 \nabla f_2(x,y,z,w)^T + v_3\,(X,Y,Z,W)^T \\ L_1(x,y,z,w) \\ L_2(X,Y,Z,W) \\ L_3(v_1,v_2,v_3) \end{bmatrix} = 0$$

correspond to C and the dual variety $\mathcal{D}$ of C, respectively. In particular, the degree of the dual variety $\mathcal{D}$ of C is the degree of $\pi_2(S)$.

Using one core of a 2.5 GHz Intel Xeon E5420 processor, Bertini v1.2.0 [2] employing regeneration [6] with intrinsic slicing computed a witness set for S, a two dimensional irreducible surface of degree 54, in 8 seconds. Upon computing the dimensions using Lemma 3, it took Bertini about a second to track the 54 paths each for π_1 and π_2 using the homotopy H defined by Equation 1. The results are summarized in Table 1. In particular, this computation verifies that C is a sextic curve and shows that the dual variety $\mathcal{D}$ of C is a degree 18 surface.

Table 1: Summary for $\pi_1(S)$ and $\pi_2(S)$

i	$\dim \pi_i(S)$	$\deg \pi_i(S)$	$\dim_{gf}(S, \pi_i)$	$\deg_{gf}(S, \pi_i)$	n_∞
1	1	6	1	1	48
2	2	18	0	1	36

References

[1] D.J. Bates, J.D. Hauenstein, C. Peterson, and A.J. Sommese. Numerical decomposition of the rank-deficiency set of a matrix of multivariate polynomials. In *Approximate Commutative Algebra*, ed. by L. Robbiano and J. Abbott, Texts and Monographs in Symbolic Computation, Vol. 14, Springer (2010), 55–77.

[2] D.J. Bates, J.D. Hauenstein, A.J Sommese, and C.W. Wampler. Bertini: software for numerical algebraic geometry. Available at `www.nd.edu/∼sommese/bertini`.

[3] M. Beltrametti and A.J. Sommese. *The Adjunction Theory of Complex Projective Varieties*. Expositions in Mathematics, 16, Walter De Gruyter, Berlin (1995).

[4] B.H. Dayton and Z. Zeng. Computing the multiplicity structure in solving polynomial systems. In *ISSAC'05*, 116–123 (electronic), ACM, New York, 2005. Available at `dx.doi.org/10.1145/1073884.1073902`.

[5] I.M. Gel'fand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, Boston, 1994.

[6] J.D. Hauenstein, A.J. Sommese, and C.W. Wampler. Regeneration homotopies for solving systems of polynomials. To appear in *Mathematics of Computation*.

[7] A. Leykin, J. Verschelde, and A. Zhao. Higher-order deflation for polynomial systems with isolated singular solutions. In *IMA Volume 146: Algorithms in Algebraic Geometry*, edited by A. Dickenstein, F.-O. Schreyer, and A.J. Sommese, Springer (2008), 79–97.

[8] A. Leykin, J. Verschelde, and A. Zhao. Newton's method with deflation for isolated singularities of polynomial systems. *Theor. Comp. Sci.* 359 (2006), pp. 111–122.

[9] D. Mumford. *Algebraic Geometry I: Complex Projective Varieties*. Springer, Berlin, 1995.

[10] T. Ojika. Modified deflation algorithm for the solution of singular problems. I. A system of nonlinear algebraic equations, *J. Math. Anal. Appl.* 123 (1987), pp. 199–221.

[11] T. Ojika, S. Watanabe, and T. Mitsui. Deflation algorithm for the multiple roots of a system of nonlinear equations, *J. Math. Anal. Appl.* 96 (1983), pp. 463–479.

[12] K. Ranestad and B. Sturmfels. The convex hull of a variety, to appear in the Julius Borcea Memorial Volume, (eds. P. Branden, M. Passare and M. Putinar), *Trends in Mathematics*, Birkhauser Verlag.

[13] A.J. Sommese and J. Verschelde. Numerical Homotopies to compute generic points on positive dimensional Algebraic Sets. *Journal of Complexity* 16 (2000), 572–602.

[14] A.J. Sommese, J. Verschelde and C.W. Wampler. Numerical decomposition of the solution sets of polynomials into irreducible components. *SIAM J. Numer. Anal.* 38 (2001), 2022–2046.

[15] A.J. Sommese, J. Verschelde, and C.W. Wampler. Symmetric functions applied to decomposing solution sets of polynomial systems. *SIAM J. Numer. Anal.* 40 (2002), 2026-2046.

[16] A.J. Sommese and C.W. Wampler. Numerical algebraic geometry. In *The Mathematics of Numerical Analysis: Real Number Algorithms, Park City, Utah, Summer 1995*, ed. by J. Renegar, M. Shub, and S. Smale, Lectures in Applied Math. 32 (1996), 749–763, Amer. Math. Soc.

[17] A.J. Sommese and C.W. Wampler. *The Numerical Solution to Systems of Polynomials Arising in Engineering and Science.* World Scientific, Singapore, 2005.