Binomiality testing and computing sparse polynomials via witness sets

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Abstract

Sparse polynomials that vanish on algebraic sets are preferred in many computations over the complex numbers since they are easy to evaluate and often arise from underlying structure. For example, a monomial vanishes on an algebraic set if and only if the algebraic set is contained in the union of the coordinate hyperplanes. Eisenbud and Sturmfels initiated a detailed study of binomial ideals 25 years ago and showed that they had many special properties including that each component is rational. Given a general point on a component and its tangent space, this paper exploits rationality to develop a local approach that decides if the component is defined by binomials or not. When a component is not defined by binomials, one often is interested in computing sparse polynomials that vanish on the component. Thus, this paper also develops an approach for computing sparse polynomials using a witness set for the component. Our approach relies on using numerical homotopy methods to sample points on the algebraic set along with incorporating multiplicity information using Macaulay dual spaces. If the algebraic set is defined by polynomials with rational coefficients, exactness recovery such as lattice based methods can be used to find exact representations of the sparse polynomials. Several examples are presented demonstrating the new methods.

1 Introduction

The number of terms in a polynomial is one measure of its complexity in which a polynomial is said to be sparse if it has relatively few terms. Sparse polynomials can be used to identify

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underlying structure. In terms of sparsity, monomials (polynomials with one term) have classically been exploited. For example, the corresponding solution set of a collection of monomials is contained in the union of the coordinate hyperplanes and flat degenerations to monomial ideals such as through Gröbner bases demonstrate the usefulness of monomial ideals in computational algebraic geometry. In 1996, Eisenbud and Sturmfels [12] initiated the study of ideals generated by binomials (polynomials with at most two terms) and showed that they have many remarkable properties. For example, every component of a binomial ideal is rational and sparsity-preserving operations can be performed on them. See the books [7, 13] for more information about toric varieties and binomial ideals.

Since monomials and binomials are extremely useful in computational algebraic geometry, several methods have been proposed to search for them. Monomials can be determined using Gröbner basis methods, e.g., see [21]. Gröbner-free approaches for deciding if a given ideal is generated by binomials using linear algebra computations are presented in [6, 23]. For some polynomial systems arising from chemical reaction networks and biological models, this approach can be more efficient than Gröbner basis methods [17, 22, 23]. An approach for detecting binomiality after an ambient automorphism was proposed in [19]. A theoretical approach based on tropical geometry, matrix theory, and computational number theory is presented in [18] for deciding if an ideal contains a binomial.

Our approach for testing for binomiality of a component is a local numerical computation where the input is a general point on the component along with its tangent space. Then, one simply compares the computed tangent space with what one must have if the component was defined by binomials as summarized in Algorithm 1.

To extend beyond binomials, this article considers computing all polynomials of degree at most d with at most t terms that vanish on an algebraic set represented by a witness set. Witness sets allow sample points and multiplicity information to be computed. From this data, we propose two methods to compute sparse polynomials: checking all minors of a matrix constructed from the degree d Veronese embedding of the points or using an ℓ_1-relaxation. If the algebraic set is defined over $\mathbb{Q}$, then exactness recovery techniques, such as lattice based methods [1], can be used to obtain exact representations of the sparse polynomials.

The rest of the article proceeds as follows. Section 2 summarizes necessary background information on witness sets, sampling, multiplicity, exactness recovery, and ℓ_1-relaxation. The method for testing binomiality is described in Section 3 and computing sparse polynomials is described in Section 4. Several examples are presented in Section 5.

2 Background

The following provides necessary background on topics from numerical algebraic geometry and sparsity. For more details regarding numerical algebraic geometry, see [3, 24].

2.1 Witness sets and sampling

For a system of polynomials

$$f(x) = \begin{bmatrix} f_1(x_1, \ldots, x_N) \\ \vdots \\ f_n(x_1, \ldots, x_N) \end{bmatrix},$$

the corresponding *algebraic set* (or *variety*) is

$$\mathcal{V}(f) = \{x \in \mathbb{C}^N \mid f(x) = 0\} \subset \mathbb{C}^N.$$

Each algebraic set can be decomposed uniquely (up to reordering) into a union of *irreducible components* creating the *irreducible decomposition*, say

$$\mathcal{V}(f) = \bigcup_{i=1}^{k} A_i$$

where each A_i is irreducible. Let $\dim A_i$ be the dimension of A_i, which is equal to the minimum of the dimension of the tangent space at each point of A_i.

A *witness set* for an irreducible algebraic set $X \subset \mathbb{C}^N$ is $\{g, \mathcal{L}, W\}$ which provides a geometric representation of X where:

- *witness system* g is a polynomial system in which X is an irreducible component of $\mathcal{V}(g)$;
- *witness slice* $\mathcal{L}$ is a linear space with $\operatorname{codim} \mathcal{L} = \dim X$ that intersects X transversely;
- *witness point set* $W = X \cap \mathcal{L}$ with $\#W = \deg X$.

One key aspect of witness sets is the ability to localize computations to the component described by the witness set. That is, for the witness system g, $\mathcal{V}(g)$ can have other irreducible components besides X, but the witness point set W localizes all further computations to X without needing to require $X = \mathcal{V}(g)$ nor having to compute the other components of $\mathcal{V}(g)$. An example of this is by deforming the linear space $\mathcal{L}$ to compute other points on X, i.e., *sample* points from X. If $\mathcal{L}' \subset \mathbb{C}^N$ is a general linear space with $\operatorname{codim} \mathcal{L}' = \operatorname{codim} \mathcal{L} = \dim X$, then one can consider the homotopy deforming $\mathcal{L}$ to $\mathcal{L}'$ along X, namely

$$X \cap (t \cdot \mathcal{L} + (1-t) \cdot \mathcal{L}'). \tag{1}$$

At $t = 1$, one starts with $W = X \cap \mathcal{L}$ which deforms to $W' = X \cap \mathcal{L}'$ at $t = 0$. By selecting various linear spaces, one is able to sample as many generic points on X as needed.

Example 1. *To illustrate, consider* $X = \mathcal{V}(x^2 + y^2 - 1) \subset \mathbb{C}^2$. *Clearly,* $\dim X = 1$ *and* $\deg X = 2$. *An example of a witness set for* X*, as illustrated in Figure 1, is* $\{g, \mathcal{L}, W\}$ *where:*

- $g = x^2 + y^2 - 1$,

- $\mathcal{L} = \mathcal{V}(x + 2y - 1)$ *with* $\operatorname{codim} \mathcal{L} = \dim X = 1$,
- $W = X \cap \mathcal{L} = \{(1, 0), (-3/5, 4/5)\}$ *with* $\#W = \deg X = 2$.

For the witness set in Ex. 1, the coefficients of the linear space $\mathcal{L}$ were chosen to be small integers for illustration purposes. In practice, the coefficients are chosen to be random complex numbers to ensure, with probability one, that $\mathcal{L}$ is transverse to X. Moreover, we also selected X in Ex. 1 to be a hypersurface in the plane for illustration purposes. When X is a hypersurface, the witness slice $\mathcal{L}$ is a line that can be deformed to provide information about the Newton polynomial of X [4, 15]. Thus, the focus of the present work is on components X for which the codimension is larger than one so that the ideal of polynomials vanishing on X, denoted $I(X)$, is not principal.

2.2 Multiplicity

If g is a witness system for X, then X could have multiplicity greater than 1 as a component of $\mathcal{V}(g)$. There are two aspects associated to this that one needs to consider. First, standard predictor-corrector path tracking methods for (1) can be employed when the multiplicity is one. However, one often wants to capture the multiplicity structure imposed by a given polynomial system in order to compute sparse polynomials satisfying that multiplicity structure. These aspects are addressed using *deflation* and *Macaulay dual spaces*, respectively.

Suppose that $X \subset \mathbb{C}^N$ is an irreducible algebraic set with witness set $\{g, \mathcal{L}, W\}$. Considering (1), let ℓ and ℓ' be linear systems such that $\mathcal{L} = \mathcal{V}(\ell)$ and $\mathcal{L}' = \mathcal{V}(\ell')$. Then, one can translate the geometric deformation (1) to the algebraic homotopy

$$H(x, t) = \begin{bmatrix} g(x) \\ t \cdot \ell(x) + (1 - t) \cdot \ell'(x) \end{bmatrix} = 0. \tag{2}$$

The multiplicity of X with respect to g is the multiplicity of $w \in W$ with respect to $\{g, \ell\}$. Hence, if X has multiplicity one with respect to g, then $\dim \operatorname{null} Jg(w) = \dim X$ where $Jg(w)$

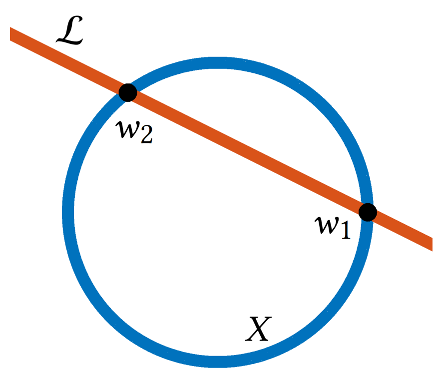

Figure 1: Illustration of witness set for X with witness point set $W = X \cap \mathcal{L} = \{w_1, w_2\}$.

is the Jacobian matrix of g evaluated at w, showing that the solution path defined by (2) starting at w is nonsingular for $0 \leq t \leq 1$.

When the multiplicity is greater than one, *deflation* is a process of removing multiplicity to return to the multiplicity one case, e.g., see [9, 16, 20]. The following describes using *isosingular deflation* [16] in order to construct a witness system g' from g such that X has multiplicity 1 with respect to g'. Define $g_0 = g$ and consider the deflation operator $\mathcal{D}$ with

$$(g_{i+1}, w) = \mathcal{D}(g_i, w)$$

where g_{i+1} consists of g_i and all $(r+1) \times (r+1)$ minors of $Jg_i(x)$ where $r = \operatorname{rank} Jg_i(w)$. The sequence $s_i = \dim \operatorname{null} Jg_i(w)$ is a nonincreasing sequence of nonnegative integers that limits to $\dim X$, i.e., there exists i^* such that $s_i = \dim X$ for all $i \geq i^*$. Hence, one can take $g' = g_{i^*}$ to be used for path tracking along X.

Next, we aim to encode the multiplicity structure of w with respect to $f = \{g, \ell\}$ using a Macaulay dual space, e.g., see [9]. For $\alpha \in \mathbb{Z}^N_{\geq 0}$, consider the differential

$$\partial_\alpha = \frac{1}{\alpha!} \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where $\alpha! = \alpha_1! \cdots \alpha_N!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Evaluating at w yields the linear functional $\partial_\alpha[w]$ defined by

$$\partial_\alpha[w](p) = (\partial_\alpha p)(w).$$

Consider the infinite dimensional complex vector space

$$D_w = \operatorname{span}\left(\left\{\partial_\alpha[w] \;\middle|\; \alpha \in \mathbb{Z}^N_{\geq 0}\right\}\right). \tag{3}$$

The *Macaulay dual space* of f at w is the vector space

$$D_w[f] = \left\{\partial \in D_w \;\middle|\; \partial((x-w)^\beta f_j) = 0 \text{ for all } \beta \in \mathbb{Z}^N_{\geq 0} \text{ and all } j\right\}.$$

Since w is an isolated solution of $f = 0$, $D_w[f]$ is a finite dimensional vector space whose dimension is equal to the multiplicity of w with respect to f.

Example 2. *Consider the polynomial system*

$$g(x,y,z) = \begin{bmatrix} 2xy + 2xz - 2yz - 1 \\ x^2 + y^2 + z^2 - 1 \end{bmatrix} \tag{4}$$

for which $X = \mathcal{V}(g)$ is an irreducible curve of degree 2 which has multiplicity 2 with respect to g. For simplicity of presentation, consider the line $\mathcal{L} = \mathcal{V}(\ell)$ where $\ell = x + y + z - 1$ and the point $w = (2, 1+\sqrt{5}, 1-\sqrt{5})/4 \in X \cap \mathcal{L}$. Since $\operatorname{rank} Jg(w) = 1$, one iteration of isosingular deflation yields

$$g' = \begin{bmatrix} g \\ 4(x+y)(y-x+z) \\ 4(x+z)(y-x+z) \\ 4(y-z)(y-x+z) \end{bmatrix}$$

with $\dim \operatorname{null} Jg'(w) = 1 = \dim X$*. Additionally, for* $f = \{g, \ell\}$*,* w *has multiplicity* 2 *with respect to* f *with Macaulay dual space*

$$D_w[f] = \operatorname{span}\left\{\partial_{(0,0,0)}[w],\quad 2\sqrt{5}\partial_{(1,0,0)}[w] - (1+\sqrt{5})\partial_{(0,1,0)}[w] + (1-\sqrt{5})\partial_{(0,0,1)}[w]\right\}.$$

2.3 Exactness recovery

For algebraic sets which are defined over the rational numbers, one often aims to recover exact rational numbers from numerical data. The exactness recovery considered in [1] is: given a numerical approximation $\widetilde{w}$ of a generic point $w \in X \subset \mathbb{C}^N$, compute exact polynomials p of degree at most d which vanish on X using lattice based methods. Thus, one aims to compute integer vectors c such that $c \cdot \nu_d(\widetilde{w}) \approx 0$ where $\nu_d(z)$ is the degree d Veronese embedding of z, namely

$$\nu_d(z) = (1, z_1, \ldots, z_N, z_1^2, z_1 z_2, \ldots, z_N^2, \ldots, z_N^d). \tag{5}$$

Example 3. *Consider the numerical approximation*

$$\widetilde{w} = (0.5510119638, -0.2463075497, 0.7973195135)$$

of a point $w \in X \cap \mathcal{V}(x + \sqrt{2}y + z - 1)$ *where* $X \subset \mathbb{C}^3$ *as in Ex. 2. Using* `PSLQ` *in* `Maple` *for* $d = 1$ *yields* $c = [0, -1, 1, 1]$ *corresponding with* $y + z - x$ *which does indeed vanish on* X *and is contained in* $\sqrt{\langle g \rangle}$ *where* g *is as in* (4)*. To search for other linear polynomials, we can repeat the* `PSLQ` *computation without the* x*-coordinate (since it is dependent on* y *and* z*) yielding* $[2798, 5601, -1779]$ *suggesting that there are no other vanishing linears with integer coefficients. Repeating with a* 50*-digit numerical approximation yields*

$$[-47223690816349078, 4761102144861194, 60698860867498949].$$

2.4 Sparsest nonzero null vector

In Section 2.3, lattice based methods are used to search for integer null vectors. To compute sparse polynomials, one is looking to compute sparse null vectors as described in Section 4.

For a vector $x \in \mathbb{R}^N$, define

$$\|x\|_0 = \#\{i \mid x_i \neq 0\}.$$

For a matrix $A \in \mathbb{R}^{m \times k}$, the *sparsest* nonzero null vector solves

$$\min\{\|x\|_0 \mid Ax = 0,\ x \neq 0\}.$$

Since one can arbitrarily rescale null vectors, one can pick a general coordinate patch to fix a scaling. That is, for a general vector $v \in \mathbb{R}^N$, one can consider

$$\min\{\|x\|_0 \mid Ax = 0,\ v \cdot x = 1\}. \tag{6}$$

Following a common technique in compressed sensing, e.g., see [5, 10, 11], one replaces the nonconvex optimization problem (6) with the convex optimization problem

$$\min\{\|x\|_1 \mid Ax = 0,\ v \cdot x = 1\} \tag{7}$$

where $\|x\|_1 = |x_1| + \cdots + |x_N|$. The optimization problem (7) is an ℓ_1-*relaxation* of (6). In fact, under suitable conditions on the matrix A, e.g., see [5, 10, 11], the solution to (7) solves (6). Since one aims to find many sparse polynomials in Section 4, we employ ℓ_1-relaxation without needing guaranteed theoretical results on the recovery of the sparsest vector.

3 Binomiality testing

For an irreducible algebraic set $X \subset \mathbb{C}^N$, it is natural to ask whether the ideal of polynomials vanishing on X, namely $I(X) \subset \mathbb{C}[x_1, \ldots, x_N]$, is generated by binomials or not. The new spin on this question is to answer this binomiality question using only a given generic point $w \in X$ and the tangent space $T_w(X) \subset \mathbb{C}^N$ of X at w so that $d := \dim X = \dim T_w(X)$.

First, consider relabeling the coordinates with $w = (w_1, \ldots, w_n, 0, \ldots, 0) \in (\mathbb{C}^*)^n \times \mathbb{C}^{N-n}$ where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ such that the first d variables are linearly independent in $T_w(X)$. With this relabeling, one immediately has a monomiality test.

Remark 4. After this relabeling, $\langle x_{n+1}, \ldots, x_N \rangle \subset I(X)$. Thus, ideal $I(X)$ is generated by monomials if and only if $d = n$ which would yield $I(X) = \langle x_{n+1}, \ldots, x_N \rangle$.

One more simplification is to ignore the last $N - n$ coordinates which are identically zero on X via projection. For the projection map $\pi : \mathbb{C}^N \to \mathbb{C}^n$ defined by $\pi(x) = (x_1, \ldots, x_n)$, it is easy to verify that $I(X) = I(\pi(X)) + \langle x_{n+1}, \ldots, x_N \rangle$ so that binomiality of $I(X)$ is equivalent to binomiality of $I(\pi(X))$. Therefore, by trivially projecting from $\mathbb{C}^N$ to $\mathbb{C}^n$ and easily updating X, w, and $T_w(X)$ accordingly, we can assume without loss of generality that $X \subset \mathbb{C}^n$ is not contained in any coordinate hyperplane, $w \in (\mathbb{C}^*)^n$, and $T_w(X) \subset \mathbb{C}^n$ where the first d coordinates are linearly independent in $T_w(X)$.

The key aspect of our approach is to exploit the rationality of irreducible algebraic sets defined by binomials [12]. In particular, $I(X) \subset \mathbb{C}[x_1, \ldots, x_n]$ is a binomial ideal if and only if there exists vectors $a_1, \ldots, a_n \in \mathbb{Z}^d$ such that

$$X = \overline{\{(w_1 t^{a_1}, \ldots, w_n t^{a_n}) \mid t \in (\mathbb{C}^*)^d\}}.$$

Since this yields a total of $d \cdot n$ unknown integers, we consider a reparameterization based on the linear independence of the first d variables. Thus, this results in $d \cdot (n - d)$ unknown rational numbers, namely $b_{d+1}, \ldots, b_n \in \mathbb{Q}^d$, such that

$$X = \overline{\{(w_1 s_1, \ldots, w_d s_d, w_{d+1} s^{b_{d+1}}, \ldots, w_n s^{b_n}) \mid s \in (\mathbb{C}^*)^d\}}. \tag{8}$$

In particular, treating a_i and b_j as column vectors and letting I_d be the $d \times d$ identity matrix, one simply has that the row reduced echelon form of

$$\begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \in \mathbb{Z}^{d \times n} \quad \text{is} \quad \begin{bmatrix} I_d & b_{d+1} & \cdots & b_n \end{bmatrix} \in \mathbb{Q}^{d \times n}.$$

Example 5. *For* $X = \mathcal{V}(x^3 - y^2) \subset \mathbb{C}^2$ *with* $w = (1, 1)$*, the classical parameterization is*

$$X = \overline{\{(t^2, t^3) \mid t \in \mathbb{C}^*\}}$$

corresponding with $a_1 = 2$ *and* $a_2 = 3$*. Thus, one has* $b_2 = 3/2$ *with*

$$X = \overline{\{(s, s^{3/2}) \mid s \in \mathbb{C}^*\}}.$$

Since w corresponds with $s = (1, \dots, 1) \in (\mathbb{C}^*)^d$ in the parameterization in (8), the tangent space of the parameterization at w is spanned by the rank d matrix

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ \frac{b_{d+1,1}w_{d+1}}{w_1} & \cdots & \frac{b_{d+1,d}w_{d+1}}{w_d} \\ \vdots & \ddots & \vdots \\ \frac{b_{n,1}w_n}{w_1} & \cdots & \frac{b_{n,d}w_n}{w_d} \end{bmatrix} \in \mathbb{C}^{n \times d}. \tag{9}$$

From $T_w(X)$, one can also express it in its reduced row echelon form, say

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ u_{d+1,1} & \cdots & u_{d+1,d} \\ \vdots & \ddots & \vdots \\ u_{n,1} & \cdots & u_{n,d} \end{bmatrix} \in \mathbb{C}^{n \times d}. \tag{10}$$

Since reduced row echelon form is classically known to be unique, one has that $I(X)$ is generated by binomials if and only if, for every $i = d+1, \dots, n$ and $j = 1, \dots, d$,

$$u_{i,j} \cdot \frac{w_j}{w_i} \in \mathbb{Q} \tag{11}$$

corresponding with $b_{i,j}$.

Algorithm 1 summarizes the approach which is justified by the preceding statements and demonstrated in the following.

Example 6. *To illustrate the steps in Algorithm 1 both numerically and symbolically simultaneously, the following considers* $X = \mathcal{V}(x^3 - y^2)$ *from Ex. 5 and* $Y = \mathcal{V}(x^3 - y^2 - y^3)$ *where* $d = 1$ *and* $n = 2$ *for both. Let* $i = \sqrt{-1}$*.*

For X*, consider the following general point (rounded to four decimal places)*

$$w = (w_1, w_2) \approx (0.2636 + 1.7272i, -1.2233 + 1.9589i).$$

Input : For an irreducible variety $X \subset \mathbb{C}^N$, a general point $w \in \mathbb{C}^N$ and the tangent space $T_w(X)$ of X at w.
Output: Array of numbers as in (11) and a boolean which is *True* if $I(X)$ is generated by binomials, otherwise *False*.

```
Set d := dim T_w(X) and let n be the number of nonzero coordinates in w.
Reorder the coordinates so that the first n are nonzero and the first d ≤ n are
  linearly independent in T_w(X).
Compute the reduced row echelon form for T_w(X) dropping the last N − n rows
  which are identically zero. This yields u_{i,j} for i = d+1, ..., n and j = 1, ..., d as in
  (10).
for i = d+1, ..., n do
  | for j = 1, ..., d do
  |   | Set b_{i,j} := u_{i,j} · w_j / w_i
  | end
end
if every b_{i,j} ∈ Q then
  | return True
else
  | return False
end
```

Algorithm 1: Binomiality Test

The reduced row echelon form of $T_w(X)$ *is*

$$\begin{bmatrix} 1 \\ \dfrac{3w_1^2}{2w_2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1.5040 + 1.2919i \end{bmatrix}.$$

Hence, since $w_1^3 = w_2^2$ *on* X, (11) *with* $(i,j) = (2,1)$ *yields*

$$\left(\frac{3w_1^2}{2w_2}\right) \cdot \frac{w_1}{w_2} = \frac{3w_1^3}{2w_2^2} = \frac{3}{2}.$$

Therefore, $I(X)$ *is generated by binomials with* $b_{2,1} = 3/2$ *as in Ex. 5.*

Now, for Y, *consider the following general point (rounded to four decimal places)*

$$w = (w_1, w_2) \approx (-1.1620 + 0.1969i, -1.6037 + 0.1764i).$$

The reduced row echelon form of $T_w(Y)$ *is*

$$\begin{bmatrix} 1 \\ \dfrac{3w_1^2}{2w_2 + 3w_2^2} \end{bmatrix} \approx \begin{bmatrix} 1 \\ 0.9022 - 0.0361i \end{bmatrix}.$$

Hence, since $w_1^3 = w_2^2 + w_2^3$ *on* Y, (11) *with* $(i,j) = (2,1)$ *yields*

$$\left(\frac{3w_1^2}{2w_2 + 3w_2^2}\right) \cdot \frac{w_1}{w_2} = \frac{3w_1^3}{2w_2^2 + 3w_2^3} = \frac{3w_2^2 + 3w_2^3}{2w_2^2 + 3w_2^3} \approx 0.6564 - 0.0647i.$$

Numerically, one sees that this is clearly not equal to a rational number since it is nonreal. Symbolically, one sees that this expression is not constant and therefore cannot be identically equal to some rational number on Y. *This confirms that* $I(Y)$ *is not generated by binomials.*

When $I(X)$ is indeed generated by binomials, the last step in our method is to compute a witness system g for X consisting of $n-d$ binomials. For $i = 1, \ldots, n-d$, let $c_{d+i} \in \mathbb{Z}$ be the least common multiple of the denominators of the entries in $b_{d+i} \in \mathbb{Q}^d$. If I_{n-d} is the $(n-d) \times (n-d)$ identity matrix and $B = \left[\begin{array}{ccc} b_{d+1} & \cdots & b_n \end{array}\right] \in \mathbb{Q}^{d \times (n-d)}$, the columns of

$$M = \left[\begin{array}{c} -B \\ I_{n-d} \end{array}\right] \cdot \left[\begin{array}{ccc} c_{d+1} & & \\ & \ddots & \\ & & c_n \end{array}\right] \in \mathbb{Z}^{n \times (n-d)}$$

are integer vectors that form a basis of the null space of $\left[\begin{array}{cccc} I_d & b_{d+1} & \cdots & b_n \end{array}\right]$. Each column of M corresponds to a binomial whose two monomials arise from the positive and negative entries, respectively. In particular, for the i^{th} column of M, consider

$$m_{i,1}(x) = \prod_{j=1}^{n} x_j^{\max\{0, M_{i,j}\}} \quad \text{and} \quad m_{i,2}(x) = \prod_{j=1}^{n} x_j^{\max\{0, -M_{i,j}\}}.$$

Hence, given the support of a binomial in $I(X)$, all that remains is to compute the coefficients which is accomplished by evaluating the monomials at w and computing a vector $q_i \in \mathbb{C}^2 \setminus \{0\}$ in the null space of

$$\left[\begin{array}{cc} m_{i,1}(w) & m_{i,2}(w) \end{array}\right].$$

For $i = 1, \ldots, n-d$, this yields a binomial

$$g_i(x) = q_{i,1} m_{i,1}(x) + q_{i,2} m_{i,2}(x)$$

which vanishes on X. If one expects X to be defined over $\mathbb{Q}$, then one can use Section 2.3 to compute $q_i \in \mathbb{Z}^2 \setminus \{0\}$ yielding binomials with integer coefficients vanishing on X.

Example 7. *Continuing with Ex. 6, one has* $c_2 = 2$ *so that*

$$M = \left[\begin{array}{c} -3/2 \\ 1 \end{array}\right] \cdot \left[\begin{array}{c} 2 \end{array}\right] = \left[\begin{array}{c} -3 \\ 2 \end{array}\right].$$

Hence, $m_{1,1}(x,y) = y^2$ *and* $m_{1,2}(x,y) = x^3$. *Since* $m_{1,1}(w) = m_{1,2}(w) \approx -2.3407 - 4.7927i$, *the null vector* $q_1 = \left[\begin{array}{cc} 1 & -1 \end{array}\right]^T$ *yields the binomial* $g_1(x,y) = y^2 - x^3$ *that vanishes on* X.

The relabeling and simplifying assumptions ensure that the system $g(x)$ consisting of $n-d$ binomials vanishes on X such that the Jacobian matrix of g evaluated at w is full rank. Therefore, X is an irreducible component of $\mathcal{V}(g)$, i.e., g is a witness system of X. The following demonstrates this on different orderings of the variables for the twisted cubic.

Example 8. *Consider the twisted cubic curve* $X = \mathcal{V}(x^2 - y, xy - z, xz - y^2) \subset \mathbb{C}^3$. *The traditional ordering of the variables, namely* (x, y, z), *gives the traditional parameterization with exponents* $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ *so that*

$$M = \begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \textit{yielding} \quad g(x,y,z) = \begin{bmatrix} y - x^2 \\ z - x^3 \end{bmatrix}.$$

Ordering the variables as (y, x, z) *gives exponents* $\begin{bmatrix} 1 & 1/2 & 3/2 \end{bmatrix}$ *so that*

$$M = \begin{bmatrix} -1 & -3 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \textit{yielding} \quad g(y,x,z) = \begin{bmatrix} x^2 - y \\ z^2 - y^3 \end{bmatrix}.$$

With this ordering, $\mathcal{V}(g)$ *consists of two cubic curves, one of which is* X.

Ordering the variables as (z, x, y) *gives exponents* $\begin{bmatrix} 1 & 1/3 & 2/3 \end{bmatrix}$ *so that*

$$M = \begin{bmatrix} -1 & -2 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} \quad \textit{yielding} \quad g(z,x,y) = \begin{bmatrix} x^3 - z \\ y^3 - z^2 \end{bmatrix}.$$

With this ordering, $\mathcal{V}(g)$ *consists of three cubic curves, one of which is* X.

4 Computing sparse polynomials

In Section 3, once one found the support monomials of a binomial, the coefficients could be computed essentially by interpolating at one general point. For the arbitrary case when not defined by binomials, the following extends this idea by first locating support monomials and then interpolating using general points. To provide a finite bound on the search space, the following aims to compute polynomials of degree at most d which have at most t terms. The method can be extended to utilize multiplicity information in the form of Macaulay dual bases as described in Section 2.2. For simplicity of presentation, we assume that the algebraic set $X \subset \mathbb{C}^N$ under consideration is irreducible, but the method trivially extends to the reducible case by considering each irreducible component.

4.1 Multiplicity one

Suppose that X is described by a witness set $\{g, \mathcal{L}, W\}$ where X has multiplicity one with respect to g. Then, the key observation is that knowing t general points on X is enough to compute vanishing polynomials with at most t terms. This is well known in the case of $t = 1$, i.e., monomials, by simply looking to see which coordinates of one general point vanish.

In the following, a polynomial $p(x) = p(x_1, \ldots, x_N)$ of degree at most d consisting of at most t monomials is written as $p(x) = \sum_{i=1}^{t} c_i x^{\alpha_i}$ with coefficients $c_i \in \mathbb{C}$ and exponent vectors $\alpha_i \in \mathbb{Z}_{\geq 0}^N$ such that $\sum_{j=1}^{N} \alpha_{i,j} \leq d$. The vector $\nu_d(z)$ is defined in (5).

Theorem 9. *Suppose that $q_1, \ldots, q_t \in X$ and*

$$A = \begin{bmatrix} \nu_d(q_1) \\ \vdots \\ \nu_d(q_t) \end{bmatrix}. \tag{12}$$

Assume that $p(x) = \sum_{i=1}^{t} c_i x^{\alpha_i}$ is a nonzero polynomial with at most t terms and $\deg p \leq d$ which vanishes on X. Then, the determinant of the $t \times t$ submatrix of A whose columns correspond to the monomials $x^{\alpha_1}, \ldots, x^{\alpha_t}$ vanishes.

Proof. Let $A_{\alpha_1,\ldots,\alpha_t}$ be the $t \times t$ submatrix of A whose columns correspond to $x^{\alpha_1}, \ldots, x^{\alpha_t}$ and $c = [c_1, \ldots, c_t]^T$. Since p vanishes on X, it follows that c is a nonzero null vector of $A_{\alpha_1,\ldots,\alpha_t}$. Hence, $A_{\alpha_1,\ldots,\alpha_t}$ is rank deficient yielding $\det A_{\alpha_1,\ldots,\alpha_t} = 0$. □

The following provides a method for finding vanishing polynomials with precisely t terms of degree at most d.

Theorem 10. *Suppose that $q_1, \ldots, q_t \in X$ are general and A as in (12). Let $A_{\alpha_1,\ldots,\alpha_t}$ be the $t \times t$ submatrix of A whose columns correspond to $x^{\alpha_1}, \ldots, x^{\alpha_t}$. If $\operatorname{rank} A_{\alpha_1,\ldots,\alpha_t} = t - 1$ and $c \in (\mathbb{C}^*)^t$ is a null vector of $A_{\alpha_1,\ldots,\alpha_t}$, then $p(x) = \sum_{i=1}^{t} c_i x^{\alpha_i}$ has precisely t terms with $\deg p \leq d$ and vanishes on X.*

Proof. Let $w \in X$ and $B = \begin{bmatrix} A \\ \nu_d(w) \end{bmatrix}$. Assume that $B_{\alpha_1,\ldots,\alpha_t}$ is the $(t+1) \times t$ submatrix of B whose columns correspond to $x^{\alpha_1}, \ldots, x^{\alpha_t}$. Since $\operatorname{rank} A_{\alpha_1,\ldots,\alpha_t} = t - 1$ and $q_1, \ldots, q_t$ are general, it immediately follows that $\operatorname{rank} B_{\alpha_1,\ldots,\alpha_t} = t - 1$ so that $\operatorname{null} A_{\alpha_1,\ldots,\alpha_t} = \operatorname{null} B_{\alpha_1,\ldots,\alpha_t}$. Hence, $B_{\alpha_1,\ldots,\alpha_t} \cdot c = 0$ yields $p(w) = 0$ showing that p vanishes on X. □

Example 11. *As in Ex. 8, let $X \subset \mathbb{C}^3$ be the twisted cubic curve with witness system*

$$g(x, y, z) = \begin{bmatrix} y - x^2 \\ xy - z \\ xz - y^2 \end{bmatrix}.$$

Clearly, X has multiplicity one with respect to g. Consider searching for all polynomials of degree at most $d = 2$ with at most $t = 2$ terms that vanish on X. One could utilize sampling (see Section 2.1) to compute 2 general points on X. For illustrative purposes, consider

$$q_1 = (\sqrt[5]{2}, \sqrt[5]{4}, \sqrt[5]{8}) \quad \text{and} \quad q_2 = (-\sqrt[3]{3}, \sqrt[3]{9}, -\sqrt[3]{27}).$$

Input : For an irreducible variety $X \subset \mathbb{C}^N$, t general points $q_i \in X$ for $i = 1, \dots t$ and d the upper bound on the degree. Additionally, if desired, multiplicity information associated with each point.
Output: List of polynomials of degree at most d vanishing on X having exactly t terms.

Initialize list of polynomials $P := \{\}$.
Construct matrix A from points q_i without multiplicity information using (12) or with multiplicity information using (14).
for *every collection of* t *columns of* A *corresponding with* $\alpha_1, \dots, \alpha_t$ **do**
    if $\det(A_{\alpha_1,\dots,\alpha_t}) = 0$ **then**
        if $\text{rank}(A_{\alpha_1,\dots,\alpha_t}) = t - 1$ *and there is a null vector* $c \in (\mathbb{C}^*)^t$ *of* $A_{\alpha_1,\dots,\alpha_t}$ **then**
            Add $\sum_{i=1}^t c_i x^{\alpha_i}$ to P.
        end
    end
end
return P

Algorithm 2: Compute sparse polynomials

Since each coordinate is nonzero, we know that there are no monomials that vanish on X*. Hence, to search for all binomials of degree at most* 2*, we consider the* 2×10 *matrix*

$$A = \left|\begin{array}{cccccccccc} 1 & x & y & z & x^2 & xy & xz & y^2 & yz & z^2 \\ \hline 1 & \sqrt[5]{2} & \sqrt[5]{4} & \sqrt[5]{8} & \sqrt[5]{4} & \sqrt[5]{8} & \sqrt[5]{16} & \sqrt[5]{16} & \sqrt[5]{32} & \sqrt[5]{64} \\ 1 & -\sqrt[3]{3} & \sqrt[3]{9} & -\sqrt[3]{27} & \sqrt[3]{9} & -\sqrt[3]{27} & \sqrt[3]{81} & \sqrt[3]{81} & -\sqrt[3]{243} & \sqrt[3]{729} \end{array}\right|$$

Searching over the 2×2 *minors yields* 3 *that vanish, corresponding to columns* $\{y, x^2\}$, $\{z, xy\}$, *and* $\{xz, y^2\}$*. Each corresponding* 2×2 *submatrix has rank* 1 *with null vector* $c = [1, -1]^T$*. This shows that the* 3 *polynomials in* g *are the* 3 *binomials of degree at most* 2 *that vanish on* X*.*

As alluded to in the proof of Theorem 10, adding rows to A arising from other points on X does not change the rank of any submatrix obtained by taking all rows and at most t columns since the first t rows of A arise from t general points of X. However, adding rows to A does increase the rank of the overall matrix until it reaches the value of the Hilbert function of X in degree d, e.g., see [8, 14]. After stabilization of the rank, the null space of the corresponding matrix is indeed the vector space of all polynomials of degree at most d that vanish on X.

One can utilize a modification of the (real) ℓ_1-relaxation in (7) to compute sparse null vectors by reducing from complex linear algebra to real linear algebra by taking real and imaginary parts as needed. If the rank of the matrix is less than the corresponding value of the Hilbert function and this process computes a corresponding polynomial with more terms than general points utilized, there is no longer a guarantee that this polynomial vanishes identically on X. Hence, one needs to test this resulting polynomial at one additional general point to determine if it does indeed vanish identically on X. If, say, a binomial is found, one can then search to find other sparse polynomials by repeating with the matrix obtained by removing one of the columns corresponding to a monomial in the binomial.

Example 12. *Reconsider the setup from Ex. 11 and let A_k be the $k \times 10$ matrix with rows $\nu_2(x_i)$ for $i = 1, \ldots, k$ where $x_i \in X$ are general points. Since we are searching for binomials and the Hilbert function of X in degree 2 is 7, i.e., a 3 dimensional linear space of quadratics vanish on X, we consider $k = 2, \ldots, 7$. As the solution obtained by (7) can be sensitive to the choice of the random patch, we performed 100 random trials for each value of k. Table 1 summarizes the results of this experiment using* `linprog` *in* `Matlab` *with "success" indicating that one of the three binomials was found. This table shows that the success rate increased as k increased, i.e., as more points were utilized.*

Table 1: Frequency of ℓ_1-relaxation successfully computing a binomial using k points out of 100 random trials for the twisted cubic.

k	2	3	4	5	6	7
successes	3	4	7	31	44	57

4.2 Multiplicity greater than one

When X has multiplicity greater than one with respect to a witness system g, one can utilize deflation (see Section 2.2) to produce a witness system which can be used for sampling (see Section 2.1). The following describes how to modify the setup from Section 4.1 to recover sparse polynomials vanishing on X with the multiplicity structure imposed by g via [14]. The key piece is to extend the definition of the Veronese embedding ν_d from (5) to one that depends upon both a point and a linear functional.

Let $z \in \mathbb{C}^N$ and $\partial \in D_z$ be a corresponding linear functional (as defined in (3)). Then, define the degree d Veronese embedding of z with respect to ∂ as (by abuse of notation)

$$\nu_d(z, \partial) = \partial(\nu_d(x)). \tag{13}$$

Thus, one applies ∂ (which includes evaluating at z) to the vector of monomials obtained from the Veronese embedding of degree d of the vector of variables $x = (x_1, \ldots, x_N)$. This is a generalization of the Veronese embedding since

$$\nu_d(z, \partial_{(0,\ldots,0)}[z]) = \nu_d(z).$$

With this setup, all items from Section 4.1 naturally extend to the case when the multiplicity is greater than one. In particular, the following are natural extensions of Theorems 9 and 10 whose proofs follow in the same manner and are thus omitted.

Theorem 13. *Suppose that* $q_1, \ldots, q_t \in X$, $\partial_1, \ldots, \partial_t$ *such that* $\partial_i \in D_{q_i}[f_{q_i}]$ *with polynomial system* $f_{q_i} = \{g, \ell_{q_i}\}$ *where* ℓ_{q_i} *consists of* $\dim X$ *linear polynomials with* $\ell_{q_i}(q_i) = 0$, *and*

$$A = \begin{bmatrix} \nu_d(q_1, \partial_1) \\ \vdots \\ \nu_d(q_t, \partial_t) \end{bmatrix}. \tag{14}$$

Assume that $p(x) = \sum_{i=1}^{t} c_i x^{\alpha_i}$ *is a nonzero polynomial with at most* t *terms and* $\deg p \leq d$ *which vanishes on* X *and on the multiplicity structure imposed by* g. *Then, the determinant of the* $t \times t$ *submatrix of* A *whose columns correspond to the monomials* $x^{\alpha_1}, \ldots, x^{\alpha_t}$ *vanishes.*

Theorem 14. *Suppose that* $q_1, \ldots, q_t \in X$ *are general,* $\partial_1, \ldots, \partial_t$ *such that* $\partial_i \in D_{q_i}[f_{q_i}]$ *is general where* $f_{q_i} = \{g, \ell_{q_i}\}$ *and* ℓ_{q_i} *consists of* $\dim X$ *general linear polynomials with* $\ell_{q_i}(q_i) = 0$, *and* A *as in* (14). *Let* $A_{\alpha_1,\ldots,\alpha_t}$ *be the* $t \times t$ *submatrix of* A *whose columns correspond to* $x^{\alpha_1}, \ldots, x^{\alpha_t}$. *If* $\operatorname{rank} A_{\alpha_1,\ldots,\alpha_t} = t-1$ *and* $c \in (\mathbb{C}^*)^t$ *is a null vector of* $A_{\alpha_1,\ldots,\alpha_t}$, *then* $p(x) = \sum_{i=1}^{t} c_i x^{\alpha_i}$ *has precisely* t *terms with* $\deg p \leq d$ *and vanishes on* X *as well as the multiplicity structure imposed by* g.

Example 15. *To illustrate, consider* $X = \mathcal{V}(g) \subset \mathbb{C}^2$ *where* X *has multiplicity* 2 *with respect to* $g(x,y) = (x-y)^2$. *Every point* $q \in X$ *is of the form* $q = (a,a)$ *where* $a \in \mathbb{C}$. *The corresponding linear function* ℓ_q *has the form* $\ell_q = b(x-a) + c(y-a)$. *Thus, for the polynomial system* $f_q = \{g, \ell_q\}$,

$$D_q[f_q] = \operatorname{span}\{\partial_{(0,0)}[q],\ c\partial_{(1,0)}[q] - b\partial_{(0,1)}[q]\}.$$

Hence, a general point $q \in X$ *with general element* $\partial \in D_q[f_q]$ *can be written as*

$$q = (a,a) \text{ and } \partial = d\partial_{(0,0)}[q] + e(c\partial_{(1,0)}[q] - b\partial_{(0,1)}[q])$$

where $a, b, c, d, e \in \mathbb{C}$ *are general with*

$$\nu_2(q, \partial) = (d, ad + ce, ad - be, a^2d + 2ace, a^2d - abe + ace, a^2d - 2abe).$$

It is easy to verify that the only linear relation amongst the entries of $\nu_2(q,\partial)$ *that vanishes for all* $a, b, c, d, e \in \mathbb{C}$ *corresponds with* $x^2 - 2xy + y^2 = (x-y)^2$ *as expected.*

5 Examples

In the following examples, `Bertini` [2] is used to generate sample points and `Matlab` is used to determine binomiality or search for sparse polynomials either exhaustively searching minors or utilizing the ℓ_1-relaxation in (7). Section 5.1-5.3 concern finding sparse polynomials while Section 5.4 considers binomiality testing for two systems arising from chemical reaction networks.

5.1 Random parameterization of twisted cubic

The classical twisted cubic parameterized by $t \mapsto (t, t^2, t^3)$ was considered in Examples 8, 11, and 12. As a demonstration that our numerical methods work with varieties defined by polynomials with arbitrary complex coefficients, consider the twisted cubic parameterized by $t \mapsto (at, bt^2, ct^3)$ where $i = \sqrt{-1}$ and

$$a = \sqrt{2} - i\sqrt{3}, \quad b = 1 + i, \quad \text{and} \quad c = 0.1653 - 0.9302i.$$

Hence, we consider the irreducible algebraic set $X = \mathcal{V}(g) \subset \mathbb{C}^3$ where

$$g(x, y, z) = \begin{bmatrix} bx^2 - a^2y \\ cxy - abz \\ b^2xz - acy^2 \end{bmatrix}.$$

With the goal of finding all polynomials of degree at most $d = 2$ with at most $t = 2$ terms, applying Algorithm 2 which searches over all 2×2 submatrices yields the three binomials in g.

Similar to Ex. 12, we then utilized `linprog` in `Matlab` to solve (7) using 100 random trials for each $k = 2, \ldots, 7$ points. Table 2 summarizes the results of this experiment with "success" indicating that one of the three binomials was found. As with Table 1, the success rate increased as k increased.

Table 2: Frequency of ℓ_1-relaxation successfully computing a binomial using k points out of 100 random trials for the random twisted cubic

k	2	3	4	5	6	7
successes	0	9	30	52	59	71

5.2 Multiple component

The advantage of our approach based on sample points is that it is easy to switch between using the multiplicity structure and ignoring the multiplicity structure thereby computing sparse polynomials in the corresponding radical ideal. To demonstrate, consider the polynomial system

$$g(x, y, z) = \begin{bmatrix} 2xy - 2y^2 + z^2 + 2x - y - 1 \\ x^2 - y^2 + z^2 + 2x - y - 1 \end{bmatrix}.$$

The algebraic set $X = \mathcal{V}(g)$ is a curve of degree 2 that has multiplicity 2 with respect to g. In order to sample points on X, we utilize g' constructed using isosingular deflation (see Section 2.2) where

$$g' = \begin{bmatrix} g \\ 2(x - y)(2x - 2y + 1) \\ 4z(x - y) \end{bmatrix}.$$

Thus, consider searching for all polynomials of degree at most $d = 2$ with at most $t = 3$ terms for X with respect to g (multiplicity 2) and g' (multiplicity 1).

Since no coordinates are zero at a general point on X, no monomial vanishes. Utilizing the multiplicity structure with respect to g, searching over the 2×2 and 3×3 submatrices as described by Theorem 14 found a single trinomial, namely $x^2 - 2xy + y^2$, which is easily observed to be the difference of the polynomials in g.

In the multiplicity one case, we first considered linear polynomials. This produced the binomial $x-y$. Since x is dependent on y, we then searched for trinomials of degree at most 2 in variables y and z. This search produced the trinomial $z^2 + y - 1$. Table 3 summarizes the results.

Table 3: Summary of sparse polynomials computed.

t	Without multiplicity	With multiplicity
2	$x - y$	–
3	$z^2 + y - 1$	$x^2 - 2xy + y^2$

We now turn to utilizing the ℓ_1-relaxation (7) to attempt to find sparse polynomials using the degree 2 Veronese embedding with k points for $k = 2, \ldots, 9$. A summary of the results of this experiment using 100 random trials for each k with `linprog` in `Matlab` utilizing multiplicity structure with respect to g is provided in Table 4. This table summarizes the number of terms of the corresponding polynomial when the computation successfully computed an optimizer and the frequency of the polynomial vanishing on X. For example, using $k = 3$ points, out of the 100 trials, a trinomial that vanished on X, namely, $x^2 - 2xy + y^2$, was computed 11 times. A 7-term polynomial was computed 4 times, but since the number of terms, namely 7, is more than the number of points, namely 3, one needs to check to see if this polynomial vanishes on X by testing at an additional general point. In this case, 2 out of the 4 vanished on X. The vanishing 7-term polynomial computed is

$$x^2 + 2xy - 3y^2 + 2z^2 + 4x - 2y - 2$$

which is the sum of the 6-term polynomials in g.

Finally, we repeat the same experiment without multiplicity with the results summarized in Table 5. Some examples of the binomials found in this experiment are

$$x^2 - y^2, \quad x^2 - yz, \quad xz - yz$$

with the trinomial $x^2 - 2xy + y^2$. Some quadrinomials and 5-term polynomials had the form

$$\alpha(x - y) + xy - y^2 \text{ and } \beta(z^2 + y - 1) - xy + y^2$$

for some $\alpha, \beta \in \mathbb{C}$.

5.3 Cubic-centered 12-bar

Consider the cubic-centered 12-bar spherical linkage shown in Figure 2 and first presented in [25]. This linkage consists of rotational joints at the center of a cube and its vertices. Links

Table 4: Summary of 100 random trials using $k = 2, \ldots, 9$ points with multiplicity structure. Reported as frequency of polynomials that vanish on X out of the number of times a polynomial was found for each k and number of terms.

	# of terms			total
k	3	6	7	successes
2	1/1	0	0	1/100
3	11/11	0	2/4	13/100
4	25/25	0	9/13	34/100
5	35/35	14/14	48/50	97/100
6	43/43	8/8	47/47	98/100
7	42/42	4/4	53/53	99/100
8	28/28	15/15	57/57	100/100
9	46/46	3/3	46/46	95/100

Table 5: Summary of 100 random trials using $k = 2, \ldots, 9$ points without multiplicity structure. Reported as frequency of polynomials that vanish on X out of the number of times a polynomial was found for each k and number of terms.

	# of terms				total
k	2	3	4	5	successes
2	3/3	7/8	1/11	0	11/100
3	10/10	6/6	10/10	2/9	28/100
4	14/14	17/17	24/24	4/4	58/100
5	29/29	19/19	34/34	5/5	87/100
6	21/21	25/25	37/37	16/16	99/100
7	21/21	24/24	34/34	19/19	98/100
8	29/29	20/20	35/35	14/14	98/100
9	29/29	24/24	33/33	12/12	98/100

connect along the edges of the cube and from each vertex to the center. To remove trivial motion in space, the center point, p_0, is fixed at the origin and two adjacent vertices are fixed, say $p_7 = (-1, 1, -1)$ and $p_8 = (-1, -1, -1)$. This results in 18 variables arising from the three coordinates in each $p_1, \ldots, p_6$, say $p_i = (x_i, y_i, z_i)$, with 17 polynomial constraints, say $g = 0$ with:

$$\begin{array}{ll} \|p_i - p_j\|^2 - 4 = 0, & (i,j) \in \left\{ \begin{array}{c} (1,2),(1,3),(1,5),(2,4),(2,6), \\ (3,4),(3,7),(4,8),(5,6),(5,7),(6,8) \end{array} \right\}, \\ \|p_k\|^2 - 3 = 0, & k = 1, \ldots, 6. \end{array}$$

There are two types of curves in $\mathcal{V}(g)$. First, there are two degree 6 curves which are complex conjugates of each other with the linkage shown in Figure 2 being a point of intersection of these curves. In the interest of locating sparse polynomials, we consider one,

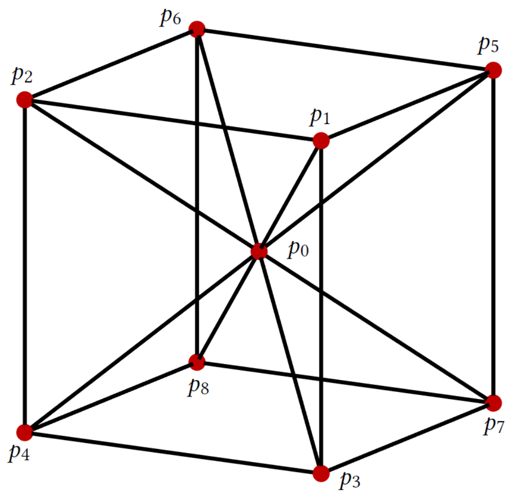

Figure 2: Cubic-centered 12-bar linkage

say X, of the second type being a degree 4 complex curve that contains a real curve. A real point on X is shown in Figure 3. We note that the multiplicity of X with respect to g is one.

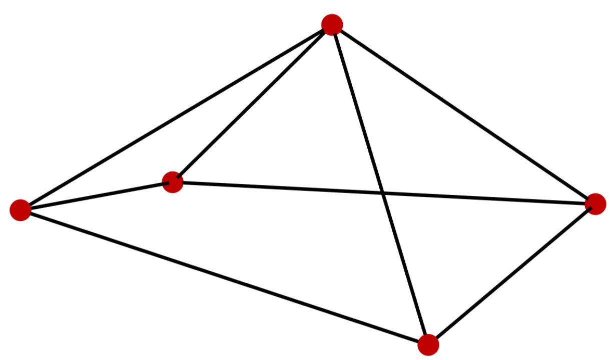

Figure 3: Cubic-centered 12-bar linkage on degree 4 curve

Since each coordinate of a general point on X is nonzero, no monomial vanishes on X. Hence, searching for linear binomials yields the following 12:

$$\begin{array}{lll} x_1 - x_4, & y_1 - y_4, & z_1 - 1, \\ x_2 + 1, & y_2 + 1, & z_2 + 1, \\ x_3 - x_5, & y_3 - y_5, & z_3 - z_5, \\ x_6 + 1 & y_6 - 1, & z_6 + 1. \end{array}$$

Due to these 12 linear binomials, we only need to consider higher degree polynomials in, say, $1, x_4, x_5, y_4, y_5, z_4, z_5$. The search for degree 2 binomials using Theorem 10 located one:

$$z_4 z_5 - x_4 x_5.$$

By removing the column corresponding to z_4z_5 and searching for trinomials using Theorem 10, we found the following 9:

$$\begin{array}{lll} 2x_4x_5 + y_4y_5 - 1, & 2x_5 + z_4 + x_5y_4, & x_5 + 2z_4 - y_5z_4, \\ 3x_5 + 2x_5y_4 + y_5z_4, & 2x_4 + z_5 - x_4y_5, & 3z_4 - x_5y_4 - 2y_5z_4, \\ 3x_4 - 2x_4y_5 - y_4z_5, & x_4 + 2z_5 + y_4z_5, & 3z_5 + x_4y_5 + 2y_4z_5. \end{array}$$

The search for quadrinomials found 61, some of which are:

$$\begin{array}{ll} x_5 - y_5 + z_5 - 1, & x_4z_5 + x_5 + z_4 + 1, \\ x_5^2 + y_5z_5 + x_5 - 1, & x_4z_5 - x_5y_4 - x_5 - 1, \\ x_4z_5 - x_4y_4 + y_5z_4 + z_4^2, & x_5z_4 + x_4 - x_5 + y_5. \end{array}$$

5.4 Two chemical reaction networks

Finally, consider testing binomiality of the steady-state points for two chemical reaction networks. The first system consists of 9 polynomials in 9 variables from [22, Ex. 3.15] considered in [6, Ex. 4.3] and the other systems consists of 29 polynomials in 29 variables from [6, Ex. 4.4]. Each system depends upon parameters and we test generic binomiality by randomly fixing the parameters. In both cases, we only focus on testing the binomiality of the unique irreducible component X not contained in any coordinate hyperplane. We first construct a real point on X by selecting a random real point and applying the dynamical system associated with the chemical reaction network to yield a real steady-state. Then, starting at this point, we track along the component over the complex numbers to yield a general point with coordinates in $\mathbb{C}^*$ on X.

First, randomly selecting parameters

$$\begin{array}{c} k_{12} = 41, \quad k_{21} = 8, \quad k_{1112} = 22, \quad k_{1211} = 46, \quad k_{1213} = 40, \quad k_{23} = 48, \quad k_{32} = 33, \\ k_{67} = 2, \quad k_{34} = 43, \quad k_{89} = 47, \quad k_{910} = 34, \quad k_{98} = 38, \quad k_{56} = 38, \quad k_{65} = 20, \end{array}$$

we consider the system

$$\begin{array}{c} f_1 = -k_{12}x_1 + k_{21}x_2 - k_{1112}x_1x_7 + (k_{1211} + k_{1213})x_9, \\ f_2 = k_{12}x_1 - k_{21}x_2 - k_{23}x_2 + k_{32}x_3 + k_{67}x_6, \\ f_3 = k_{23}x_2 - k_{32}x_3 - k_{34}x_3 - k_{89}x_3x_7 + k_{910}x_8 + k_{98}x_8, \\ f_4 = k_{34}x_3 - k_{56}x_4x_5 + k_{65}x_6, \quad f_5 = -k_{56}x_4x_5 + k_{65}x_6 + k_{910}x_8 + k_{1213}x_9, \\ f_6 = k_{56}x_4x_5 - (k_{65} + k_{67})x_6, \quad f_7 = k_{67}x_6 - k_{1112}x_1x_7 - k_{89}x_3x_7 + k_{98}x_8 + k_{1211}x_9, \\ f_8 = k_{89}x_3x_7 - (k_{910} + k_{98})x_8, \quad f_9 = k_{1112}x_1x_7 - (k_{1211} + k_{1213})x_9. \end{array}$$

With $i = \sqrt{-1}$, consider the general point (rounded to four decimal places)

$$\begin{aligned} w &= (w_1, \ldots, w_9) \\ &\approx (-0.0141 + 0.0378i, -0.0723 + 0.1938i, -0.0457 + 0.1224i, 1.0396 - 0.2729i, \\ &-0.8712+1.2364i, -0.9816+2.6309i, 1.6959-0.0000i, -0.0505+0.1355i, -0.0061+0.0164i). \end{aligned}$$

In order to ensure the first $d = 2$ coordinates are linearly independent, we relabel the coordinates as

$$w' = (w_1, w_4, w_2, w_3, w_5, w_6, w_7, w_8, w_9)$$

yielding the reduced row echelon form of the tangent space $T_{w'}(X)$ of X at w' as

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 5.1250 & 0 \\ 3.2368 & 0 \\ 36.2556 + 9.5168i & 1.0760 - 0.9068i \\ 69.5921 & 0 \\ 0 & 0 \\ 3.5833 & 0 \\ 0.4338 & 0 \end{bmatrix}. \tag{15}$$

In this case, since each $b_{i,j}$ computed in (11) is either 1, -1, or 0, namely

$$\begin{bmatrix} I_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

the corresponding irreducible algebraic set is defined by binomials. In fact, with

$$M = \begin{bmatrix} -1 & -1 & -1 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

seven binomials (in terms of the original ordering of variables) forming a witness system are

$$8x_2 - 41x_1, \quad 38x_3 - 123x_1, \quad 193x_4x_5 - 7776x_1, \quad 76x_6 - 5289x_1,$$
$$799x_7 - 1355, \quad 1279x_8 - 4583x_1, \quad 2093x_9 - 908x_1.$$

This agrees with the results described in [6, Ex. 4.3].

Next, randomly selecting the parameters

$$k_1 = 82, \quad k_2 = 91, \quad k_3 = 13, \quad k_4 = 92, \quad k_5 = 64, \quad k_6 = 10, \quad k_7 = 28, \quad k_8 = 55,$$
$$k_9 = 96, \quad k_{10} = 97, \quad k_{11} = 16, \quad k_{12} = 98, \quad k_{13} = 96, \quad k_{14} = 49, \quad k_{15} = 81, \quad k_{16} = 15,$$
$$k_{17} = 43, \quad k_{18} = 92, \quad k_{19} = 80, \quad k_{20} = 96, \quad k_{21} = 66, \quad k_{22} = 4, \quad k_{23} = 85, \quad k_{24} = 94,$$
$$k_{25} = 68, \quad k_{26} = 76, \quad k_{27} = 75, \quad k_{28} = 40, \quad k_{29} = 66, \quad k_{30} = 18, \quad k_{31} = 71, \quad k_{32} = 4,$$
$$k_{33} = 28, \quad k_{34} = 5, \quad k_{35} = 10, \quad k_{36} = 83, \quad k_{37} = 70, \quad k_{38} = 32, \quad k_{39} = 96,$$
$$k_{40} = 4, \quad k_{41} = 44, \quad k_{42} = 39, \quad k_{43} = 77, \quad k_{44} = 80, \quad k_{45} = 19, \quad k_{46} = 49,$$

we consider the system

$$f_1 = -k_1x_1x_2 + k_2x_3 + k_6x_6, \quad f_2 = -k_1x_1x_2 + k_2x_3 + k_3x_3, \quad f_3 = k_1x_1x_2 - k_2x_3 - k_3x_3,$$
$$f_4 = k_{11}x_{10} + k_{12}x_{10} + k_{38}x_{25} + k_{42}x_{27} + k_3x_3 - k_{37}x_{18}x_4 - k_4x_4x_5 + k_5x_6 - k_7x_4x_7 + k_8x_8 + k_9x_8 - k_{10}x_4x_9,$$
$$f_5 = k_{14}x_{12}+k_{15}x_{12} + k_{17}x_{13} + k_{18}x_{13} + k_{35}x_{24} + k_{36}x_{24} + k_{41}x_{27} + k_{42}x_{27} - k_{13}x_{11}x_5 - k_{34}x_{16}x_5 - k_{40}x_{26}x_5 - k_4x_4x_5 + k_5x_6 + k_6x_6 - k_{16}x_5x_9,$$
$$f_6 = k_4x_4x_5 - k_5x_6 - k_6x_6, \quad f_7 = k_{18}x_{13} - k_7x_4x_7 + k_8x_8, \quad f_8 = k_7x_4x_7 - k_8x_8 - k_9x_8,$$
$$f_9 = k_{11}x_{10} + k_{15}x_{12} + k_{17}x_{13} + k_9x_8 - k_{10}x_4x_9 - k_{16}x_5x_9, \quad f_{10} = -k_{11}x_{10} - k_{12}x_{10} + k_{10}x_4x_9,$$
$$f_{11} = k_{12}x_{10} + k_{14}x_{12} - k_{19}x_{11}x_{14} + k_{20}x_{15} + k_{21}x_{15} - k_{22}x_{11}x_{16} + k_{23}x_{17} + k_{24}x_{17} - k_{13}x_{11}x_5,$$
$$f_{12} = -k_{14}x_{12} - k_{15}x_{12} + k_{13}x_{11}x_5, \quad f_{13} = -k_{17}x_{13} - k_{18}x_{13} + k_{16}x_5x_9,$$
$$f_{14} = -k_{19}x_{11}x_{14} + k_{20}x_{15} + k_{30}x_{21} + k_{36}x_{24}, \quad f_{15} = k_{19}x_{11}x_{14} - k_{20}x_{15} - k_{21}x_{15},$$
$$f_{16} = k_{21}x_{15} - k_{22}x_{11}x_{16} + k_{23}x_{17} - k_{28}x_{16}x_{19} + k_{27}x_{20} + k_{29}x_{21} + k_{33}x_{23} + k_{35}x_{24} - k_{34}x_{16}x_5,$$
$$f_{17} = k_{22}x_{11}x_{16} - k_{23}x_{17} - k_{24}x_{17},$$
$$f_{18} = k_{24}x_{17} - k_{25}x_{18}x_{19} + k_{26}x_{20} - k_{31}x_{18}x_{22} + k_{32}x_{23} + k_{38}x_{25} + k_{39}x_{25} - k_{43}x_{18}x_{28} + k_{44}x_{29} + k_{45}x_{29} - k_{37}x_{18}x_4,$$
$$f_{19} = -k_{46}x_{19} - k_{28}x_{16}x_{19} - k_{25}x_{18}x_{19} + k_{26}x_{20} + k_{27}x_{20} + k_{29}x_{21} + k_{30}x_{21} + k_{45}x_{29},$$
$$f_{20} = k_{25}x_{18}x_{19} - k_{26}x_{20} - k_{27}x_{20}, \quad f_{21} = k_{28}x_{16}x_{19} - k_{29}x_{21} - k_{30}x_{21},$$
$$f_{22} = -k_{31}x_{18}x_{22} + k_{32}x_{23} + k_{33}x_{23}, \quad f_{23} = k_{31}x_{18}x_{22} - k_{32}x_{23} - k_{33}x_{23},$$
$$f_{24} = -k_{35}x_{24} - k_{36}x_{24} + k_{34}x_{16}x_5, \quad f_{25} = -k_{38}x_{25} - k_{39}x_{25} + k_{37}x_{18}x_4,$$
$$f_{26} = k_{39}x_{25} + k_{41}x_{27} - k_{40}x_{26}x_5, \quad f_{27} = -k_{41}x_{27} - k_{42}x_{27} + k_{40}x_{26}x_5,$$
$$f_{28} = k_{46}x_{19} - k_{43}x_{18}x_{28} + k_{44}x_{29}, \quad f_{29} = k_{43}x_{18}x_{28} - k_{44}x_{29} - k_{45}x_{29}.$$

With $i = \sqrt{-1}$, consider the general point (rounded to four decimal places)

$$\begin{aligned} w &= (w_1, \ldots, w_{29}) \\ &\approx (1.5158 - 0.5153i, 1.3106 - 0.9109i, 1.1962 - 1.6212i, 0.5785 - 0.3621i, \\ &2.8712 - 1.1329i, 1.5551 - 2.1076i, 8.3981 + 3.4910i, 1.1354 - 0.1894i, \\ &3.4249 + 0.7317i, 1.9114 - 0.6951i, 1.0792 + 0.0292i, 2.3126 - 0.8410i, \\ &1.1847 - 0.1977i, -1.8542 + 2.7217i, -1.0274 + 1.4238i, -6.7909 - 9.0614i, \\ &- 0.1579 - 0.2229i, 0.4784 - 0.3800i, -1.8506 - 0.5498i, -0.4928 + 0.1982i, \\ &3.6117 + 9.7631i, 1.0433 - 0.3766i, 0.7899 - 1.2794i, -1.6002 - 0.9851i, \\ &0.0761 - 0.2150i, 2.4771 - 2.8467i, 0.1873 - 0.5291i, -6.0087 - 8.5835i, -4.7725 - 1.4180i). \end{aligned}$$

In order to ensure the first $d = 7$ coordinates are linearly independent, we relabel the coordinates as

$$w' = (w_1, w_2, w_4, w_7, w_{14}, w_{16}, w_{18}, w_3, w_5, w_6, w_8, w_9, w_{10}, w_{11}, w_{12}, w_{13}, w_{15}, w_{17}, w_{19}, w_{20}, w_{21}, w_{22}, w_{23}, w_{24}, w_{25}, w_{26}, w_{27}, w_{28}, w_{29}).$$

yielding the reduced row echelon form of the tangent space $T_{w'}(X)$ of X at w' as

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1.0333 - 0.7182i & 1.1952 - 0.4063i & 0 & 0 & 0 & 0 & 0 \\
1.9257 - 0.0927i & 1.8823 + 0.4438i & -4.4464 - 0.8251i & 0 & 0 & 0 & 0 \\
1.3433 - 0.9337i & 1.5537 - 0.5282i & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.5573 + 0.6473i & 0.1073 - 0.0672i & 0 & 0 & 0 \\
-1.8782 - 1.1213i & -1.5004 - 1.6012i & 7.3689 + 7.1423i & 0.3786 - 0.0703i & 0 & 0 & 0 \\
-1.2701 + 0.0268i & -1.2320 - 0.3259i & 8.7424 + 1.8679i & 0.1647 - 0.1512i & 0 & 0 & 0 \\
-1.2646 - 0.4684i & -1.0896 - 0.8018i & 5.2702 + 3.5006i & 0.1108 - 0.0426i & 0 & 0 & 0 \\
-1.5367 + 0.0324i & -1.4905 - 0.3943i & 10.5773 + 2.2599i & 0.1993 - 0.1830i & 0 & 0 & 0 \\
0 & 0 & 1.6250 + 0.6755i & 0.1119 - 0.0701i & 0 & 0 & 0 \\
1.7876 - 1.2708i & 2.0754 - 0.7303i & -9.5308 + 3.8781i & -0.0442 + 0.1879i & 0.5329 + 0.0144i & 0 & 0 \\
0.0971 + 0.3272i & 0.0030 + 0.3423i & -0.0909 - 1.5984i & -0.0254 - 0.0160i & 0 & 0.0241 + 0.0007i & 0 \\
-1.0400 + 1.5392i & -1.4289 + 1.1960i & 4.0913 - 6.3381i & -0.0842 - 0.1007i & -0.2252 + 0.2841i & -0.0994 - 0.0352i & 0 \\
0.0393 + 0.5096i & -0.1032 + 0.5022i & -0.2031 - 2.0655i & -0.0354 - 0.0073i & 0.0001 + 0.0997i & -0.0274 + 0.0094i & -0.8334 - 0.2476i \\
10.0048 - 0.4899i & 9.7816 + 2.2980i & -40.5786 + 2.8422i & -0.1621 + 0.6890i & 1.9541 + 0.0528i & -0.7118 + 0.2809i & 0 \\
0.2497 - 0.0528i & 0.2554 + 0.0182i & 0.0615 + 0.2059i & 0.0211 - 0.0154i & 0.1224 - 0.1545i & 0.0998 + 0.0576i & -0.7355 + 0.8279i \\
0.2205 - 0.2666i & 0.2864 - 0.1960i & 0.2388 + 0.1667i & 0.0094 - 0.0341i & -0.0003 - 0.2672i & 0.1544 - 0.0230i & 2.2322 + 0.6632i \\
-0.7482 - 0.9043i & -0.4710 - 1.0791i & 1.2215 + 2.4674i & 0 & 0 & 0.1544 - 0.0609i & 0 \\
0 & 0 & 0.2616 - 0.2078i & 0 & 0 & 0 & 0.3164 - 0.1980i \\
-2.0372 + 1.1854i & -2.2924 + 0.5788i & 10.5785 - 3.2192i & 0 & 0 & 0 & 6.0730 - 1.1269i \\
0 & 0 & 0.6440 - 0.5115i & 0 & 0 & 0 & 0.7788 - 0.4875i \\
-9.6157 + 3.0309i & -10.1101 + 0.2596i & 38.7821 - 13.1258i & -0.0180 - 0.7122i & -1.9160 + 0.4474i & -0.3035 - 0.4851i & -1.0368 + 17.1187i \\
-2.6821 + 3.9696i & -3.6851 + 3.0845i & 10.5512 - 16.3455i & -0.2172 - 0.2597i & -0.5807 + 0.7327i & -0.2563 - 0.0908i & 0
\end{bmatrix}.$$

In this case, the values of $b_{h,j}$ as in (11) for $h = 8, \ldots, 29$ and $j = 1, \ldots, 7$ form the matrix

$$\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
-1 & -1 & 2 & 1 & 0 & 0 & 0 \\
-1 & -1 & 3 & 1 & 0 & 0 & 0 \\
-2 & -2 & 4 & 1 & 0 & 0 & 0 \\
-1 & -1 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
-2 & -2 & 4 & 1 & 1 & 0 & 0 \\
-2 & -2 & 4 & 1 & 0 & 1 & 0 \\
-0.0344 - 1.5402i & -0.0344 - 1.5402i & 0.7240 + 2.5670i & 0.3448 + 0.5134i & 0.3448 + 0.5134i & -0.3448 - 0.5134i & 0 \\
-0.0344 - 1.5402i & -0.0344 - 1.5402i & 0.7240 + 2.5670i & 0.3448 + 0.5134i & 0.3448 + 0.5134i & -0.3448 - 0.5134i & 1 \\
-0.0344 - 1.5402i & -0.0344 - 1.5402i & 0.7240 + 2.5670i & 0.3448 + 0.5134i & 0.3448 + 0.5134i & 0.6552 - 0.5134i & 0 \\
0.3618 - 0.0695i & 0.3618 - 0.0695i & 0.0637 + 0.1158i & 0.2127 + 0.0232i & -0.0256 + 0.5846i & 0.2640 - 1.1460i & -0.2384 + 0.5614i \\
0.3618 - 0.0695i & 0.3618 - 0.0695i & 0.0637 + 0.1158i & 0.2127 + 0.0232i & -0.0256 + 0.5846i & 0.2640 - 1.1460i & 0.7616 + 0.5614i \\
1 & 1 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
-1 & -1 & 2 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
-0.0344 - 1.5402i & -0.0344 - 1.5402i & 0.7240 + 2.5670i & 0.3448 + 0.5134i & 0.3448 + 0.5134i & -0.3448 - 0.5134i & -1 \\
-0.0344 - 1.5402i & -0.0344 - 1.5402i & 0.7240 + 2.5670i & 0.3448 + 0.5134i & 0.3448 + 0.5134i & -0.3448 - 0.5134i & 0
\end{bmatrix}^T.$$

Clearly, not all of the entries are rational which shows that the corresponding irreducible algebraic set is not defined by binomials. Note that the results in [6, Ex. 4.4] are for the entire set of steady-state points while this computation is focused on the irreducible component of steady-state points not contained in any coordinate hyperplane.

6 Conclusion

This paper described an approach based on using sample points to compute sparse polynomials that vanish on an algebraic set. A binomiality test utilized a sample point and corresponding tangent space. For computing sparse polynomials with at most t terms, t sample points were used. One advantage of using sample points is that multiplicity structure can be easily utilized via Macaulay dual spaces or ignored thereby computing sparse polynomials in the corresponding radical ideal. When the ideal is defined over the rational numbers, exactness recovery techniques can be used to find exact representations.

Since the number of entries of the Veronese embedding is combinatorial in the number of variables and degree, the bottleneck in our exhaustive submatrix search is the growth in the number of submatrices that one needs to consider. One approach for overcoming this is to solve an ℓ_1-relaxation (7). Experiments showed that using more points often resulted in successfully computing sparse polynomials using the ℓ_1-relaxation.

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