On computing local monodromy and the numerical local irreducible decomposition

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Abstract

Similarly to the global case, the local structure of a holomorphic subvariety at a given point is described by its local irreducible decomposition. Geometrically, the key requirement for obtaining a local irreducible decomposition is to compute the local monodromy action of a generic linear projection at the given point, which is always well-defined on any small enough neighborhood. We characterize some of the behavior of local monodromy actions of linear projection maps under analytic continuation thereby allowing computations to be performed beyond a local neighborhood. Using this characterization, we then present an algorithm to compute the local monodromy action following the paradigm of numerical algebraic geometry. A germ of an algebraic subvariety at a point is represented by a numerical local irreducible decomposition comprised of a local witness set for each local irreducible component. The results are illustrated using several examples facilitated by an implementation in an open source software package.
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1 Introduction

Theories to understand the geometry and topology of a space at its singular points comprise a major and ongoing area of mathematical study. Some classically studied aspects of singularity theory include local invariants, local monodromy groups, and integration over singular spaces (for a general overview, see the book [1]). Computational methods directed towards identifying singularities, understanding local structure, and stratifying spaces increasingly arise in applications as well, e.g., [8, 13, 16, 21, 23]. We will focus here on a classical setting: given a system of algebraic or analytic equations and a point which satisfies them as input, compute information about the local structure of the solution set at that point in $\mathbb{C}^N$.

Understanding the *global* structure of a solution set of a system of algebraic equations is the foundational problem of algebraic geometry. Globally, the solution set can be decomposed into finitely many irreducible components. The set of regular points of each irreducible component is connected.

For a germ of a complex algebraic subvariety, there is a natural analog yielding the local irreducible decomposition such that the set of regular points of each local irreducible component is connected. In fact, the theory contained here applies to germs of holomorphic subvarieties, though we will consider examples with algebraic inputs. Holomorphic germs also have a unique local irreducible decomposition (e.g., see [14, Thm. II.B.7]). Although well-defined locally, challenges remain regarding the local neighborhood on which to perform the necessary computations. Our main theoretical result is summarized in the following and allows for computations to be performed beyond such a local neighborhood.

Theorem. If $\mathbf{V}$ is a reduced germ of a holomorphic subvariety of $\mathbb{C}^N$ with pure dimension d, there is a Zariski open dense set of linear projections $\mathbb{C}^N \to \mathbb{C}^d$ where, if $\widetilde{\pi}$ is a member, the projection map germ $\widetilde{\boldsymbol{\pi}}|_{\mathbf{V}} : \mathbf{V} \to \mathbf{C}^{\mathbf{d}}$, has a well-defined local monodromy action. Moreover, if $\widetilde{\pi}|_V : V \to \mathbb{C}^d$ is a proper projective holomorphic map with pure $(d-1)$-dimensional critical locus representing $\widetilde{\boldsymbol{\pi}}|_{\mathbf{V}}$, then the local monodromy action is a sub-action of the monodromy action of $\widetilde{\pi}|_V$. The corresponding local monodromy group decomposes into orbits with one orbit for each local irreducible component.

This theoretical result permits computation using the numerical algebraic geometric paradigm (for a general overview, see the books [5, 25]). Since irreducibility need not be stable under perturbations, the input polynomial system is assumed to be known exactly.[1] For example, $xy = 0$ is reducible consisting of two lines while $xy - \epsilon = 0$ is an irreducible curve of degree 2 for all $\epsilon \in \mathbb{C} - \{0\}$. Globally, each irreducible component is represented by a witness set yielding a corresponding numerical irreducible decomposition. One key property is that global irreducibility is maintained under intersection by a general hyperplane for irreducible components of dimension at least two. Therefore, all computations associated with deciding irreducibility of positive-dimensional components can be reduced down to the complex curve case.

An analog following the numerical algebraic geometric paradigm via local witness sets and a numerical local irreducible decomposition was described in [9]. As with the global case, assume that the input polynomial system and input point are known exactly so that, for example, proper decisions regarding which homotopy solution paths limit to the point can be made correctly using adaptive precision path tracking [4] and endgames (e.g., see [5, Chap. 3] and [25, Chap. 10]). Although [9] posited the existence of a numerical local irreducible decomposition, it did not actually consider how to compute such a decomposition. Using the main theoretical result above, we show how to compute a numerical local irreducible decomposition in Algorithm 2. Practically speaking, this sidesteps theoretical difficulties with finding small enough restrictions for monodromy computations at the cost of deciding which paths limit to the point using endgames. A software package implementing the algorithm is available at `https://github.com/P-Edwards/LocalMonodromy.jl`.

One challenge with the local case is that, in contrast with the global case, one can not always simplify reducibility questions down to curves by intersecting with general hyperplanes. To illustrate, consider the cone defined by $x_1^2 + x_2^2 - x_3^2 = 0$, which is an irreducible surface

[1] See [12] for *robust* numerical algebraic geometry allowing perturbations in the input system.

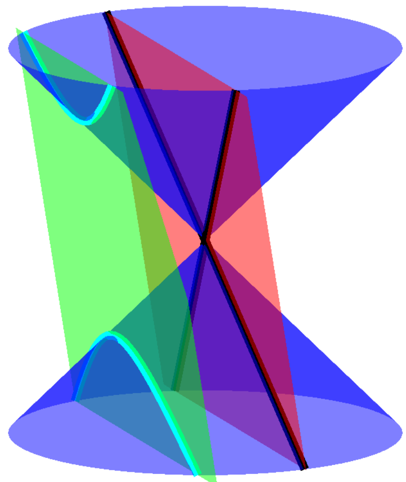


Figure 1: Intersecting the cone (blue) with a general slice (green) yields an irreducible curve (cyan), while intersecting with a general slice through the origin (red) yields two lines (black).

in $\mathbb{C}^3$ whose real part is illustrated in Figure 1. For a general $\alpha \in \mathbb{C}^3$, one can intersect this surface with a general hyperplane defined by $x_1 = \alpha_1 x_2 + \alpha_2 x_3 + \alpha_3$ yielding an irreducible curve. Equivalently, $(\alpha_1 x_2 + \alpha_2 x_3 + \alpha_3)^2 + x_2^2 - x_3^2$ is an irreducible polynomial for general $\alpha \in \mathbb{C}^3$. The origin is a singular point of the cone at which the cone is locally irreducible as will be demonstrated below. However, for a general hyperplane of the form $x_1 = \alpha_1 x_2 + \alpha_2 x_3$ passing through the origin, $(\alpha_1 x_2 + \alpha_2 x_3)^2 + x_2^2 - x_3^2$ is no longer irreducible since every singular quadratic plane curve is simply a pair of intersecting lines as shown in Figure 1.

Reprising the notation and conditions of the above theorem, we first define a local monodromy action for any representative $\widetilde{\pi}|_V$ by restricting loops used for generating the action to a complex line in $\mathbb{C}^d$. This construction is motivated by a theorem of Hamm and Lê Dũng Tráng [15] on the fundamental group of an analytic hypersurface and is a localized version of a construction for global monodromy actions taken in [19] using a theorem of Zariski [26]. Although the global case can always be reduced to curves, the theorem in [15] permits a reduction down to surfaces in the local case.

Restricting $\widetilde{\pi}|_V$ to small enough open balls in the domain and codomain around a point of interest, one has from the local parameterization theorem that orbits of the monodromy action correspond to local irreducible components (Lemma 2.1). The theorem of [15] permits one to restrict loops to a complex line in the codomain to recover this information, though possibly requiring even smaller balls. Unfortunately, it is unclear how or whether appropriately small balls may be computed in practice. Our main theoretical result avoids this issue completely by instead characterizing how the local monodromy action obtained by restricting $\widetilde{\pi}|_V$ includes in a simple way into the monodromy action of the unrestricted cover $\widetilde{\pi}|_V$. Equivalently, if $\widetilde{\pi}|_V$ is an appropriate analytic continuation of a different representative of its germ $\boldsymbol{\widetilde{\pi}|_V}$ at a point, we show that domain and codomain of monodromy computations may be extended to those of $\widetilde{\pi}|_V$. The following example gives an overview of this approach.

Illustrating example Reconsider the cone $C \subset \mathbb{C}^3$ defined by $f(x) = x_1^2 + x_2^2 - x_3^2 = 0$. At the origin, its germ in $\mathbb{C}^3$ is reduced with pure dimension 2. For illustrative purposes, consider

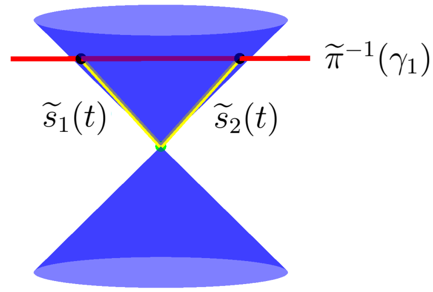
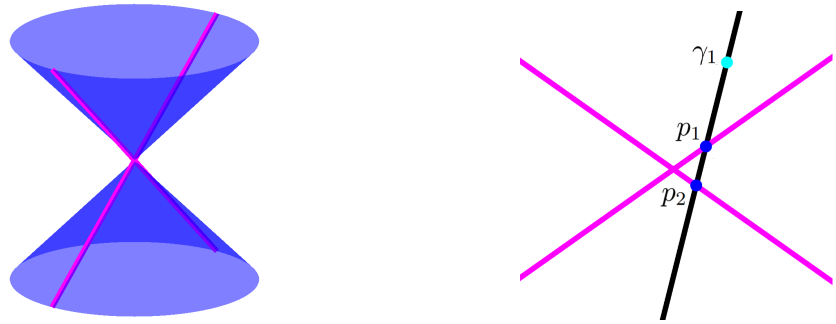


Figure 2: Intersection of the cone (blue) with line (red) yields two points (black) which are the start points of two paths (yellow) that limit to the origin (green).



(a) (b)

Figure 3: (a) Critical points (magenta) with respect to $\widetilde{\pi}$ on the cone (blue); (b) critical locus (magenta) in the image of $\widetilde{\pi}$ intersected with a line (black) passing through γ_1 (cyan) yielding two points (blue).

the sufficiently general linear projection $\widetilde{\pi}(x) = (x_1 + x_2, x_3)$. Thus, for the sufficiently general point $\gamma_1 = (1/2, 1) \in \mathbb{C}^2$, note that $\widetilde{\pi}^{-1}(\gamma_1)$ is a line in $\mathbb{C}^3$ and the fiber $\widetilde{\pi}^{-1}(\gamma_1) \cap C$ of $\widetilde{\pi}|_C$ consists of two points. Along the segment $\gamma(t) = t\gamma_1$ for $t \in [0, 1]$, the intersection $\widetilde{\pi}^{-1}(\gamma(t)) \cap C$ defines two solution paths $\widetilde{s}_1, \widetilde{s}_2 : [0, 1] \to C$ starting at these two fiber points. We first "localize" the fiber by retaining those fiber points $\widetilde{s}_i(1)$ whose solution paths limit to the origin, i.e., with $\widetilde{s}_i(0) = 0$. In this example, all the fiber points localize in this way as illustrated in Figure 2. We denote them $\widetilde{s}_1 = \widetilde{s}_1(1)$ and $\widetilde{s}_2 = \widetilde{s}_2(1)$, abusing notation.

Next, we need to consider the critical points of $\widetilde{\pi}|_C$, which comprise a hypersurface on the cone as illustrated in Figure 3(a). Once the critical points are removed, $\widetilde{\pi}_C$ is a covering map of its image. We again localize these critical points and their images similar to how we localized the fiber points. To compute the critical locus in terms of the projection $\widetilde{\pi}$, consider another sufficiently general linear projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ with $z \mapsto z_1 - z_2/4$. Let $\theta(1) = \pi(\gamma_1)$ and consider the linear space $\mathcal{L}_{\theta(1)} \subset \mathbb{C}^2$ defined by $\pi(z) = \theta(1)$. Since the determinant of the Jacobian matrix of f and $\widetilde{\pi}$ is $2x_1 - 2x_2$, the critical points and critical

Figure 4: (a) Illustration of a basic loop starting at γ_1 and only encircling p_1 once counterclockwise; (b) pictorial illustration of monodromy action interchanging $\widetilde{s}_1$ and $\widetilde{s}_2$ with the vertical direction corresponding to arc length around the basic loop in (a).

locus of $\widetilde{\pi}|_C$ are defined the system

$$\begin{bmatrix} f(x) \\ \hline \widetilde{\pi}(x) - z \\ \hline \det J(f, \widetilde{\pi})(x) \\ \hline \pi(z) - \theta(1) \end{bmatrix} = \begin{bmatrix} x_1^2 + x_2^2 - x_3^2 \\ \hline x_1 + x_2 - z_1 \\ x_3 - z_2 \\ \hline 2x_1 - 2x_2 \\ \hline z_1 - z_2/4 - 1/4 \end{bmatrix} = 0. \tag{1}$$

In terms of the z coordinates, there are two points in the intersection of $\pi^{-1}(\theta(1))$ with the critical locus in $\mathbb{C}^2$ as illustrated in Figure 3(b). Replacing $\theta(1)$ with $\theta(t) = \pi(\gamma(t)) = t/6$ in (1), yields two solution paths $p_1, p_2 : [0,1] \to \mathbb{C}^2$ lifting θ starting at these two points. Similarly to our previous localization, both paths limit to the origin, i.e., $p_1(0) = p_2(0) = 0$. We retain the two points and denote them $p_1 = p_1(1)$ and $p_2 = p_2(1)$, abusing notation.

Our theoretical results show that one can compute the local monodromy action of the germ of the cone at the origin by considering the local monodromy action arising from loops in $\mathcal{L}_{\theta_1} - \{p_1, p_2\}$ lifted to the cone. In particular, one can identify $\mathcal{L}_{\theta_1} - \{p_1, p_2\}$ with the plane $\mathbb{R}^2$ with two points removed and observe that the action of a basic loop starting at γ_1 that only encircles p_1 once counterclockwise, as illustrated in Figure 4(a), suffices to generate the local monodromy action. Such a loop lifts to two paths in C starting at $\widetilde{s}_1$ and $\widetilde{s}_2$. In fact, the corresponding path starting at $\widetilde{s}_1$ ends at $\widetilde{s}_2$ and vice versa as pictorially illustrated in Figure 4(b). Note that a basic loop starting at γ_1 that only encircles p_2 once counterclockwise performs the same monodromy action as a basic loop encircling p_1 clockwise. Hence, the local monodromy action has a single orbit and the local monodromy group is the symmetric group on two elements. Transitivity of this action corresponds to the germ of the cone at the origin being irreducible.

The rest of the paper justifies this process and is organized as follows. Section 2 recalls some essential results and definitions about germs of holomorphic subvarieties and their local irreducible decompositions. Section 3 lays out our theory of local monodromy actions and groups for projection maps, which is used in Section 4 to justify an algorithm for computing these objects. Section 5 contains several examples computed using an implementation of the algorithm. A short conclusion is provided in Section 6.

2 Background

We review local parameterization of holomorphic subvarieties, give an overview of homotopy continuation and numerical algebraic geometry, and state Hamm and Lê Dũng Tráng's theorem on fundamental groups of hypersurface complements [15].

2.1 Local irreducible decomposition and local parameterization

A germ $\mathbf{V}$ of a holomorphic subvariety of an open set $\widetilde{U} \subseteq \mathbb{C}^N$ at $x^* \in \widetilde{U}$ has an irreducible decomposition. More precisely, using Gunning's notation [14, Thm. II.B.7], $\mathbf{V}$ can be written as a finite union of germs $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2 \cup \cdots \cup \mathbf{V}_m$ where each $\mathbf{V}_j$ is an irreducible germ, $\mathbf{V}_j \neq \mathbf{V}$ if $m > 1$, $\mathbf{V}_j \not\subseteq \mathbf{V}_k$ for $j \neq k$, and the germs $\mathbf{V}_j$ are uniquely determined up to relabeling. Following [9], the local irreducible decomposition of an irreducible subvariety (algebraic or holomorphic) V at $x^* \in V$ is the decomposition given by the germ of V at x^*. Moreover, if V is reducible, then the local irreducible decomposition of V at x^* is the union of local irreducible decompositions of its (global) irreducible components.

Since one may always translate x^* to the origin, it suffices to consider germs at the origin, which we will do going forward to simplify notation except where otherwise indicated. While holomorphic subvarieties may exhibit more complicated global behavior, they exhibit the structure of finite branched coverings locally. The following collects several standard results making this precise in a useful format for our purposes.

Lemma 2.1. Let $\mathbf{V}$ be a pure d-dimensional germ of a holomorphic subvariety of $\mathbb{C}^N$ with V a representative. There exists an (algebraically) Zariski open set of linear projections $\mathbb{C}^N \to \mathbb{C}^d$ where the following holds provided that $\pi : \mathbb{C}^N \to \mathbb{C}^d$ is a member. For all small enough open balls $\widetilde{B} \subseteq \mathbb{C}^N$ and $B \subseteq \mathbb{C}^d$ at the origin, $\hat{V} := V \cap \widetilde{B} \cap \pi^{-1}(B)$ has the form $\hat{V} = \hat{V}_1 \cup \cdots \cup \hat{V}_m$ where the $\hat{V}_i$ are irreducible holomorphic subvarieties representing the irreducible components of $\mathbf{V}$. Furthermore:

1. $\pi|_{\hat{V}}$ and $\pi|_{\hat{V}_i}$ are finite branched holomorphic coverings of B with $0 \in \mathbb{C}^N$ the only element of the fiber over $0 \in \mathbb{C}^d$ for both maps.

2. The image of the branch locus for $\pi|_{\hat{V}}$ and $\pi|_{\hat{V}_i}$ is a holomorphic subvariety of B with the same dimension as the branch locus.

3. If $\widetilde{R}$ and $\widetilde{R}_i$ are the branch loci of $\pi|_{\hat{V}}$ and $\pi|_{\hat{V}_i}$ respectively, the monodromy action on any fiber of $\pi|_{\hat{V}_i - \widetilde{R}_i}$ is transitive and the monodromy action of $\pi|_{\hat{V} - \widetilde{R}}$ on any fiber partitions the fiber into orbits, one for each local irreducible component.

Proof. The first statement follows from the local parameterization theorem, e.g., see [14, Lem. II.E.12]. Noting that the branch locus and its image are holomorphic subvarities for a finite holomorphic branched covering, e.g., see [14, Thm. II.C.13,14], the second statement follows from the first using Remmert's proper mapping theorem, e.g., see [14, II.N.1]. Since the germ of $\hat{V}_i$ at the origin is irreducible, possibly shrinking B further, we have that $\hat{V}_i - \widetilde{R}_i$ is path connected [14, II.E.13] and the corresponding monodromy action is transitive. Any point in the intersection of two distinct irreducible components of $\hat{V}$ is a branch point of $\pi|_{\hat{V}}$, so, in particular, every point in a fiber over a regular point of $\pi|_{\hat{V}}$ is contained in one and only one irreducible component $\hat{V}_i$. □

Definition 2.2. Reprising the notation of the above lemma, if $\mathbf{V}$ fulfills its conditions and $\pi, \widetilde{B}, B, \hat{V}$ fulfill the conclusions, $\pi|_{\hat{V}}$ is a *local parametrization* of $\mathbf{V}$.

Remark 2.3. Taking the balls $\widetilde{B}$ and B in the above lemma small enough, we may assume that $\hat{V}$ is defined as the zero set of a system F of holomorphic functions on $\widetilde{U} \subseteq \mathbb{C}^N$. If $\hat{V}$ is a reduced complete intersection with respect to such a system, the critical points of $\pi|_{\hat{V}}$ and $\pi|_{\hat{V}_i}$ are either empty or holomorphic subvarieties of dimension $d-1$ which contain the corresponding branch points. In this case, Lemma 2.1 remains true replacing branch loci with the corresponding critical loci. Critical loci are commonly used to facilitate numerical algebraic geometric computations involving branch loci, e.g., see [6, 19]. Equations for branch loci are generally challenging to obtain from F, whereas critical loci admit a straightforward formulation using rank-dropping conditions via determinants or null space conditions [2].

Remark 2.4. An essential observation in numerical algebraic geometry is that if $V \subseteq \mathbb{C}^N$ is algebraic of pure dimension d, then there is a Zariski open subset of linear projections $\mathbb{C}^N \to \mathbb{C}^d$ where, if $\pi : \mathbb{C}^N \to \mathbb{C}^d$ is a member, then $\pi|_V$ is a finite branched covering map. This follows from the Noether normalization theorem. For symmetry, call such a map a *global parametrization* for V. A core part of the theoretical justification for the localization heuristics presented in [9] can then be summarized using Lemma 2.1 as follows. A linear projection π is generically a global parametrization for V which, after restricting the domain and codomain, is a local parametrization of $\mathbf{V}$ at the origin.

2.2 Witness sets and homotopy continuation

Numerical algebraic geometry (for a general overview, see the books [5, 25]) represents an irreducible algebraic variety via a witness set. Suppose that $f : \mathbb{C}^N \to \mathbb{C}^n$ is a polynomial system and $\mathcal{V}(f) = \{x \in \mathbb{C}^N \mid f(x) = 0\}$. If $V \subset \mathcal{V}(f)$ is irreducible of dimension d, then for a general codimension d linear space $\mathcal{L}$, the intersection $V \cap \mathcal{L}$ is finite and the number of such points in the intersection is equal to $\deg V$. The set $\{f, \mathcal{L}, V \cap \mathcal{L}\}$ is called a *witness set* for V and $V \cap \mathcal{L}$ is called a *witness point set* for V. If V is of pure dimension but reducible, a witness point set for V is a union of witness point sets for each irreducible component of V. A single codimension d linear space $\mathcal{L}$ is used to intersect all irreducible components. The following considers a local version of these definitions [9].

Definition 2.5 (Definition 1 from [9])**.** Let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a system of functions which are holomorphic in a neighborhood of $x^* \in \mathbb{C}^N$ with $f(x^*) = 0$. Let $V \subseteq \mathbb{C}^N$ be a local irreducible component of $\mathcal{V}(f)$ at x^* of dimension d and $\widetilde{L}_1, \ldots, \widetilde{L}_d : \mathbb{C}^N \to \mathbb{C}$ be general linear polynomials such that $\widetilde{L}_i(x^*) = 0$. For $u \in \mathbb{C}^d$, consider the linear space $\mathcal{L}_u = V(\widetilde{L}_1 - u_1, \ldots, \widetilde{L}_d - u_d) \subset \mathbb{C}^N$. A *local witness set* for V is the triple $\{f, \mathcal{L}_{u^*}, W\}$ defined in a sufficiently small neighborhood $U \subset \mathbb{C}^d$ of the origin for general $u^* \in U$ where W is the finite subset of points of $V \cap \mathcal{L}_{u^*}$ which converge to x^* as $t \to 0$ along any path defined by $V \cap \mathcal{L}_{u(t)}$ such that $u : [0, 1] \to U$ with $u(0) = 0$ and $u(1) = u^*$.

Definition 2.6 ([9]). A *numerical local irreducible decomposition* of a holomorphic subvariety V at $x^* \in V$ is a formal union of local witness sets, one for a representative of each irreducible component of the germ $\mathbf{V}$ at x^*.

The set W in Definition 2.5 is called a *local witness point set*. In analogy with the global case, a local witness point set for a reducible pure dimension germ $\mathbf{V}$ is a union of local witness point sets for the irreducible components of $\mathbf{V}$. Some conditions which [9] leaves implicit are required for Definition 2.5 to make sense. It is sufficient for the projection map $\pi : \mathbb{C}^N \to \mathbb{C}^d$ defined by the linear forms $\widetilde{L}_1, \ldots, \widetilde{L}_d$ to have $\pi|_{V \cap \pi^{-1}(U)}$ be an (unbranched) covering map. For example, if V is algebraic, the Noether normalization theorem shows that this is true for a Zariski open set of linear forms with $U = \mathbb{C}^d - R$ the complement of the image of the branch locus of $\pi|_V$. If V is reduced with respect to f, then one may instead complement by the image of the critical locus of $\pi|_V$.

Definition 2.5 is dependent upon computing start points of paths which converge to x^*. This is an example of the use of homotopy continuation in numerical algebraic geometry. In particular, suppose that $\pi : V \to Y$ is a finite branched holomorphic covering map and $\gamma : [0, 1] \to Y$ is a path with $\gamma|_{(0,1]}$ a smooth path into the regular part of Y. Then, $\pi^{-1}(\gamma(1))$ is finite with, say, k points and γ lifts through π to a set of k paths $\widetilde{\gamma}_i : [0, 1] \to V$, each having $\widetilde{\gamma}_i(1)$ a distinct point of $\pi^{-1}(\gamma(1))$. See, e.g., [24, Thm. 3] for more details. These lifts are smooth and disjoint in the sense that each restriction $\widetilde{\gamma}_i|_{(0,1]}$ is smooth and none of the images of the $\widetilde{\gamma}_i|_{(0,1]}$ intersect. Each lift $\widetilde{\gamma}_i$ is a solution to an initial value problem with initial values given by $\pi^{-1}(\gamma(1))$ and implementations of numerical homotopy continuation methods are designed to track the solutions numerically. The lifts $\widetilde{\gamma}_i$ are *solution paths*.

Assume for the moment that solutions to lifting problems of this type can be solved exactly using homotopy continuation. Also assume that, given algebraic V and a generic codimension d linear space $\mathcal{L}$, we can compute the witness point set $V \cap \mathcal{L}$. Computing such an intersection using homotopy continuation is a standard maneuver. We immediately obtain an algorithm for computing a local witness point set for V at a point $x^* \in V$.

Algorithm 1: LOCAL WITNESS POINT SETS

Input : A polynomial system $f : \mathbb{C}^N \to \mathbb{C}^{N-d}$ defining a reduced complete intersection $V = \mathcal{V}(f) \subseteq \mathbb{C}^N$ of dimension d.
Input : A point $x^* \in V$.
Input : A generic linear map $\pi : \mathbb{C}^N \to \mathbb{C}^d$ with components $\widetilde{L}_i$.
Input : A generic smooth path $\gamma : [0, 1] \to \mathbb{C}^d$ with $\gamma(0) = \pi(x^*)$.
Output: A local witness point set W^l for the germ of V at x^*.

1 Compute the witness point set $W^g := V \cap \mathcal{L}_{\gamma(1)} = \{v_1, \ldots, v_k\}$;
2 Compute solution paths $\widetilde{\gamma}_i : [0, 1] \to V$ for the map $\pi|_V$ corresponding to initial conditions $\widetilde{\gamma}_i(1) = v_i$;
3 `Return` the subset W^l of points $\widetilde{\gamma}_i(1)$ where $\widetilde{\gamma}_i(0) = x^*$;

Note that this algorithm differs slightly from Definition 2.5 in that it does not assume $\pi(x^*) = 0$. This will be useful in discussing practical numerical issues in localization, which we take up in Section 4. It is straightforward to translate to the case $\pi(x^*) = 0$. As long as W^g is finite in Step 1 above and can be computed, one may otherwise relax the requirement that f be polynomial to f holomorphic.

Using homotopy continuation to move linear slices in a witness set is a powerful tool in numerical algebraic geometry. In the current context, we can consider general loops in the corresponding Grassmannian to induce a monodromy action that can be used to identify the global and local irreducible components by partitioning the fiber into orbits, one for each global and local irreducible component, respectively.

Example 2.7. Globally, the cone $V = \mathcal{V}(x_1^2+x_2^2-x_3^2) \subset \mathbb{C}^3$ is irreducible of dimension 2 and degree 2. Consider the sufficiently general line $L_{(1/2,1)} = \mathcal{V}(x_1 + x_2 - 1/2, x_3 - 1) \subset \mathbb{C}^3$ as in the Introduction. Equivalently, as input to Algorithm 1, consider $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ defined by $\pi = (x_1+x_2, x_3)$ with $L_{(1/2,1)} = \pi^{-1}(1/2, 1)$. Figure 2 illustrates the witness point set $V \cap L$.

Define $\gamma : [0, 1] \to \mathbb{C}^2$ by $\gamma(t) = (t/2, t)$. Let $L_t = L_{\gamma(t)} = \mathcal{V}(x_1 + x_2 - t/2, x_3 - t) \subset \mathbb{C}^3$ so that $V \cap L_t$ defines 2 paths lifting γ starting from the two points in $V \cap L_{(1/2,1)}$. As also shown in Figure 2, both lifts converge to the origin. Thus, both points in $V \cap L_{(1/2,1)}$ are elements of the local witness point set for the V at the origin output by Algorithm 1. As discussed in the Introduction, the cone is irreducible at the origin so that the corresponding local monodromy group is a transitive group on two elements, i.e., the symmetric group S_2.

2.3 Heuristic numerical local irreducible decomposition

On the one hand, the localized definitions in the previous section are at least somewhat geometrically intuitive. On the other, one can also motivate them via the local parametrization theorem. The following result is a direct consequence of Lemma 2.1. In less precise terms, it states that if one translates a local witness point set for V close enough to the point of localization, the induced monodromy action of small loops near the point of localization encodes a numerical local irreducible decomposition. The property of a local parametrization having a unique element in the fiber over $0 \in \mathbb{C}^d$ is essential.

Proposition 2.8. Let V be a pure d-dimensional holomorphic subvariety of $\mathbb{C}^N$ defined by $f : \mathbb{C}^N \to \mathbb{C}^n$ at the origin and let $\pi : \mathbb{C}^N \to \mathbb{C}^d$ be a global parametrization of V that restricts to a local parametrization $\pi|_{\hat{V}}$ of $\mathbf{V}$. Set $U = \mathbb{C}^d - \pi|_V(\widetilde{R})$ where $\widetilde{R} \subseteq V$ is the branch locus of $\pi|_V$. Let $\gamma : [0, 1] \to \mathbb{C}^d$ be a smooth path beginning at $\gamma(0) = 0$. Then:

- The set W^l obtained from Algorithm 1 with inputs as indicated in this proposition is a local witness point set for $\mathbf{V}$.
- Suppose that $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_k : [0, 1] \to V$ are the lifts of γ corresponding to the points in W^l and that $\widetilde{\gamma}_i(t) \in \hat{V}$ for all i and some $t \in (0, 1]$. Denote $W_t^l := \{\widetilde{\gamma}_i(t)\}_{i=1}^k$. There exists a ball $B \subseteq \mathbb{C}^d$ containing the origin where the orbits of W_t^l under the monodromy action of $\pi_1(B - U, \gamma(t))$ through $\pi|_{\hat{V}}$ partition W_t^l into local witness point sets for the irreducible components of $\mathbf{V}$. There is exactly one orbit per local irreducible component.

This suggests the heuristic algorithm for computing a numerical local irreducible decomposition implicitly proposed in [9]. First, choose a projection $\widetilde{\pi} : \mathbb{C}^N \to \mathbb{C}^d$ uniformly at random. With probability 1, it fulfills the conditions necessary for Proposition 2.8 and is a global parametrization for V. Next, choose a path $\gamma : [0, 1] \to \mathbb{C}^d$ starting at $\widetilde{\pi}(x^*)$ and ending nearby at a point chosen uniformly at random. A straight line path suffices with probability 1. Use Algorithm 1 to compute a local witness point set for V at x^* using this data. Finally, compute the orbits of the monodromy action induced by random "small" loops in $\mathbb{C}^d$ based at $\gamma(1)$. Here, "small" means loops whose image contains only points with small distance to $\gamma(0)$.

There are several obstructions which render this a heuristic rather than theoretically complete algorithm for computing a numerical local irreducible decomposition for V at x^*. One does not know *a priori* how small the loops used for computing monodromy actions must be to capture only local behavior. The points in the local witness set may also be relatively far away from x^*. It therefore no longer follows directly from Lemma 2.1 that computed orbits correspond to local irreducible components, though it follows from the theory we will develop in Section 3 that this is true. Finally, in the global case, one uses *trace tests* which provide a stopping criteria for when additional loops will stop reducing the number of observed orbits. Analogous tests do not exist in the local case.

2.4 Hyperplane sections and fundamental groups of complements

As mentioned in the Introduction, Zariski's theorem [26] was used in [19] to compute global monodromy groups via a surjection of fundamental groups under slicing. In order to consider the local case, let $\widetilde{B}_r$ denote the ball of radius r in $\mathbb{C}^N$ centered at the origin and B_r denote the same in $\mathbb{C}^d$ for $r > 0$. A "Zariski theorem of Lefschetz type" due to Hamm and Lê Dũng Tráng [15] shows that the map induced by inclusion of fundamental groups $\pi_1((B_\rho - R) \cap \mathcal{L}_\theta, b) \to \pi_1(B_\rho - R, b)$ is surjective for a holomorphic hypersurface R, sufficiently small ρ, and any well-behaved complex line $\mathcal{L}_\theta \subseteq \mathbb{C}^d$. For any hyperplane $L \subseteq \mathbb{C}^d$, by abuse of notation, let L also denote some fixed linear form such that $L = 0$ defines the hyperplane L. Let $L^\theta = \mathcal{V}(L - \theta)$ for $\theta \in \mathbb{C}$.

Lemma 2.9. Let R be a pure $(d-1)$-dimensional holomorphic subvariety of $\mathbb{C}^d$ containing the origin with $d \geq 2$. There exists a Zariski open subset of hyperplanes $\mathcal{U} \subseteq \mathrm{Gr}(d-1, d)^{d-1}$, depending on R, with the following property. If $(L_1, L_2, \ldots, L_{d-1}) \in \mathcal{U}$, there exists a maximum radius $A > 0$ for balls B_ρ at the origin such that, where if $0 < \rho \leq A$, then the fundamental group of $B_\rho - R$ is generated by homotopy classes of loops restricted to $\cap_{i=1}^{d-2} L_i \cap L_{d-1}^\theta$ for all small enough θ. Note that, when $d = 2$, $\cap_{i=1}^{d-2} L_i$ is defined to be $\mathbb{C}^2$.

More precisely, there is $\theta(\rho) > 0$ such that if $0 < |\theta| \leq \theta(\rho)$, the homomorphism induced by inclusion

$$\pi_1((B_\rho - R) \cap_{i=1}^{d-2} L_i \cap L_{d-1}^\theta, b) \to \pi_1(B_\rho - R, b)$$

is surjective for any $b \in (B_\rho - R) \cap_{i=1}^{d-2} L_i \cap L_{d-1}^\theta$.

Proof. This is simply repeated application of [15, Thm. 0.2.1]. Each application requires that R be defined as the vanishing locus of a single holomorphic function. This is true for small enough ρ for pure $(d-1)$-dimensional R, e.g., see [14, II.G.5]. ∎

Remark 2.10. In the case of curves, i.e., when $d = 1$, then R is 0-dimensional. Thus, $B_\rho - R$ is a real 2-dimensional disc with a finite number of punctures for any ρ. There is at least one puncture at the origin as we have assumed R contains the origin. The fundamental group of $B_\rho - R$ is generated by a set of homotopy classes of based loops, each winding around a distinct puncture once counterclockwise. Slicing is no longer necessary.

Let $d \geq 2$. Continuing the same notation with $\mathcal{L}_\theta := \cap_{i=1}^{d-2} L_i \cap L_{d-1}^\theta$ for sufficiently small θ, note that $(B_\rho - R) \cap \mathcal{L}_\theta = (B_\rho \cap \mathcal{L}_\theta) - (B_\rho \cap R \cap \mathcal{L}_\theta)$. For general $\mathcal{L}_\theta$, possibly shrinking B_ρ further, one has from Lemma 2.1 that $B_\rho \cap R \cap \mathcal{L}_\theta$ is finite. Subsequently, $(B_\rho - R) \cap \mathcal{L}_\theta$ is homeomorphic to a real plane with the finitely many points $B_\rho \cap R \cap \mathcal{L}_\theta$ missing. Its fundamental group is generated by a set of homotopy classes of based loops, one per missing point. Each loop in such a set encircles exactly one point in $B_\rho \cap R \cap \mathcal{L}_\theta$ once counterclockwise.

3 Local monodromy actions and groups

Our major objective in this section is to establish how the monodromy action of a local parametrization behaves under continuation. This will illuminate whether and how local witness point sets can be used with homotopy continuation for local monodromy computations. Given that our development of the corresponding theory is somewhat more abstract than is typical in numerical algebraic geometry, it is helpful to frame it with the following somewhat informal "theorem" as an objective. The full version is Theorem 3.26. Effectively, it says that local witness point sets for a variety and its critical point locus suffice to compute local irreducible components. Equivalently, local monodromy computations need not be performed close to a singular point, provided one first computes local witness point sets.

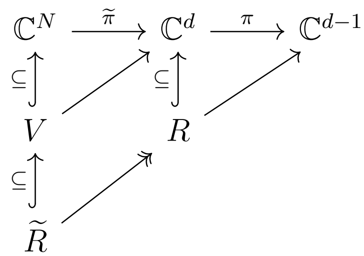
Theorem. Let $0 \in V \subseteq \mathbb{C}^N$ with V pure d-dimensional. Let $\widetilde{\pi} : \mathbb{C}^N \to \mathbb{C}^d$, $\pi : \mathbb{C}^d \to \mathbb{C}^{d-1}$, and $\gamma : [0,1] \to \mathbb{C}^d$ be generic with $\gamma(0) = 0$. Define $R := \widetilde{\pi}(\mathrm{crit}(\widetilde{\pi}|_V))$, $\theta = \pi \circ \gamma$, and $\mathcal{L}_1 = \pi^{-1}(\theta(1))$. Let $\widetilde{W}^l$ be the local witness point set for V computed from Algorithm 1 with $\widetilde{\pi}$, γ, and 0 as input. Let W^l be the local witness point set for R computed from Algorithm 1 using π, θ and 0 as input. Then, the monodromy action through $\widetilde{\pi}$ on $\widetilde{W}^l$ from loops encircling the points in W^l restricted to $\mathcal{L}_1$ is well-defined and the orbits are local witness point sets for the local irreducible components of V at 0.

To state and prove this rigorously, we require two types of ingredients. First, we must carefully define the data required to use both homotopy continuation and slicing with Lemma 2.9 simultaneously, as well as how to restrict that data. Second, we must prove that monodromy actions constructed from such data change in a continuous way under change-of-basepoint isomorphisms.

3.1 Monodromy representatives

When $d \geq 2$, denote any intersection of the form $\cap_{i=1}^{d-2} L_i \cap L_{d-1}^\theta$ as in that Lemma 2.9 by $\mathcal{L}_\theta$. When $d = 1$, let $\mathcal{L}_\theta$ denote $\mathbb{C}$ for all θ. We will require some machinery to define a limiting process that behaves well with respect to homotopy continuation methods. The following setup definition is motivated by the requirements of one of Morgan and Sommese's foundational parameter homotopy continuation theorems [24, Thm. 3].

Definition 3.1. A *monodromy representative* for a germ of a holomorphic map $\widetilde{\boldsymbol{\pi}} : \mathbf{V} \to \mathbf{C}^d$ is comprised of the following sets, maps, and commutative diagram, where $\widetilde{\pi}$ is a representative of $\widetilde{\boldsymbol{\pi}}$:



Furthermore:

- V is a pure d-dimensional holomorphic subvariety of $\mathbb{C}^N$ containing the origin.
- $\widetilde{R}$ is a pure $(d-1)$-dimensional holomorphic subvariety of V containing the origin.
- $\widetilde{\pi}|_V$ is a proper projective holomorphic map and the restriction $\widetilde{\pi}|_{V-\widetilde{R}}$ is a finite unbranched covering of its image.
- $R = \widetilde{\pi}(\widetilde{R})$ is a pure $(d-1)$-dimensional holomorphic subvariety of $\mathbb{C}^d$.
- The restriction $\pi|_R$ is a proper projective holomorphic map.

Remark 3.2. We will reuse the notation in Definition 3.1 going forward to refer to the components of any given monodromy representative.

Remark 3.3. Let $\mathbf{V}$ be a pure d-dimensional germ of a holomorphic subvariety at the origin in $\mathbb{C}^N$. As discussed in the previous section, for a linear projection $\widetilde{\pi} : \mathbb{C}^N \to \mathbb{C}^d$ satisfying Lemma 2.1, there is a representative $\hat{V}$ of $\mathbf{V}$ where $\widetilde{\pi}|_{\hat{V}}$ is a finite branched holomorphic covering of a ball B centered at $0 \in \mathbb{C}^d$ and $0 \in \mathbb{C}^N$ is the unique element in $\widetilde{\pi}|_{\hat{V}}^{-1}(0)$. If $\mathbf{V}$ is reducible then the origin is necessarily a singular point of $\hat{V}$ (the contrapositive follows from the implicit function theorem). The map $\widetilde{\pi}|_{\hat{V}}$ therefore has a nonempty branch locus $\widetilde{R} \subseteq \hat{V}$ which has codimension at least 1 in $\hat{V}$. If the codimension is larger than 1, simple connectedness yields trivial local monodromy. Moreover, as in Remark 2.3, one can always replace the branch locus with the critical locus which, by abuse of notation, we will also call $\widetilde{R}$. Therefore, without loss of generality, we can assume that $\widetilde{R}$ has codimension 1 in $\hat{V}$. Denote $R = \widetilde{\pi}(\widetilde{R})$.

If $d \geq 2$, possibly shrinking B further, let $L_1, \ldots, L_{d-1} \subseteq \mathbb{C}^d$ be hyperplanes for which Lemma 2.9 applies to R and B, and such that the projection $\pi : \mathbb{C}^d \to \mathbb{C}^{d-1}$ defined by $(L_1, L_2, \ldots, L_{d-1})$ has $\pi|_R$ a finite proper branched covering of its image. This is true for generic choices of hyperplanes since Lemma 2.9 and the local parameterization theorem both apply generically. If $d = 1$, let $\pi : \mathbb{C}^1 \to \mathbb{C}^0$ be the obvious map and shrink B so that $B \cap R = \{0\}$. Then, the diagram in Definition 3.1 with V replaced by $\hat{V}$ and other notation referring to the specific choices in this remark's setup is a monodromy representative for $\widetilde{\boldsymbol{\pi}}|_{\mathbf{V}}$.

Definition 3.4. Given a pure d-dimensional germ $\mathbf{V}$ of holomorphic subvariety of $\mathbb{C}^N$, call any monodromy representative of a linear projection $\widetilde{\boldsymbol{\pi}} : \mathbf{V} \to \mathbf{C}^d$ which fulfills the conditions in Remark 3.3 a *localized monodromy representative* for $\mathbf{V}$.

Example 3.5. Let $V = \mathcal{V}(f) \subseteq \mathbb{C}^N$ be a pure d-dimensional algebraic complete intersection where $f : \mathbb{C}^N \to \mathbb{C}^{N-d}$ is algebraic and $d \geq 1$. Let $Jf(x)$ denote the Jacobian matrix of f at $x \in \mathbb{C}^N$. Assume that V is reduced with respect to f in the sense that $N - \text{rank}(Jf(v)) = d$ at every regular point $v \in V$ and suppose that V has a singular point at the origin. Let $\widetilde{\pi} : \mathbb{C}^N \to \mathbb{C}^d$ be a linear projection where $\widetilde{\pi}|_V$ represents the germ of a local parameterization at the origin in the sense of Lemma 2.1. Hence, the critical locus of $\widetilde{\pi}$, which contains the branch locus, is either empty or of pure dimension $d-1$ since it is characterized by satisfying the critical equations: $f(z) = 0$ and $\det J(f, \widetilde{\pi})(z) = 0$.

Typically, it is challenging to compute equations defining the image of the critical locus under $\widetilde{\pi}$ directly. Instead, consider the graph $G := \{(\widetilde{v}, v) \in \mathbb{C}^N \times \mathbb{C}^d \mid f(\widetilde{v}) = 0, \widetilde{\pi}(\widetilde{v}) = v\}$. The critical points of $\widetilde{\pi}|_V$ correspond to the points on this graph defined by

$$CG := \{(\widetilde{v}, v) \in G \mid \text{rank}\, J(f, \widetilde{\pi})(\widetilde{v}) < N\}.$$

Define $\widetilde{R}$ to be the image of the projection of CG onto the first factor with $R = \widetilde{\pi}(\widetilde{R})$ being the projection onto the second factor. Note that $\widetilde{R}$ is the critical locus of $\widetilde{\pi}|_V$ which contains the branch locus. The elements of Definition 3.1 with notation referring to the specific choices in this example are a monodromy representative for $\boldsymbol{\widetilde{\pi}|_V}$.

Our limiting procedure for defining local monodromy groups in this context will proceed by tracking the monodromy action defined by a monodromy representative along a continuous path $\gamma : [0, a] \to \mathbb{C}^d$ that goes from 0 to some other point. The constraints we need to place on γ are determined again by parameter homotopy continuation considerations. For convenience, the following lemma collects those constraints and is a direct corollary of [24, Thm. 3-(2,4)]. The solution paths can be tracked using homotopy continuation.

Lemma 3.6. Given a monodromy representative as in Definition 3.1, let $\gamma : [0, a] \to \mathbb{C}^d$ be a continuous path with $\gamma|_{(0,a]}$ smooth, $\text{im}(\gamma) \subseteq \widetilde{\pi}(V)$. There exist Zariski open dense subsets $V_0 \subseteq \widetilde{\pi}(V)$ and $R_0 \subseteq \pi(R)$ where:

1. If K_1 is the line containing $\gamma(0)$ and $\gamma(a)$, then there are at most countably many points in $K_1 - (K_1 \cap V_0)$ and they are geometrically isolated. If also $\gamma((0, a]) \subseteq K_1 \cap V_0$, then γ lifts to V through $\widetilde{\pi}|_V$ as a disjoint set of finitely many smooth paths $[0, a] \to V$.

2. If additionally $\text{im}(\pi \circ \gamma) \subseteq \pi(R)$ and K_2 is the line in $\mathbb{C}^{d-1}$ containing $\pi(\gamma(0))$ and $\pi(\gamma(a))$, then there are at most countably many points in $K_2 - (K_2 \cap R_0)$ and they are geometrically isolated. If also $(\pi \circ \gamma)((0, a]) \subseteq K_2 \cap R_0$, then $\pi \circ \gamma$ lifts through $\pi|_R$ as a disjoint set of finitely many smooth paths $[0, a] \to R$.

Remark 3.7. When $d = 1$ in the above lemma, $\mathbb{C}^{d-1} = \mathbb{C}^0$ and $\pi \circ \gamma$ lifts through $\pi|_R$ as constant maps.

Definition 3.8. Fix a monodromy representative as in Definition 3.1. A path $\gamma : [0, a] \to \mathbb{C}^d$ starting at 0 is a *limiting path* for that representative if:

- γ fulfills all the conditions from Lemma 3.6, including those in Items 1 and 2;
- $\|\gamma\| : [0, a] \to \mathbb{R}$ is an increasing function; and

- $\text{im}(\gamma) \subseteq \cap_{i=1}^{d-2} L_i$ if $d \geq 3$.

Definition 3.9. Let $\widetilde{\epsilon}, \rho > 0$. The *restriction* of a monodromy representative with limiting path $\gamma : [0,a] \to \mathbb{C}^d$ to $\widetilde{B}_{\widetilde{\epsilon}}$ and B_ρ is obtained by replacing:

- V with $V \cap \widetilde{B}_{\widetilde{\epsilon}} \cap \widetilde{\pi}^{-1}(B_\rho)$
- $\widetilde{R}$ with $\widetilde{R}' = \widetilde{R} \cap \widetilde{B}_{\widetilde{\epsilon}} \cap \widetilde{\pi}^{-1}(B_\rho)$
- R with $R' = \widetilde{\pi}(\widetilde{R}')$
- γ with $\gamma|_{[0,\beta]}$ where $\beta = \sup(\{t \in [0,a] \mid \gamma([0,t]) \subseteq B_\rho \cap \pi^{-1}(\pi(R'))\})$. One has $\beta > 0$ since $\|\gamma\|$ is increasing.

To fix some notation, for a monodromy representative and a limiting path $\gamma : [0,a] \to \mathbb{C}^d$, note that there exists by Lemma 3.6 a disjoint set of paths $\widetilde{s}_1, \ldots, \widetilde{s}_k : [0,a] \to V$ lifting γ through $\widetilde{\pi}|_V$ and similarly a disjoint set of paths $p_1, \ldots, p_j : [0,a] \to R$ lifting $\pi \circ \gamma$ through $\pi|_R$. At any $t \in [0,a]$, denote $\{\widetilde{s}_i(t)\}_{i=1}^k$ by $\widetilde{S}(t)$ and similarly $\{p_i(t)\}_{i=1}^j$ by $P(t)$.

Remark 3.10. Suppose that γ is a limiting path for a monodromy representative for a germ of dimension $d \geq 2$. For any $t \in [0,a]$, if $L_1, L_2, \ldots, L_{d-1}$ are hyperplanes in $\mathbb{C}^d$ defined by the vanishing of each corresponding component function of $\pi = (L_1, L_2, \ldots, L_{d-1})$ and $\gamma(t) \in \cap_{i=1}^{d-2} L_i$, then $P(t) = R \cap \mathcal{L}_{\theta(t)}$ where $\theta(t) = L_{d-1}(\gamma(t))$. In particular, the indicated intersection of R with hyperplanes is finite, and $\mathcal{L}_{\theta(t)}$ is homeomorphic to a plane in 2 real dimensions. The fundamental group $\pi_1((\widetilde{\pi}(V) - R) \cap \mathcal{L}_{\theta(t)}, \gamma(t))$ is subsequently generated by homotopy classes of j loops in $(\widetilde{\pi}(V) - R) \cap \mathcal{L}_{\theta(t)}$ based at $\gamma(t)$, each of which encircles exactly one distinct point in $P(t)$, and this fundamental group acts on the fiber $\widetilde{S}(t) = \widetilde{\pi}|_V^{-1}(\gamma(t))$ by monodromy.

In the case $d = 1$, recall that $\mathcal{L}_{\theta(t)} = \mathbb{C}^d = \mathbb{C}^1$ for all $\theta(t)$ and $\pi : \mathbb{C}^1 \to \mathbb{C}^0$ is constant. The paths $p_1, \ldots, p_j$ are therefore also constant. Equivalently, $R = P(t)$ for all t. The fundamental group $\pi_1((\widetilde{\pi}(V) - R) \cap \mathcal{L}_{\theta(t)}, \gamma(t))$ is generated by j loops in $(\widetilde{\pi}(V) - R) \cap \mathcal{L}_{\theta(t)}$ based at $\gamma(t)$ and this fundamental group acts on the fiber $\widetilde{S}(t) = \widetilde{\pi}|_V^{-1}(\gamma(t))$ by monodromy.

Definition 3.11. With notation as in Remark 3.10, call a loop $\ell_i : [0,1] \to (\widetilde{\pi}(V) - R) \cap \mathcal{L}_{\theta(t)}$ a *basic loop* for $p_i(t)$ if it is the concatenation of a straight line path, a path winding once counterclockwise around a circle centered at $p_i(t)$ which encircles no other point in $P(t)$, and the reverse of the first straight line path.

Remark 3.12. Given a localized monodromy representative for a germ $\mathbf{V}$ with projections $\widetilde{\pi}|_V$ and $\pi|_R$, and limiting path γ, note that since $0 \in \mathbb{C}^N$ and $0 \in \mathbb{C}^d$ are the only elements in the fibers of $\widetilde{\pi}|_V$ and $\pi|_R$ over 0 respectively, we must have that $\widetilde{S}(0) = \{0\}$ and $P(0) = \{0\}$.

3.2 Monodromy functors

Consider a monodromy representative together with limiting path γ. As alluded to above, for any t, we can consider two monodromy actions: the action of $\pi_1(\widetilde{\pi}(V) - R, \gamma(t))$ on the fiber $\widetilde{\pi}|_V^{-1}(\gamma(t))$, and the sliced version as in Remark 3.10. It is natural, and will prove computationally useful, to characterize how actions at different values of t are related.

Indeed, the unsliced case is an instance of standard covering space theory. For different values of t, γ induces a change-of-basepoint isomorphism between corresponding unsliced monodromy actions. We cannot generally compute unsliced monodromy actions directly, however, so our primary interest is in understanding the sliced case.

Our setup for approaching this question is category theoretic. The advantage is largely organizational, as it allows us to fully specify important relationships concisely. We will see that the unsliced and sliced monodromy actions behave in essentially the same way, and a functor-based framework suggests where proof is necessary to show this in the sliced case. As a useful guide, our main technical observation (Lemma 3.17) is that the sliced monodromy actions for the same monodromy representative are isomorphic at any two values of t.

A monodromy representative together with a limiting path can be used to define a functor of monodromy actions along the path. For any interval $I \subseteq \mathbb{R}$, let, by abuse of notation, I also denote the corresponding category obtained from the poset $(I, \leq)$ where $\leq$ is the standard order. Let **Act** be the category of group actions on sets. In the following, we suppress giving an explicit symbol for a group action where it is clear from context. More precisely, **Act** is the category where:

- If G is any group and S any set, any group action of G on S is an object of **Act**, which is denoted by (G, S).
- An arrow $(G_1, S_1) \to (G_2, S_2)$ in **Act** is a pair (h, ι) where $h : G_1 \to G_2$ is a homomorphism and $\iota : S_1 \to S_2$ is a map such that $h(g)\iota(s) = \iota(gs)$ for all $g \in G_1, s \in S_1$.
- Composition and identities are component-wise.

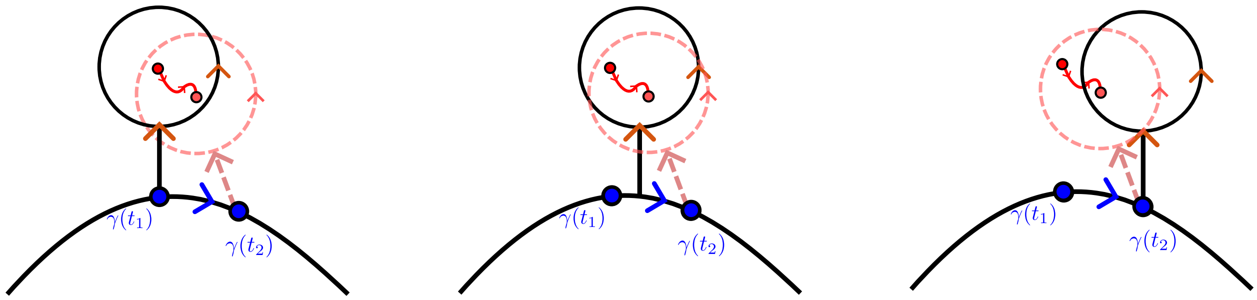
Remark 3.13. There is a functor $\mathrm{im} : \mathbf{Act} \to \mathbf{Group}$. For any object (G, S) of **Act**, view the group action as a homomorphism $\nu : G \to \mathrm{Aut}(S)$. Then, $\mathrm{im}(G, S)$ is $\mathrm{im}(\nu)$. Given an arrow $(h, \iota) : (G_1, S_1) \to (G_2, S_2)$, the homomorphism $\mathrm{im}(h, \iota) : \mathrm{im}(\nu_1) \to \mathrm{im}(\nu_2)$ is $\nu_2 \circ h \circ \nu_1^{-1}$.

Definition 3.14. Given a monodromy representative with a limiting path γ, the (unsliced) *monodromy functor* for this representative, denoted by $\mathrm{Mon}_\gamma : (0, a] \to \mathbf{Act}$ where the monodromy representative is clear from context, is defined by the monodromy action $\mathrm{Mon}_\gamma(t) = (\pi_1(\widetilde{\pi}(V) - R, \gamma(t)), S(t))$. For any $t_1 \leq t_2 \in (0, a]$, the map $\mathrm{Mon}_\gamma(t_1 \leq t_2)$ is defined on the fiber by $s_i(t_1) \mapsto s_i(t_2)$ for $s_1(t_1), \ldots, s_k(t_1) \in S(t)$ and on the fundamental group by the standard change-of-basepoint isomorphism[2] induced by $\ell \mapsto \overline{\gamma|_{[t_1,t_2]}} \cdot \ell \cdot \gamma|_{[t_1,t_2]}$ for all loops ℓ based at $\gamma(t_1)$. The *sliced monodromy functor* for the monodromy representative, $\mathrm{Mon}^s_\gamma : (0, a] \to \mathbf{Act}$, is the same as Mon_γ but replacing $\pi_1(\widetilde{\pi}(V) - R, \gamma(t))$ with the image of the map on fundamental groups induced by the inclusion $(\widetilde{\pi}(V) \cap \mathcal{L}_{\theta(t)}) - R \hookrightarrow \widetilde{\pi}(V) - R$. When $\dim(V) = 1$, note $\mathrm{Mon}^s_\gamma = \mathrm{Mon}_\gamma$.

Proposition 3.15. Mon_γ and Mon^s_γ are functors.

Proof. We first prove the unsliced case. The only property that is non-routine to check is that $\mathrm{Mon}_\gamma(t_1 \leq t_2)$ as defined is a map of group actions for any $t_1 \leq t_2 \in (0, a]$. Denote that map as (iso, ι). For any $[\ell]$ in the fundamental group component of $\mathrm{Mon}_\gamma(t_1)$, let $\tilde{\ell}$ be the

[2] We adopt the typical convention here that an overline denotes the reverse of a path and $\cdot$ denotes concatenation of paths.



(a) (b) (c)

Figure 5: Illustrating first translation homotopy in Lemma 3.17: (a) loop $\ell_i^{t_1}$ (black, orange arrow), $\ell_i^{t_2}$ (pink, dashed), path $p_i|_{[t_1,t_2]}$ (red), and path γ (bottom, blue arrow); (b) loop $\ell_i^{t_1}$ has been translated to $\ell_i^{t_1} + c_t$ for some $t \in [t_1, t_2]$; (c) loop ends translation at $\ell_i^{t_1} + c_{t_2}$.

unique lifting of ℓ through $\widetilde{\pi}|_V$. For any $\widetilde{s}_i(t_1) \in \widetilde{S}(t_1)$, note that $\tilde{\ell}$ ends at $[\ell]\widetilde{s}_i(t_1)$, which we denote by $\widetilde{s}_z(t_1)$. Also, note that $\overline{\widetilde{s}_i|_{[t_1,t_2]}} \cdot \tilde{\ell} \cdot \widetilde{s}_z|_{[t_1,t_2]}$ lifts $\overline{\gamma|_{[t_1,t_2]}} \cdot \ell \cdot \gamma|_{[t_1,t_2]}$, starts at $\widetilde{s}_i(t_2)$, and ends at $\widetilde{s}_z(t_2)$. This yields

$$\iota([\ell]\widetilde{s}_i(t_1)) = \iota(\widetilde{s}_z(t_1)) = \widetilde{s}_z(t_2) = \mathrm{iso}([\ell])\widetilde{s}_i(t_2).$$

The sliced case follows from the unsliced case, with one caveat. We must show that $\mathrm{Mon}^s_\gamma(t_1 \le t_2)$ is well-defined for $t_1 < t_2$. I.e., we need to check that $\overline{\gamma|_{[t_1,t_2]}} \cdot \ell \cdot \gamma|_{[t_1,t_2]}$ is basepoint preserving homotopic to a loop in $\mathcal{L}_{\theta(t_2)}$ for any loop ℓ in $\mathcal{L}_{\theta(t_1)}$. This follows from the stronger Lemma 3.17, so we defer the proof to that result. □

Definition 3.16. The *local monodromy action* of a monodromy representative with a limiting path is $\varprojlim \mathrm{Mon}_\gamma$. Similarly, the *local monodromy group* is $\varprojlim \mathrm{im} \circ \mathrm{Mon}_\gamma$.

These inverse limits are the natural category-theoretic objects which "fill in" the missing value of Mon_γ at 0. As we remarked earlier, however, the change-of-basepoint maps $\mathrm{Mon}_\gamma(t_1 \le t_2)$ are isomorphisms of actions. An inverse is given by taking $\widetilde{s}_i(t_2) \mapsto \widetilde{s}_i(t_1)$ for all $i = 1, \ldots, k$ and by taking the change-of-basepoint isomorphism on fundamental groups corresponding to the reverse path $\overline{\gamma|_{[t_1,t_2]}}$. It follows directly that the local monodromy action (and group) of a monodromy representative always exists and is isomorphic to $\mathrm{Mon}_\gamma(t)$ for any $t \in (0, 1]$. *A priori*, $\varprojlim \mathrm{Mon}^s_\gamma$ does not necessarily exist. The behavior of sliced monodromy functors is similar to the unsliced case in higher dimensions, however, as the next result shows.

Lemma 3.17. For any $t_1 \le t_2$, let iso be the homomorphism component of $\mathrm{Mon}^s_\gamma(t_1 \le t_2)$. If $\ell_i^{t_1}$ is a basic loop for $p_i(t_1)$ and $\ell_i^{t_2}$ is a basic loop for $p_i(t_2)$, then $\mathrm{iso}([\ell_i^{t_1}]) = [\ell_i^{t_2}]$. In particular, $\mathrm{Mon}^s_\gamma(t_1 \le t_2)$ is an isomorphism.

Proof. For any $t \in (0, a]$, let $d(t)$ denote the minimum of both $\min_{i \ne z}\{\|p_i(t) - p_z(t)\|\}$ and $\min\{\|p_i(t)\|\}_{i=1}^j$, and let $D = \min_{t \in [t_1,t_2]} d(t)$. Note that $D > 0$ as it is the minimum of a continuous positive function on a compact interval. Without loss of generality, we can assume the radius of the loops $\ell_i^{t_1}$ and $\ell_i^{t_2}$ is less than $D/2$.

First, consider the simpler case where $p_i(t) \in B_{D/2}(p_i(t_1)+\gamma(t)-\gamma(t_1))$ for all $t \in [t_1, t_2]$. If necessary, shrink D further such that the indicated ball does not contain 0 or any point $p_z(t)$ for $z \neq i$ and any $t \in [t_1, t_2]$. In this case, there is a basepoint-preserving homotopy of loops $H : I \times I \to (\widetilde{\pi}(V) - R)$ between $\mathrm{iso}(\ell_i(t_1))$ and $\ell_i(t_2)$. It is given in two steps. To define the first homotopy H_1 with domain $I \times [t_1, t_2]$, let c_t be the constant loop at $\gamma(t) - \gamma(t_1)$. Then, set

$$H_1(\bullet, t) = \overline{\gamma|_{[t,t_2]}} \cdot (\ell_i^{t_1} + c_t) \cdot \gamma|_{[t,t_2]}.$$

See Figure 5. At the end of this homotopy, $H_1(\bullet, t_2)$ is homotopic to the translated loop $\ell_i^{t_1} + c_{t_2}$ with circular portion of radius less than $D/2$ centered at $p_i(t_1)+\gamma(t_2)-\gamma(t_1)$, the ball $B_{D/2}(p_i(t_1)+\gamma(t_2)-\gamma(t_1)) \cap \mathcal{L}_{\theta(t_2)}$ contains $p_i(t_2)$, and it contains no other point $p_z(t_2)$. Note that $\ell_i^{t_1} + c_{t_2}$ and $\ell_i^{t_2}$ have images contained in $(\widetilde{\pi}(V) \cap \mathcal{L}_{\theta(t_2)}) - R \cong B - F$ for some closed ball $B \subseteq \mathbb{R}^2$ and a finite set F. It is subsequently straightforward to see that $\ell_i^{t_1} + c_{t_2}$ and $\ell_i^{t_2}$ are endpoint-preserving homotopic in $\widetilde{\pi}(V) - R$.

For the general case, notice that since $t \mapsto \|p_i(t) - (p_i(t') + \gamma(t) - \gamma(t_1))\|$ is continuous for fixed $t' \in [t_1, t_2]$, for every $t \in [t_1, t_2]$ there is an open neighborhood of t in $[t_1, t_2]$ for which the simple case applies. Since $[t_1, t_2]$ is compact, the first part of the lemma follows from finitely many applications of the simpler case. The map on actions given by $\widetilde{s}_i(t_2) \mapsto \widetilde{s}_i(t_1)$ for $i = 1, \ldots, k$ and induced by $[\ell_i(t_2)] \mapsto [\ell_i(t_1)]$ for $i = 1, \ldots, j$ is therefore an inverse to $\mathrm{Mon}^s_\gamma(t_1 \leq t_2)$. ☐

It follows directly that the inverse limit of a sliced monodromy functor Mon^s_γ exists and is isomorphic to $\mathrm{Mon}^s_\gamma(t)$ for any t in the domain of γ. We find the following observations useful.

Proposition 3.18. If $\gamma_1 : [0, a_1] \to \mathbb{C}^d$ and $\gamma_2 : [0, a_2] \to \mathbb{C}^d$ are limiting paths for the same monodromy representative, then $\varprojlim \mathrm{Mon}^s_{\gamma_1} \cong \varprojlim \mathrm{Mon}^s_{\gamma_2}$.

Proof. There exist some $t_1, t_2 \in (0, \min(a_1, a_2)]$ with $\|\gamma_1(t_1)\| \leq \|\gamma_2(t_2)\|$ since the paths are increasing in norm. Form a new limiting path $\gamma_3 : [0, a_3] \to \mathbb{C}^d$ that ends at $\gamma_2(t_2)$ and includes $\gamma_1(t_1)$ in its image. By Lemma 3.17, all the maps $\mathrm{Mon}^s_{\gamma_i}(t_1' \leq t_2')$ are isomorphisms for $i = 1, 2, 3$ with $t_1' \leq t_2'$ in the appropriate interval. The limits in question are therefore isomorphic to $\mathrm{Mon}^s_{\gamma_i}(t_i)$ for $i = 1, 2$, and there is an isomorphism between those defined by $\mathrm{Mon}^s_{\gamma_3}$. ☐

Proposition 3.19. Consider a monodromy representative with a limiting path restricted to $\widetilde{B_\epsilon}$ and B_ρ. Let $\gamma_1 : [0, a_1] \to \mathbb{C}^d$ be the original limiting path and $\gamma_2 = \gamma_1|_{[0,a_2]}$ be the restriction of the original limiting path so obtained. If $\mathrm{Mon}^r_{\gamma_2}$ is the (sliced or unsliced) monodromy functor of the restricted monodromy representative and Mon_{γ_2} for the unrestricted representative with path γ_2, there is a natural transformation $\mathrm{Mon}^r_{\gamma_2} \Rightarrow \mathrm{Mon}_{\gamma_2}$ induced by inclusion. The natural transformation induces a map $\varprojlim \mathrm{Mon}^r_{\gamma_2} \to \varprojlim \mathrm{Mon}_{\gamma_2}$.

Proof. This follows from a routine checking of definitions. To be somewhat more explicit, for any fixed t there is an inclusion $\widetilde{S}(t) \cap \widetilde{B}_\epsilon \subseteq \widetilde{S}(t)$ of the restricted fiber into the unrestricted fiber. Similarly, there is an inclusion on fundamental groups induced by the inclusion $B_\rho \subseteq B$, where $B \subseteq \mathbb{C}^d$ is the ball covered by the unrestricted monodromy representative. The indicated map between inverse limits is equal to this inclusion for any t up to isomorphism. ☐

3.3 Local monodromy actions of holomorphic subvarieties

We are now in position to state and prove our collection of main results, which will allow for computations with sliced monodromy functors rather than unsliced ones. Using the notation of Definition 3.1, we will assume throughout this subsection that limiting paths are restricted to codomain $\cap_{i=1}^{d-2} L_i$ for dimensions $d \geq 3$ in order to fulfill the requirements of the local Lefschetz-Zariski theorem in Lemma 2.9. If $d = 2$ the following results and proofs also hold replacing $\cap_{i=1}^{d-2} L_i$ with $\mathbb{C}^2$. If $d = 1$ the results also hold with simpler arguments that do not appeal to Lemma 2.9 and minor corresponding notational adjustments.

Theorem 3.20. Given any localized monodromy representative for a pure d-dimensional germ $\mathbf{V}$ of a holomorphic subvariety of $\mathbb{C}^N$ with limiting path $\gamma_1 : [0, a_1] \to \cap_{i=1}^{d-2} L_i$, the local monodromy action of this representative exists and is isomorphic to the sliced limit $\varprojlim \mathrm{Mon}^s_{\gamma_2}$ for some restriction $\gamma_2 = \gamma_1|_{[0,a_2]}$.

Proof. By Lemma 2.9 and the definition of a localized monodromy representative, there exists $\theta > 0$ such that, if $|L_{d-1}(\gamma_1(t))| < \theta$, then $\mathrm{Mon}^s_{\gamma_1}(t) \cong \mathrm{Mon}_{\gamma_1}(t)$. Since γ_1 is an increasing path, $|L_{d-1} \circ \gamma_1|$ is a continuous increasing function. It follows that there exists $a_2 > 0$ with $|L_{d-1}(\gamma_1(t))| < \theta$ for all $t \in [0, a_2]$. From Lemma 3.17 one immediately has that $\varprojlim \mathrm{Mon}^s_{\gamma_2}$ exists and is isomorphic to $\mathrm{Mon}^s_{\gamma_2}(t)$ for any $t \in [0, a_2]$. The result now follows from Proposition 3.18. □

Definition 3.21. The *local monodromy action* of a pure d-dimensional holomorphic germ $\mathbf{V}$ of an open subset of $\mathbb{C}^N$ is the local monodromy action of any localized monodromy representative for $\mathbf{V}$ along any limiting path γ. Define the *local monodromy group* of $\mathbf{V}$ similarly.

In principle, this definition depends on the generic choices of data used to construct a localized monodromy representative for $\mathbf{V}$, and we must check that different choices yield isomorphic local monodromy actions. The following corollary first justifies referring to *the* local monodromy action and group of a localized monodromy representative.

Corollary 3.22. For any localized monodromy representative with limiting paths

$$\gamma_1, \gamma_2 : [0, a_i] \to \cap_{i=1}^{d-2} L_i$$

for a pure d-dimensional germ $\mathbf{V}$ of a holomorphic subvariety of $\mathbb{C}^N$:

1. The local monodromy action is isomorphic to $\mathrm{Mon}^s_{\gamma_1}(t)$ for any $t \in (0, a_1]$.
2. The local monodromy actions defined by γ_1 and γ_2 are isomorphic.
3. The local monodromy actions defined by any two restrictions of the same localized monodromy representative for $\mathbf{V}$ with limiting path γ are isomorphic.

Proof. The first statement follows from Theorem 3.20 and Lemma 3.17. The second follows from Theorem 3.20 and Proposition 3.18. For the third statement, note that for any two restrictions of the same localized monodromy representative, there exists a common restriction of both with monodromy action Mon_1. Let $\mathrm{Mon}_2, \mathrm{Mon}_3$ be the monodromy actions of the other two restrictions. From the first statement, we know that there is some t_0 such that $\varprojlim \mathrm{Mon}_i \cong \mathrm{Mon}_i^s(t_0)$ for $i = 1, 2, 3$. We can further choose t_0 small enough that the corresponding fibers $\widetilde{S}_i(t_0)$ and $P_i(t_0)$ have the same number of points for $i = 1, 2, 3$. From Proposition 3.19, we have induced maps $\mathrm{Mon}_1^s(t_0) \to \mathrm{Mon}_i^s(t_0)$ for $i = 2, 3$, and one may observe that these are isomorphisms from their definition. □

It is also natural to ask whether the choice of generic projections $\widetilde{\pi}$ and π for the localized monodromy representative impact the action. Handling π is straightforward, but $\widetilde{\pi}$ takes some care since this determines the branch locus.

Proposition 3.23. The local monodromy actions of any two localized monodromy representatives which differ only in the linear projections $\pi^1, \pi^2 : \mathbb{C}^d \to \mathbb{C}^{d-1}$ have isomorphic local monodromy actions.

Proof. By Theorem 3.20, there is a limiting path γ for both monodromy representatives and $t_0 > 0$ where, denoting the corresponding sliced monodromy actions by Mon_i^s for $i = 1, 2$, one has that the corresponding local monodromy actions are isomorphic to $\mathrm{Mon}_i^s(t_0)$ for $i = 1, 2$, respectively. By standard parameter homotopy continuation results, e.g., see [24], given a generic path $\Gamma : [0, 1] \to \mathrm{Gr}(d-1, d)^{d-1}$, i.e., $\Gamma(T) = (L_1(T), \ldots, L_{d-1}(T))$, and denoting

$$\mathcal{L}_T := \cap_{i=1}^{d-2} \{L_i(T) = 0\} \cap \{L_{d-1}(T) = [L_{d-1}(T)](\gamma(t_0))\},$$

the set $\{R \cap \mathcal{L}_T\}_{T \in [0,1]}$ can be parameterized as a disjoint set of smooth paths $\delta_i : [0, 1] \to \mathbb{C}^d$ for $i = 1, \ldots, j$. If Γ starts at the hyperplanes defining π^1 and ends at those defining π^2, a similar loop-translating argument to Lemma 3.17 produces an isomorphism of group actions $\mathrm{Mon}_1^s(t_0) \to \mathrm{Mon}_2^s(t_0)$. This isomorphism is the identity on the fiber component. On the fundamental group component, it is induced by a map which takes a basic loop for $\delta_i(0)$ to a basic loop for $\delta_i(1)$ for each $i = 1, \ldots, j$. □

Proposition 3.24. The local monodromy actions of any two localized monodromy representatives which differ only in the linear projections $\widetilde{\pi}^1, \widetilde{\pi}^2 : \mathbb{C}^N \to \mathbb{C}^d$ have isomorphic local monodromy actions.

Proof. One can follow a similar argument as in Proposition 3.23, but now apply parameter homotopy continuation results, e.g., see [24], to a parameter homotopy between $\widetilde{\pi}^1$ and $\widetilde{\pi}^2$ so that the local monodromy actions along the path between them are isomorphic. □

Taken together, Corollary 3.22 and Propositions 3.23 and 3.24 show that the various generic choices made when forming a localized monodromy representative yield isomorphic local monodromy actions.

Theorem 3.25. The local monodromy actions of a pure d-dimensional holomorphic germ $\mathbf{V}$ of an open subset of $\mathbb{C}^N$ defined by any two sets of localized monodromy representative data are isomorphic.

Our last order of business is to use the structural results in this section to see how a monodromy representative, rather than localized monodromy representative, encodes information about the corresponding germ's local monodromy action. Recall from Remark 3.12 that the fiber points $\widetilde{S}(t)$ and critical points $P(t)$ limit to 0 (in $\mathbb{C}^N$ and $\mathbb{C}^d$ respectively) as $t \to 0$ for a localized monodromy representative. This observation provides a constructive way to filter "local" and "non-local" points in these sets.

Theorem 3.26. Given any monodromy representative which restricts to a localized monodromy representative with limiting path $\gamma_1 : [0, a_1] \to \cap_{i=1}^{d-2} L_i$ of a pure d-dimensional germ $\mathbf{V}$ of a holomorphic subvariety of $\mathbb{C}^N$, consider $\widetilde{S}^l(t)$ and $P^l(t)$ which are the subsets of fiber points and sliced branch points respectively for $t \in [0, a_1]$ with corresponding solution paths beginning at 0. Then, for any $t \in (0, a_1]$, the local monodromy action of $\mathbf{V}$ is isomorphic to the sub-action of $\mathrm{Mon}^s_{\gamma_1}(t)$ which is comprised of the subgroup generated by homotopy classes of basic loops around the points of $P^l(t)$ acting on the points of $\widetilde{S}^l(t)$.

Proof. By assumption, there is some restriction of the monodromy representative which is a localized monodromy representative for $\mathbf{V}$, say with restricted limiting path $\gamma_2 = \gamma_1|_{[0,a_2]}$, sliced monodromy action $\mathrm{Mon}^s_{\gamma_1}$ for the original representative, and (sliced) monodromy functor $\mathrm{Mon}^r_{\gamma_2}$ for the restriction of the monodromy representative. For any $t \in (0, a_2]$, one may observe directly by definition that the image of the arrow $\mathrm{Mon}^r_{\gamma_2}(t) \to \mathrm{Mon}^s_{\gamma_2}(t) = \mathrm{Mon}^s_{\gamma_1}(t)$ as in Proposition 3.19 is the described sub-action of $\mathrm{Mon}^s_{\gamma_1}(t)$ and that the arrow is monic. For $t \geq a_2$, compose the arrow $\mathrm{Mon}^r_{\gamma_2}(a_2) \to \mathrm{Mon}^s_{\gamma_1}(a_2)$ with the isomorphism $\mathrm{Mon}^s_{\gamma_1}(a_2 \leq t)$. □

4 Computing local monodromy actions

The theoretical results of Section 3 yield an approach for computing local monodromy actions. In particular, this theory shows that one can use analytic continuation to extend beyond the small enough neighborhood restriction for localizing monodromy computations. We will discuss the corresponding numerical local irreducible decomposition algorithm based on this theory in two parts: a main theoretical description followed by considerations when trying perform such computations.

4.1 Algorithm

In the following, we specialize the setup to the situation in Example 3.5. Recalling that notation, assume $f : \mathbb{C}^N \to \mathbb{C}^{N-d}$ is a system of polynomial equations defining a pure d-dimensional algebraic complete intersection $V = \mathcal{V}(f)$ and that V is reduced with respect to f. We assume that f is specified exactly as input. Similar to statements in the beginning of Section 3.3, the case of $d \geq 3$ is considered for notational convenience though the algorithm is easily adjusted for the $d = 2$ and $d = 1$ cases. Let $x^* \in V$ be a point, which we assume is specified exactly as input. We will not assume $x^* = 0$ in this section as this will be more convenient when discussing potential sources of difficulty with practical considerations.

Consider a linear projection map $\widetilde{\pi} : \mathbb{C}^N \to \mathbb{C}^d$ constructed by filling the entries of a $d \times N$ matrix uniformly at random, say with real and imaginary parts of entries between 0 and 1 to be definite. Then, with probability 1, $\widetilde{\pi}|_V$ is a global parametrization of V that (1) restricts to a local parameterization of V at x^* and (2) fulfills parameter homotopy continuation conditions. Let $\widetilde{R}$ denote the critical locus of $\widetilde{\pi}$ and $R = \widetilde{\pi}(\widetilde{R})$ denote its image. Choosing a linear projection $\pi : \mathbb{C}^d \to \mathbb{C}^{d-1}$ in a similar random fashion, one obtains with probability 1 a map with $\pi|_R$ fulfilling the same properties relative to R as $\widetilde{\pi}|_V$ fulfills relative to V. Translating notation to x^*, let L_i^* denote the hypersurface $\mathcal{V}(\pi_i(x) - \pi_i(\widetilde{\pi}(x^*))) \subseteq \mathbb{C}^d$ for all $i = 1, \ldots, d-2$.

The final generic choice to be made for localization is selecting a path $\gamma : [0,1] \to \cap_{i=1}^{d-2} L_i^*$. It suffices to choose a point $\gamma(1) \in B_1^* \cap_{i=1}^{d-2} L_i^*$ where $B_1^* \subseteq \mathbb{C}^d$ is the ball of radius 1 centered at $\widetilde{\pi}(x^*)$, and set $\gamma(t) = (1-t)\widetilde{\pi}(x^*) + t\gamma(1)$. The choice of radius 1 here is arbitrary, as any other positive radius would also suffice. With probability 1, this data can be used to construct a monodromy representative with limiting path for the germ of $\widetilde{\pi}|_V$ at x^* which restricts to a localized monodromy representative for the germ of V at x^* as defined in Section 3. Denote $\theta = \pi \circ \gamma$ and denote the complex line $\mathcal{L}^*_{\theta(1)} := \cap_{i=1}^{d-2} L_i^* \cap \pi_{d-1}^{-1}(\theta(1))$.

In the following algorithm, recall that CG denotes the critical point correspondence for $\widetilde{\pi}|_V$. That is, $CG := \{(\widetilde{v}, v) \mid (\widetilde{v}, v) \in \operatorname{graph}(\widetilde{\pi}|_V), \operatorname{rank} J(f, \widetilde{\pi})(\widetilde{v}) < N\}$. The image of the critical point locus of $\widetilde{\pi}|_V$, R, is the image of CG projected onto its second factor. It may also help to note that $(x^*, \widetilde{\pi}(x^*)) \in CG$.

Algorithm 2: NUMERICAL LOCAL IRREDUCIBLE DECOMPOSITION

Input : A polynomial system $f : \mathbb{C}^N \to \mathbb{C}^{N-d}$ defining a reduced complete intersection $V = V(f) \subseteq \mathbb{C}^N$ of dimension d

Input : A point $x^* \in V$.

Output: A numerical local irreducible decomposition of V at x^*.

1 Select a linear map $\widetilde{\pi} : \mathbb{C}^N \to \mathbb{C}^d$ uniformly at random;
2 Select a linear map $\pi : \mathbb{C}^d \to \mathbb{C}^{d-1}$ uniformly at random;
3 Select $\gamma(1) \in B_1^* \cap_{i=1}^{d-2} L_i^*$ uniformly at random and set $\gamma(t) = (1-t)\widetilde{\pi}(x^*) + t\gamma(1)$ for $t \in [0,1]$. Denote $\theta := \pi \circ \gamma$;
4 Compute a local witness point set $\widetilde{W^l}$ for V at x^* using Algorithm 1 with inputs $\widetilde{\pi}, \gamma$;
5 Compute a local witness point set W' for CG at $(x^*, \widetilde{\pi}(x^*))$ using Algorithm 1 with inputs $(\widetilde{\pi}, \pi)$ and path $(\gamma, \theta) : [0,1] \to \mathbb{C}^d \times \mathbb{C}^{d-1}$;
6 Set W^l equal to $\{v\}_{(\widetilde{v}, v) \in W'}$, which forms a local witness point set for R at $\widetilde{\pi}(x^*)$;
7 Compute the partition $\widetilde{W^l} = \widetilde{w}_1 \coprod \widetilde{w}_2 \coprod \cdots \coprod \widetilde{w}_k$ given by the monodromy action on $\widetilde{W^l}$ through $\widetilde{\pi}$ of basic loops in $\mathcal{L}^*_{\theta(1)}$ based at $\gamma(1)$ encircling the points of W^l ;
8 `Return` the sets $\{f, \widetilde{\pi}, \widetilde{w}_i\}$ for $i = 1, \ldots, k$;

Theorem 4.1. Assuming generic choices for $\widetilde{\pi}$, π, and γ in Algorithm 2, the output of the algorithm is a numerical local irreducible decomposition of the germ of V at x^*.

Proof. We continue with notation as in Algorithm 2. By Theorem 3.26, the local monodromy action of $\mathbf{V}$, the germ of V at x^*, is isomorphic to the monodromy action on $\widetilde{W^l}$ by the subgroup of homotopy classes of based loops at $\gamma(1)$ generated by the set of homotopy classes of basic loops encircling the points of W^l contained in $\mathcal{L}^*_{\theta(1)}$. By Lemma 2.1 and Corollary 3.22 the orbits of the local monodromy action of $\mathbf{V}$ are in bijection with the local irreducible components of $\mathbf{V}$. It is straightforward to observe by definition that the orbits of $\widetilde{W^l}$ so computed are local witness point sets for their corresponding local irreducible components. □

4.2 Practical considerations

There are two distinct portions of Algorithm 2 which are relevant for further discussion when performing computations. The first consists of the local witness point computations in lines 4 and 5. The second is the computation of monodromy actions from lifting basic loops in line 7. Several software packages are able to perform the homotopy continuation computations in the key parts of Algorithms 1 and 2. These include `Bertini` [3], `HomotopyContinuation.jl` [10], and NAG4M2 [22]. We focus our discussion here on practical considerations which are unique to our algorithm rather than those faced by homotopy continuation methods in general.

First, note that the input polynomial system $f : \mathbb{C}^N \to \mathbb{C}^{N-d}$ is assumed in Algorithm 2 to be provided in exact form, i.e., the coefficients of the corresponding polynomials are specified exactly. This is an essential assumption unless $x^* \in V$ is nonsingular. Even small generic perturbations to the coefficients of f will result in nonsingular $\mathcal{V}(f)$.

The point $x^* \in V$ is also assumed to be exact for convenience as the localization procedure in Algorithm 1 must determine which solution paths converge to x^* and which do not. Moreover, when x^* is singular, which is the case of interest for computing a local irreducible decomposition, the Jacobian matrix is ill-conditioned in a neighborhood of a singular point and so care needs to be taken when performing path tracking nearby. Adpative precision path tracking [4] adjusts the precision based on the local conditioning to ensure enough digits are being used to accurately track the path. Endgames (see [25, Chap. 10] and [5, Chap. 3]) can be employed to accurately determine the endpoint of paths. To the best of our knowledge, no certification procedures exist for certifying the output Algorithm 1 is correct. More precisely, one expects in practice to be able to certify that some of the solution paths computed in Algorithm 1 do not limit to x^*, but not necessarily to be able to perform certification for all paths which have this property. We must always decide some small proximity or numerical threshold at which we will classify distinct solution paths as having converged to x^* rather than a different point that can be made robust using adpative precision and endgames.

The main numerical difficulty in Algorithm 2, apart from localization, is in using homotopy continuation to compute the monodromy action in line 7. In practice, this often requires tracking paths which pass near the critical locus of $\widetilde{\pi}$. See, e.g., Figure 7 in the next section. This is precisely a situation where numerical methods may be poorly conditioned. In particular, consider the critical points in the *global* witness point set W^g for R from which the local witness point set W^l is computed using Algorithm 1. If the points in W^g cluster closely together, the corresponding basic loops encircling points from W^l must necessarily pass close to the critical locus. The advantage here is that the solution paths are generically nonsingular. Hence, adaptive precision path tracking [4] with large enough precision will succeed and can be certified if desired. Options for certification include certified path tracking, e.g., see [7, 18], and *a posteriori* certification of heuristic path tracking [17]. Alternatively, it can often be computationally less expensive in practice to simply try again with different random choices for $\widetilde{\pi}$, π, and γ.

Restricting the input of Algorithm 2 to complete intersections is not strictly necessary, but is stated this way for simplicity of presentation. There are standard techniques used in numerical algebraic geometry, e.g., randomization and Bertini's theorem [25, §A.9], for reducing to this case. Note that such reduction techniques simply add pre- and post-processing steps which do not change the core procedure.

Finally, we remark that in practice there are two possible algebraic formulations for capturing the rank vanishing condition rank $J(f, \widetilde{\pi}) < N$ appearing in the definition of the critical point set CG. The matrix $J(f, \widetilde{\pi})$ is an $N \times N$ square matrix with d constant rows corresponding to $\widetilde{\pi}$, so its rank vanishing is captured by setting the determinant equal to 0. When N and the degree of f is modest, this works well. An alternative is to use a null space formulation [2] which adds the cost of additional variables that capture the null space to avoid computing large determinants.

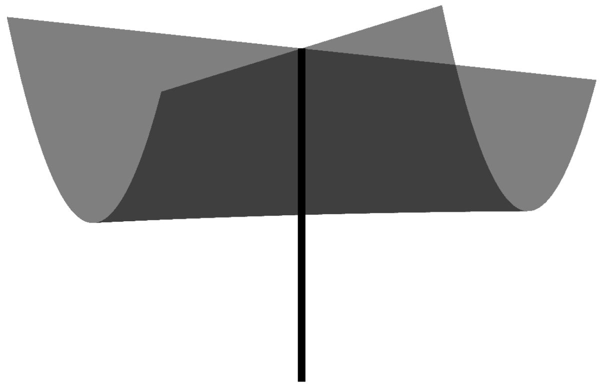


Figure 6: Illustration of the the Whitney umbrella.

5 Examples

We conclude with several examples of computing a numerical local irreducible decomposition using an implementation of Algorithm 2 available at https://github.com/P-Edwards/LocalMonodromy.jl.[3] This implementation uses `HomotopyContinuation.jl` [10] for path tracking without certification. Computation times are reported with homotopy continuation parallelized using an Intel Core i7-920 CPU with 8 CPU threads. Memory requirements were less than 2GB. To fix some convenient terminology from Algorithms 1 and 2, call the number of points in a global witness point set for V the *global fiber degree* of V, $|\widetilde{w}_i|$ the *local fiber degree* of each locally irreducible component V_i, the number of points in a global witness point set for R the *global branch degree*, and the number of points in a local witness point set for R at $\widetilde{\pi}(x^*)$ the *local branch degree*.

Example 5.1. The Whitney umbrella $V \subseteq \mathbb{C}^3$ is the surface defined by $x_1^2 - x_3x_2^2 = 0$ and has singular points along the line $x_1 = x_2 = 0$ as illustrated in Figure 6. At a point $x^* = (0, 0, \kappa)$ with $\kappa \neq 0$, there is a nontrivial factorization $(x_1 - x_2\sqrt{x_3})(x_1 + x_2\sqrt{x_3}) = 0$ in the local ring of holomorphic germs at x^*, so one expects V to be locally reducible with two locally irreducible components at x^*. When $\kappa = 0$, this factorization is not available since there is no holomorphic inverse to $z \mapsto z^2$ in a neighborhood of 0. One therefore expects V to be locally irreducible at the origin.

Algorithm 2 at the origin computes one local irreducible component of local fiber degree 2, global fiber degree 3, local branch degree 2, and global branch degree 4. The results at $x^* = (0, 0, -1)$ were similar except having two local irreducible components, each having local fiber degree 1. In these cases, the computations required took approximately 22 seconds.

This example is notable for having a critical point locus with an unreduced irreducible component, namely the line $x_1 = x_2 = 0$ which is often called the "handle" of the Whitney umbrella. It is unreduced in the sense of having generic multiplicity greater than 1. A standard approach in numerical algebraic geometry to perform computations on such components is to *deflate* the component first described in [25, §10.5] (see also [20]). Our experiments include a deflation step for this and other unreduced examples.

[3] A static version of the package together with files suitable for reproducing the examples is available at https://doi.org/10.5281/zenodo.14532556.

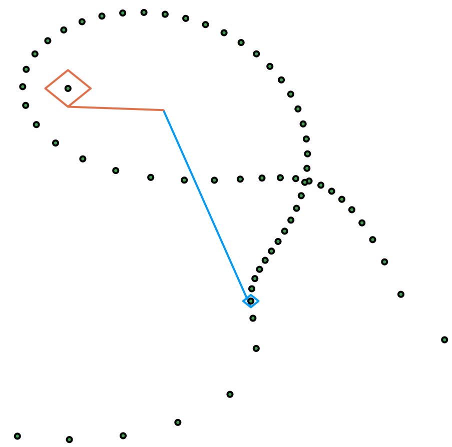


Figure 7: Branch points and corresponding localized monodromy loops of the branch locus intersected with a complex line, identified with $\mathbb{R}^2$, for the Brieskorn manifold with global fiber degree 174. For illustration purposes, only 65 of 174 global branch points are depicted.

Example 5.2. Table 1 collects data arising from various globally irreducible hypersurfaces which have an isolated singularity at the origin. In particular, the last one is related to the construction of a so-called Brieskorn manifold [11] with global branch degree 174. Figure 7 shows how increasing degree can complicate the clustering pattern of branch points.

Equation	Local fiber degrees	Global fiber degree	Local branch degree	Global branch degree
$x^2 + (y-1)y^2 = 0$	1,1	3	1	3
$(3x + y + 2z)^2 x^3 + (x-1)(y+z)^3 = 0$	1,2	5	1	10
$xy - z^3 = 0$	2	3	2	4
$x^2 + y^2 + z^2 = 0$	2	2	2	2
$x^2 + y^2 + z^2 + w^2 = 0$	2	2	2	2
$z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{59} = 0$	2	59	2	174

Table 1: Summary of results for several hypersurfaces at the origin.

Example 5.3. Our final example arises in kinematics and studied in [12, §8.1] arising from a coupler curve of a four-bar linkage. In particular, for the polynomial system

$$f(x,y,u,v,a,b,c,d) = \begin{bmatrix} x^2 + y^2 - a^2 \\ (u-b)^2 + v^2 - c^2 \\ (x-u)^2 + (y-v)^2 - d^2 \end{bmatrix},$$

the set $V = V(f) \subset \mathbb{C}^8$ is irreducible of codimension 3. The variables a, b, c, d are mechanical parameters of the four-bar linkage while x, y, u, v describe the coupler curve of the resulting four-bar linkage. We consider the local irreducible decomposition of V at the origin. Here, the critical point locus has dimension 4. The computation found that the origin is locally irreducible with local degree 8 and both the global and local branch degrees were 24. Moreover, the corresponding local monodromy group was computed to be the entire symmetric group on the 8 fiber points. In total, this computation took approximately 45 seconds.

6 Conclusion

This paper introduced a theory of local monodromy actions for germs of holomorphic subvarieties, their behavior under continuation, algorithms which leverage that theory to compute numerical local irreducible decompositions, and an open source software implementation for doing so in the algebraic case. Several examples are used to demonstrate this novel theory for computing local monodromy actions and numerical local irreducible decompositions. We conclude with brief thoughts on possible future extensions.

First, certification of routines and robustness in the style of [7] of the numerical local irreducible decomposition should be considered.

As discussed in the Introduction, a numerical local irreducible decomposition at a point $x^* \in V$ with V algebraic encodes coarse stratified topological information about V at x^*. More precisely, the number of d-dimensional local irreducible components at x^* is the minimum number of d-dimensional strata local to x^* in any stratification. There are natural questions which correspondingly arise about assigning and computing local monodromy actions for higher dimensional objects. For instance, if V is d-dimensional and there exists a stratification of V with x_1^*, x_2^* in the same $(d-1)$-dimensional stratum, one expects the local monodromy actions at those points to be isomorphic. Numerical computations related to these types of questions should now be possible, e.g., by computing maps between the corresponding local monodromy actions with homotopy continuation.

It is also natural to consider whether and how our theory could be adapted to the case of real varieties, as these are of the most interest in engineering and science applications. There is a version of the local parametrization theorem for germs of real holomorphic varieties, so elements of the theory in Section 3 translate to the real case without significant modification. Obstructions do arise, however. For instance, slicing a real holomorphic set with a complementary linear space near a point does not generally yield points on representatives for each locally irreducible component. Nevertheless, our monodromy functor formalism provides a framework for probing these questions in the context of numerical computations.

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References

[1] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vasilev. *Singularity Theory. I*. Springer-Verlag, Berlin, english edition, 1998. Translated from the 1988 Russian original by A. Iacob. 1

[2] D. J. Bates, J. D. Hauenstein, C. Peterson, and A. J. Sommese. Numerical decomposition of the rank-deficiency set of a matrix of multivariate polynomials. In *Approximate commutative algebra*, Texts Monogr. Symbol. Comput., pages 55–77. SpringerWienNewYork, Vienna, 2009. 7, 22

[3] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. Bertini: Software for numerical algebraic geometry. Available at bertini.nd.edu with permanent doi: dx.doi.org/10.7274/R0H41PB5. 21

[4] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. Adaptive multiprecision path tracking. *SIAM Journal on Numerical Analysis*, 46(2):722–746, 2008. 2, 22

[5] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. *Numerically Solving Polynomial Systems With Bertini*, volume 25 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013. 2, 7, 22

[6] D. J. Bates, C. Peterson, A. J. Sommese, and C. W. Wampler. Numerical computation of the genus of an irreducible curve within an algebraic set. *J. Pure Appl. Algebra*, 215(8):1844–1851, 2011. 7

[7] C. Beltrán and A. Leykin. Robust certified numerical homotopy tracking. *Found. Comput. Math.*, 13(2):253–295, 2013. 22, 25

[8] P. Bendich, B. Wang, and S. Mukherjee. Local homology transfer and stratification learning. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '12, page 1355–1370. Society for Industrial and Applied Mathematics, 2012. 1

[9] D. A. Brake, J. D. Hauenstein, and A. J. Sommese. Numerical local irreducible decomposition. In I. S. Kotsireas, S. M. Rump, and C. K. Yap, editors, *Mathematical Aspects of Computer and Information Sciences*, pages 124–129, Cham, 2016. Springer International Publishing. 2, 6, 7, 8, 9

[10] P. Breiding and S. Timme. HomotopyContinuation.jl: A package for homotopy continuation in Julia. In J. H. Davenport, M. Kauers, G. Labahn, and J. Urban, editors, *Mathematical Software – ICMS 2018*, pages 458–465, Cham, 2018. Springer International Publishing. 21, 23

[11] E. Brieskorn. Beispiele zur Differentialtopologie von Singularitäten. *Invent. Math.*, 2:1–14, 1966. 24

[12] E. R. Cobian, J. D. Hauenstein, and C. W. Wampler. Robust numerical algebraic geometry. *arXiv preprint arXiv:2403.18749*, 2024. 2, 24

[13] C. Fevola, S. Mizera, and S. Telen. Landau singularities revisited: Computational algebraic geometry for Feynman integrals. *Phys. Rev. Lett.*, 132:101601, 2024. 1

[14] R. C. Gunning. *Introduction to holomorphic functions of several variables. Vol. II.* The Wadsworth & Brooks/Cole Mathematics Series. Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1990. Local theory. 2, 6, 10

[15] H. A. Hamm and Lê Dũng Tráng. Un théorème de Zariski du type de Lefschetz. *Ann. Sci. École Norm. Sup. (4)*, 6:317–355, 1973. 3, 6, 10

[16] W. Hao, J. D. Hauenstein, M. H. Regan, and T. Tang. A numerical method for solving elliptic equations on real closed algebraic curves and surfaces. *J. Sci. Comput.*, 99(2):Paper No. 56, 22, 2024. 1

[17] J. D. Hauenstein, I. Haywood, and A. C. Liddell. An a posteriori certification algorithm for Newton homotopies. In *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, ISSAC '14, page 248–255, New York, NY, USA, 2014. Association for Computing Machinery. 22

[18] J. D. Hauenstein and A. C. Liddell. Certified predictor-corrector tracking for Newton homotopies. *Journal of Symbolic Computation*, 74:239–254, 2016. 22

[19] J. D. Hauenstein, J. I. Rodriguez, and F. Sottile. Numerical computation of Galois groups. *Found. Comput. Math.*, 18(4):867–890, 2018. 3, 7, 10

[20] J. D. Hauenstein and C. W. Wampler. Isosingular sets and deflation. *Found. Comput. Math.*, 13(3):371–403, 2013. 23

[21] M. Helmer and V. Nanda. Conormal spaces and Whitney stratifications. *Found. Comput. Math.*, 23(5):1745–1780, 2023. 1

[22] A. Leykin. Numerical algebraic geometry. *J. Softw. Algebra Geom.*, 3:5–10, 2011. 22

[23] U. Lim, H. Oberhauser, and V. Nanda. Hades: Fast singularity detection with local measure comparison. *arXiv preprint arXiv:2311.04171*, 2023. 1

[24] A. P. Morgan and A. J. Sommese. Coefficient-parameter polynomial continuation. *Appl. Math. Comput.*, 29(2):123–160, 1989. 8, 11, 13, 19

[25] A. J. Sommese and C. W. Wampler. *The Numerical Solution Of Systems Of Polynomials Arising In Engineering And Science.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. 2, 7, 22, 23

[26] O. Zariski. A theorem on the Poincaré group of an algebraic hypersurface. *Ann. of Math. (2)*, 38(1):131–141, 1937. 3, 10