Mixed volume and ramification points of polyhedral homotopies

Joseph Cummings[1][0000-0003-3984-0579] and

Jonathan D. Hauenstein[2][0000-0002-9252-8210]

[1] University of Edinburgh, Edinburgh EH9 3FD, UK jcummin4@ed.ac.uk
https://sites.google.com/view/josephcummingsmath/

[2] University of Notre Dame, Notre Dame, IN 46556, USA hauenstein@nd.edu
https://hauenstein.phd

Abstract. Numerical algebraic geometry utilizes homotopy continuation to solve polynomial systems via numerically tracking solution paths. The homotopies of interest here are polyhedral homotopies which arise between two polynomial systems with the same Newton polytopes. A singularity along a solution path of a homotopy is a ramification point. Building on previous work of ramification points of homotopies, this paper shows that the number of ramification points for polyhedral homotopies can be described in terms of mixed volumes. This interpretation utilizes a nef divisor which can be determined by solving a linear program. Software is described for performing such computations and utilized to demonstrate the new results.

Keywords: ramification point · polyhedral homotopy · mixed volume · homotopy continuation · numerical algebraic geometry · singularity.

1 Introduction

Path tracking in homotopy continuation is used in numerical algebraic geometry to compute solutions to system of polynomial equations (see the books [1,16] for more details). A standard assumption is that each path is trackable (see [9, Defn. 4.5]) which, for a nonsingular start point, means that one does not encounter a singularity along the path until possibly at the endpoint. Such singularities are called ramification points. Although properly constructed homotopies ensure that every path is trackable, there is a natural question regarding how many ramification points can arise in a homotopy. This question was answered in [15, Thm. 18] in terms of vector bundles, the canonical bundle, and Chern classes with special cases and their distribution considered in [8,15].

The following considers ramification points for polyhedral homotopies. To that end, for $p \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$, let $\mathrm{New}(p) \subset \mathbb{R}^N$ be the Newton polytope of p and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. A polyhedral homotopy $H : (\mathbb{C}^\times)^N \times \mathbb{C} \to \mathbb{C}^N$ has the form

$$H(x;t) = (1-t)f(x) + tg(x) \tag{1}$$

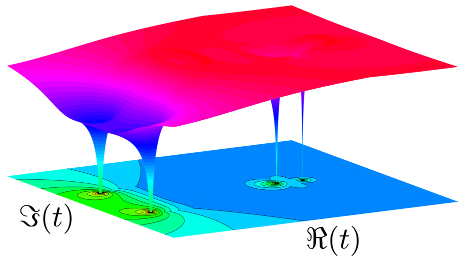

Fig. 1: Plot of the distance between the two solutions as a function of the real $\Re(t)$ and imaginary $\Im(t)$ parts of $t \in \mathbb{C}$ along with contours which suggests that H in (2) has four ramification points.

such that $\mathrm{New}(f_j) = \mathrm{New}(g_j)$ for $j = 1, \ldots, N$, that is, f and g have the same Newton polytopes. The number of paths to track for a polyhedral homotopy H is bounded above by the mixed volume of the Newton polytopes [2,10] with many algorithms available for computing mixed volume, e.g., see [3,5,11,13,17].

Example 1. To illustrate, consider the homotopy

$$H(x;t) = (1-t)\begin{bmatrix} x_1x_2 - 2 \\ -4x_1x_2^2 + 5x_1 + x_2 - 4 \end{bmatrix} + t\begin{bmatrix} 5x_1x_2 - 4 \\ 5x_1x_2^2 - 5x_1 + 4x_2 + 2 \end{bmatrix} \quad (2)$$

where

$$P_1 = \mathrm{New}(f_1) = \mathrm{New}(g_1) = \mathrm{conv}(\{(1,1),(0,0)\}),$$
$$P_2 = \mathrm{New}(f_2) = \mathrm{New}(g_2) = \mathrm{conv}(\{(1,2),(1,0),(0,1),(0,0)\}),$$

with conv() denoting the convex hull. Since $\mathrm{MV}(P_1, P_2) = 2$, the homotopy H defines two solution paths. Figure 1 plots the distance between the two solution paths over $t = \Re(t) + \Im(t)\sqrt{-1}$ suggesting that H has four ramification points.

The main theoretical result (Theorem 3) is that the number of ramification points for a polyhedral homotopy is also bounded above in terms of mixed volumes. In particular, Theorem 3 depends upon having a suitable nef divisor in which Theorem 4 shows how to compute a suitable ample divisor via solving a linear program. We have developed software for performing such computations.

The rest of the paper is structured as follows. Section 2 derives the theoretical underpinnings. Section 3 provides a short summary of the software along with an example. A short conclusion is provided in Section 4.

2 Counting ramification points for polyhedral homotopies

The following is the theoretical foundation derived from [15, Thm. 18] with the rest of this section interpreting this result in terms of mixed volumes allowing one to sidestep the need to perform computations in an intersection ring.

Theorem 1 (Rephrasing of Theorem 18 of [15]). *Let $\mathcal{E}$ be a basepoint free vector bundle over an N-dimensional smooth toric variety X_Σ and let V be the space of global sections of $\mathcal{E}$. Assume there is at least one section s of $\mathcal{E}$ with at least one isolated solution. Then, the number of ramification points of a general pencil $\ell \in \mathbb{P}(V^*)$ is given by*

$$(c_1(K_{X_\Sigma}) + c_1(\mathcal{E}))c_{N-1}(\mathcal{E}) + Nc_N(\mathcal{E}) \tag{3}$$

where K_{X_Σ} is the canonical bundle of X_Σ and $c_k(\mathcal{A})$ is the k^{th} Chern class of $\mathcal{A}$. Moreover, if $\mathcal{E} \cong \bigoplus_{j=1}^N \mathcal{L}_j$ where $\operatorname{rk}(\mathcal{L}_j) = 1$, then (3) simplifies to

$$\left(c_1(K_{X_\Sigma}) + \sum_{j=1}^N c_1(\mathcal{L}_j) \right) \sum_{j=1}^N \prod_{i \neq j} c_1(\mathcal{L}_i) + N \prod_{j=1}^N c_1(\mathcal{L}_j). \tag{4}$$

The following provides the key relationship connecting intersection products with mixed volume suggesting that (4) can be written in terms of mixed volumes.

Theorem 2. *Let X_Σ be a (smooth) toric variety of dimension N. If $D_1, \ldots, D_N$ are nef divisors with associated polytopes $P_1, \ldots, P_N$, respectively, then*

$$\prod_{j=1}^N c_1(D_j) = \operatorname{MV}(P_1, \ldots, P_N). \tag{5}$$

Proof. To each divisor D_j, let $L_j = H^0(X_\Sigma, \mathcal{O}(D_j))$ be the space of global sections. Since P_j is the polytope associated with D_j, L_j is spanned by the Laurent monomials t^α where $\alpha \in P_j$. The number of solutions to a generic system $f_1 = \cdots = f_N = 0$ where each $f_j \in L_j$ is $\operatorname{MV}(P_1, \ldots, P_N)$ [2] which, by [12, Thm. 6.8], agrees with the intersection product $\prod_{i=1}^N c_1(D_j)$.

Armed with this theoretical foundation, we now consider a polyhedral homotopy $H : (\mathbb{C}^\times)^N \times \mathbb{C} \to \mathbb{C}^N$ as in (1) such that $f, g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]^N$ where $\operatorname{New}(f_j) = \operatorname{New}(g_j) = P_j$ for $j = 1, \ldots, N$. To avoid trivialities, we assume that $\operatorname{MV}(P_1, \ldots, P_N) > 0$. As is standard in homotopy continuation, we assume that g is general in that the number of nonsingular solutions of $g = 0$ in $(\mathbb{C}^\times)^N$ is $\operatorname{MV}(P_1, \ldots, P_N)$. Hence, for all but finitely many $\theta \in [0, 2\pi)$, $H(x; t(s)) = 0$ defines $\operatorname{MV}(P_1, \ldots, P_N)$-many trackable solution paths in $(\mathbb{C}^\times)^N \times (0, 1]$ where

$$t(s) = \frac{\gamma s}{1 - s + \gamma s} \quad \text{with} \quad \gamma = e^{\theta\sqrt{-1}}, \tag{6}$$

e.g., see [16, Thm. 8.3.1]. This is called the gamma trick and ramification points arise at the failures. That is, $(x^*, t^*) \in (\mathbb{C}^\times)^N \times \mathbb{C}^\times$ is a ramification point of the polyhedral homotopy H if $H(x^*, t^*) = 0$ and $\det J_x H(x^*, t^*) = 0$ where $J_x H(x^*, t^*)$ is the Jacobian matrix of H with respect to x evaluated at (x^*, t^*).

Example 2. For H in (2), the system $\{H(x; t), \det J_x H(x; t)\} = 0$ has four solutions $(x, t) \in (\mathbb{C}^\times)^2 \times \mathbb{C}^\times$ which, to four decimal places, are:

$$(5.4946, 0.1832, 0.4904), \quad (0.5632, 2.0767, 0.3100),$$
$$(0.3378 \pm 0.0092\sqrt{-1}, -0.1689 \mp 0.9815\sqrt{-1}, -0.7497 \pm 0.3037\sqrt{-1})$$

in accordance with Figure 1.

We now connect a polyhedral homotopy H as above with Theorems 1 and 2.

Lemma 1. *Let $Q = P_1 + \cdots + P_N$ be the Minkowski sum of the polytopes and Σ_Q be the inner normal fan of Q. Let Σ be a smooth fan refining Σ_Q so that X_Σ is projective. For $j = 1, \ldots, N$, consider the divisor*

$$D_{P_j} = \sum_{\rho \in \Sigma(1)} \left(-\min_{x \in P_j} \langle x, u_\rho \rangle\right) D_\rho$$

where u_ρ is the first lattice point on the ray $\rho \in \Sigma(1)$ and D_ρ is the associated toric divisor. Let $\mathcal{L}_j = \mathcal{O}(D_{P_j})$ be the associated line bundle on X_Σ with rank N vector bundle $\mathcal{E} = \bigoplus_{j=1}^N \mathcal{L}_j$ on X_Σ. Then, the divisors $D_{P_1}, \ldots, D_{P_N}$ are nef so that $\mathcal{E}$ is globally generated. Moreover, H corresponds to a pencil of $\mathcal{E}$.

Proof. The fan Σ can always be chosen so that X_Σ is projective, e.g. one can take a regular triangulation of Σ_Q. Now, for each j, the fan Σ refines the inner normal fan of P_j. For a vertex v of P_j, let σ_v be the corresponding maximal cone of Σ_{P_j}. By construction, D_{P_j} is a Cartier divisor with Cartier data $\{(U_\sigma, t^{m_\sigma})\}_{\sigma \in \Sigma(N)}$ where $m_\sigma = v$ is the vertex of P_j such that $\sigma \subseteq \sigma_v$. Clearly, $m_\sigma \in P_j$ so that D_{P_j} is nef (see [4, Theorem 6.1.10]). Since each summand of $\mathcal{E}$ is nef, it follows that $\mathcal{E}$ is globally generated. In fact, a global section of $\mathcal{E}$ is a vector-valued function whose components are global sections of $\mathcal{L}_j$, which, by construction, the Newton polytope of each of these Laurent polynomials is P_j. In this way, we can view H as a pencil of $\mathcal{E}$.

With this setup, the following is the main theoretical result.

Theorem 3. *Let A be a nef divisor on X_Σ where $\widetilde{A} = A + K_{X_\Sigma} + \sum_{j=1}^N D_{P_j}$ is also nef. Let P_A and $P_{\widetilde{A}}$ be the polytopes associated with A and $\widetilde{A}$, respectively. Then, an upper bound on the number of ramification points for H is*

$$\sum_{j=1}^N \left(\mathrm{MV}(P_{\widetilde{A}}, P_1, \ldots, \widehat{P_j}, \ldots, P_N) - \mathrm{MV}(P_A, P_1, \ldots, \widehat{P_j}, \ldots, P_N)\right) + N \cdot \mathrm{MV}(P_1, \ldots, P_N) \quad (7)$$

where $\widehat{P_j}$ means that it is removed. That is, the first expression replaces P_j with $P_{\widetilde{A}}$ while the second replaces P_j with P_A.

Proof. The expression in (7) immediately follows from Lemma 1, Theorems 1 and 2, and the multi-linearity and symmetry of the intersection product.

Remark 1. If $K_{X_\Sigma} + \sum_{j=1}^N D_{P_j}$ is nef, then $A = 0$ and $\widetilde{A} = K_{X_\Sigma} + \sum_{j=1}^N D_{P_j}$ satisfy the assumptions of Theorem 3 so that, using multi-linearity, (7) becomes

$$\sum_{j=1}^N \mathrm{MV}(P_1, \ldots, P_{j-1}, P_{\widetilde{A}} + P_j, P_{j+1}, \ldots, P_N).$$

Example 3. Directly solving in Example 2 showed that H in (2) has 4 ramification points. The following uses mixed volume computations.

Let Σ be the inner normal fan of $P_1 + P_2$ where P_1 and P_2 are listed in Example 1. In this case, Σ is smooth so there is no need to refine Σ any further. The ray generators of Σ are given as columns in the matrix below.

$$\Sigma(1) = \begin{pmatrix} -1 & 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 & 1 \end{pmatrix}$$

We denote the i^{th} column of $\Sigma(1)$ by ρ_i, and we let D_i denote the corresponding toric divisor. One can check that $\operatorname{Pic}(X_\Sigma) \cong \mathbb{Z}^3$ and $[D_2], [D_4]$, and $[D_5]$ form a basis. In particular, we have

$$\begin{aligned} D_{P_1} &= D_1 \sim D_2 + D_4, \\ D_{P_2} &= D_1 + D_3 + D_5 \sim D_2 + 2 \cdot D_4 + 2 \cdot D_5, \\ K_{X_\Sigma} &= -\sum_{i=1}^{5} D_i \sim -2 \cdot D_2 - 3 \cdot D_4 - 2 \cdot D_5. \end{aligned}$$

Hence, the divisor $D_{P_1} + D_{P_2} + K_{X_\Sigma}$ is linearly equivalent to 0 and is thus nef. Using the notation from Theorem 3, we can therefore take $A = 0$. Moreover, since $\widetilde{A} \sim 0$, all the mixed volumes involving A and $\widetilde{A}$ vanish. Hence, an upper bound on the number of ramification points simplifies to just

$$2 \cdot \operatorname{MV}(P_1, P_2) = 2 \cdot 2 = 4.$$

in agreement with our earlier computation.

Since $(\mathbb{C}^\times)^N$ is not closed, some of the ramification points can lie on the closure and thus (7) can be a strict upper bound of ramification points in $(\mathbb{C}^\times)^N \times \mathbb{C}^\times$ even when both f and g are generic as shown in the following.

Example 4. For general coefficients $a_j, b_k \in \mathbb{C}$, consider the polyhedral homotopy

$$H(x;t) = (1-t) \begin{bmatrix} a_1x_1 + a_2x_2 + a_3 \\ a_4x_1x_2 + a_5 \end{bmatrix} + t \begin{bmatrix} b_1x_1 + b_2x_2 + b_3 \\ b_4x_1x_2 + b_5 \end{bmatrix}$$

where

$$P_1 = \operatorname{conv}\{(1,0),(0,1),(0,0)\}, P_2 = \operatorname{conv}(\{(1,1),(0,0)\}), \text{ and } \operatorname{MV}(P_1,P_2) = 2.$$

Following a similar computation as in Example 3, the computed upper bound on the number of ramification points derived from Theorem 3 is 4. However, this can observed to be a strict upper bound by considering

$$\det J_x H(x;t) = ((1-t)a_4 + tb_4)((1-t)(a_1x_1 - a_2x_2) + t(b_1x_1 - b_2x_2)). \quad (8)$$

When the first factor of (8) vanishes, namely when $t = a_4/(a_4 - b_4)$, the second polynomial in H reduces to $(1-t)a_5 + tb_5$ which does not vanish generically. In particular, as t approaches $a_4/(a_4 - b_4)$, the two solution paths of H diverge to a common point at infinity. The second factor in (8) yields 3 ramification points.

In order to utilize Theorem 3 in computations, one needs to ensure that such a nef divisor A always exists along with a method for computing it. The following shows that one can even ensure that both A and $\widetilde{A}$ are ample.

Theorem 4. *There exists an ample divisor A on X_Σ so that $A+K_{X_\Sigma}+\sum_{j=1}^N D_j$ is also ample. Moreover, such a divisor A can be computed via a linear program.*

Proof. We first need to find an interior lattice point in the nef cone, denoted $\mathrm{Nef}(X_\Sigma) \subseteq \mathrm{Pic}(X_\Sigma) \otimes \mathbb{R}$. Such a point is guaranteed to exist since X_Σ is projective. Also, since X_Σ is smooth, $\mathrm{Pic}(X_\Sigma)$ is isomorphic to $\mathbb{Z}^{|\Sigma(1)|-N}$, we can fix a basis $[E_1], \ldots, [E_{|\Sigma(1)|-N}]$ of $\mathrm{Pic}(X_\Sigma)$. Let W be a matrix whose rows are indexed by the torus-invariant curves of X_Σ and whose columns are indexed by the chosen basis of $\mathrm{Pic}(X_\Sigma)$. Fix an ordering on the maximal cones of Σ. Each torus-invariant curve C can now be written as $V(\tau)$ with $\tau = \sigma \cap \sigma'$ where σ and σ' are maximal cones in Σ and σ precedes σ' in the ordering fixed earlier. Consider the $(C, [E_j])$ entry of W where $C = V(\sigma \cap \sigma')$ and $[E_j]$ has Cartier data given by $\{(U_\sigma, t^{m_\sigma})\}_{\sigma \in \Sigma(N)}$. Fix $u \in \sigma'$ such that $\overline{u}$ generates $\overline{\sigma}'$ where $\overline{\sigma}'$ is the image of the cone σ' in $\mathbb{R}^N/\mathrm{span}_{\mathbb{R}}(\tau)$. Since Σ is smooth, one can choose u to be $u_{\rho'}$ where $\rho' \in \sigma' \setminus \sigma$. Then, this entry is given by $\langle m_\sigma - m_{\sigma'}, u\rangle$. This value is $E_j \cdot C$, e.g. see [4, Prop. 6.2.8].

Given any divisor class $[D]$, it can be written as $\sum_{j=1}^{|\Sigma(1)|-N} a_j [E_j]$ for some coefficients a_j. In particular,

$$W \cdot (a_1, \ldots, a_{|\Sigma(1)|-N})^T = (D \cdot V(\tau))_{\tau \in \Sigma(N-1)}.$$

The toric Kleiman criterion, e.g., see [4, Thm. 6.3.13], states that $[D]$ is ample if and only if all the entries of the vector on the right are positive. Thus, we can rephrase the problem of finding an ample divisor, which must exist, in terms of finding the maximizer of the following linear program:

$$\begin{aligned} \text{maximize} &: \quad q \\ \text{subject to} &: \quad W \cdot v \geq q \cdot \mathbf{1}, \\ & \quad -1 \leq v_j \leq 1, \ \forall j \end{aligned}$$

where $\mathbf{1}$ is a vector where each entry is 1. Let $(v^*, q^*) \in \mathbb{Q}^{|\Sigma(1)|-N} \times \mathbb{Q}_{>0}$ be a solution to the linear program and let d be the least common denominator among the entries of v^*. Then, $[B] = \sum_{j=1}^{|\Sigma(1)|-N} d v_j^* [E_j]$ is an ample divisor class.

Since $\mathrm{Nef}(X_\Sigma)$ is a full-dimensional cone and B is ample, there exists a large enough integer m such that $A = m \cdot B$ and $A + K_{X_\Sigma}$ are both ample, which can be checked using the matrix W. Finally, since each D_j is nef, the divisor $\widetilde{A} = A + K_{X_\Sigma} + \sum_{j=1}^N D_j$ is also ample.

3 Software

We have developed code that is freely available on GitHub. The majority of our code for finding an upper bound on the number of ramification points in a polyhedral homotopy is written in `Macaulay2` [6], specifically using the `NormalToricVarieties` package [14]. In order to solve the linear program in Theorem 4, we use `Python` along with `NumPy` [7] and `SciPy` [18]. The implementation for determining a smooth subdivision of a polyhedral fan depends upon random choices and thus may compute different smoothings of the fan in question with different runs. However, the ultimate output computed via (7) is independent of which smooth fan is chosen.

The rest of this section considers the trivariate polyhedral homotopy

$$H(x;t) = \begin{bmatrix} c_1(t) + c_2(t)x_1 + c_3(t)x_2 + c_4(t)x_3 \\ c_5(t) + c_6(t)x_1 + c_7(t)x_1^2 + c_8(t)x_2 + c_9(t)x_3 \\ c_{10}(t) + c_{11}(t)x_1 + c_{12}(t)x_1^2 + c_{13}(t)x_1^3 + c_{14}(t)x_2 + c_{15}(t)x_3 \end{bmatrix}$$

where $c_j(t) = t \cdot a_j + (1-t) \cdot b_j$ for general $a_j, b_j \in \mathbb{C}$. Reading columnwise, the Newton polytopes P_j are

$$P_1 = \operatorname{conv}\begin{pmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}, \quad P_2 = \operatorname{conv}\begin{pmatrix} 0\,2\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}, \quad \text{and} \quad P_3 = \operatorname{conv}\begin{pmatrix} 0\,3\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,1 \end{pmatrix}$$

with $\mathrm{MV}(P_1, P_2, P_3) = 3$. Hence, $H(x;t) = 0$ defines 3 solution paths.

Next, we consider the the rays of a smooth fan Σ refining the inner normal fan of $\sum_{i=1}^{3} P_i$. The first 6 columns of the following are the original rays of the inner normal fan of $\sum_{i=1}^{3} P_i$, while the last column is needed to smooth the fan:

$$\begin{pmatrix} 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 1 & -3 & -2 & -1 & 0 & -1 \\ 0 & 0 & -3 & -2 & -1 & 1 & -1 \end{pmatrix}. \tag{9}$$

There are 10 maximal cones of Σ. In the following list, $\{i, j, k\}$ corresponds to the cone whose rays are generated by columns i, j, and k of (9):

$$\{1,2,6\}, \{1,2,7\}, \{1,6,7\}, \{2,3,4\}, \{2,3,7\}, \{2,4,5\}, \{2,5,6\}, \{3,4,6\}, \{3,6,7\}, \{4,5,6\}.$$

Let D_j denote the toric divisor corresponding to the j^{th} column of the matrix above. The group of torus-invariant Cartier divisors $\mathrm{CDiv}_T(X_\Sigma)$ is free of rank 7 and is generated by $D_1, \ldots, D_7$. The divisors corresponding to P_1, P_2, and P_3 are

$$\begin{aligned} D_{P_1} &= 3 \cdot D_3 + 2 \cdot D_4 + D_5 + D_7, \\ D_{P_2} &= 3 \cdot D_3 + 2 \cdot D_4 + 2 \cdot D_5 + D_7, \\ D_{P_3} &= 3 \cdot D_3 + 3 \cdot D_4 + 3 \cdot D_5 + D_7. \end{aligned}$$

The Picard group $\mathrm{Pic}(X_\Sigma)$ is free of rank 4. The natural projection

$$\pi : \mathrm{CDiv}_T(X_\Sigma) \to \mathrm{Pic}(\mathrm{X}_\Sigma)$$

is therefore represented (with respect to the above generators) by a 4×7 matrix. To obtain an explicit basis of $\mathrm{Pic}(X_\Sigma)$ consisting of classes of torus-invariant divisors, we compute a right inverse of π. The columns of this right inverse yield 4 torus-invariant divisors whose images under π form a basis of $\mathrm{Pic}(X_\Sigma)$. The 4×7 matrix representing π and a 7×4 matrix B are:

$$\pi = \begin{pmatrix} 1 & 0 & -2 & 3 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \\ D_7 \end{array} \!\!\begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Reading B columnwise, the j^{th} column yields the torus-invariant divisor

$$E_j = \sum_{i=1}^{7} B_{ij} D_i.$$

In this new basis, we have

$$\begin{aligned} [D_{P_1}] &= [E_3], \\ [D_{P_2}] &= [E_2] + [E_3], \\ [D_{P_2}] &= 3 \cdot [E_1] + 3 \cdot [E_2] + [E_4], \\ [K_{X_\Sigma}] &= -2 \cdot [E_1] - 2 \cdot [E_2] - 2 \cdot [E_3] - [E_4]. \end{aligned}$$

Now, we construct the matrix W using this basis. Since Σ has 15 walls, W is a 15×4 matrix where the rows are indexed by the 2-dimensional faces of Σ and the columns are indexed by $[E_1], \ldots, [E_4]$. The 2-dimensional face σ_{ij} corresponds to the face whose rays are generated columns i and j of (9). We can check the nefness of a toric divisor by writing it in our chosen basis and multiplying W on the right by the corresponding coefficient vector whose transpose is:

$$W^T = \begin{array}{c} \\ [E_1] \\ [E_2] \\ [E_3] \\ [E_4] \end{array} \!\!\begin{array}{c} \begin{array}{ccccccccccccccc} \sigma_{12} & \sigma_{16} & \sigma_{17} & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{34} & \sigma_{36} & \sigma_{37} & \sigma_{45} & \sigma_{46} & \sigma_{56} & \sigma_{67} \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & -3 & 0 & 1 & 1 & 0 & 0 & 0 & -3 \end{pmatrix} \end{array}.$$

Using W, we see that $D_{P_1} + D_{P_2} + D_{P_3} + K_{X_\Sigma}$ is not nef. Thus, we can solve the linear program described in Theorem 4 to compute an ample divisor A such that $\widetilde{A} = A + D_{P_1} + D_{P_2} + D_{P_3} + K_{X_\Sigma}$ is also ample. This yields $(4/5, 1, 1, 1/5)^T$ so that, after clearing denominators, we obtain the vector $(4, 5, 5, 1)^T$. This is the coefficient vector of an ample divisor in our chosen basis. A representative of A is found by multiplying this vector by the matrix B:

$$A = 2 \cdot D_1 + 6 \cdot D_2 - D_3 + 2 \cdot D_5.$$

One can check that both A and $\widetilde{A}$ are ample. Therefore, we can apply (7) to provide an upper bound on the number of ramification points for H. The necessary mixed volumes are given in Table 1 where the sum of the right-hand column yields 12. Directly solving $\{H(x;t), \det J_x H(x;t)\} = 0$ shows this is sharp.

$\mathrm{MV}(P_{\widetilde{A}}, P_2, P_3)$	20
$\mathrm{MV}(P_1, P_{\widetilde{A}}, P_3)$	20
$\mathrm{MV}(P_1, P_2, P_{\widetilde{A}})$	15
$-\mathrm{MV}(P_A, P_2, P_3)$	-19
$-\mathrm{MV}(P_1, P_A, P_3)$	-19
$-\mathrm{MV}(P_1, P_2, P_A)$	-14
$3 \cdot \mathrm{MV}(P_1, P_2, P_3)$	9

Table 1: Values of mixed volumes used in (7)

4 Conclusion

The number of ramification points provides a natural intrinsic measure of the quality of a homotopy. For arbitrary homotopies, this quantity can be computed via intersection theory using Chern classes as in [15, Thm. 18]. Here, we focused on computing this invariant for polyhedral homotopies by deriving a formula in terms of mixed volumes of associated polytopes. In particular, we provide a practical roadmap (with a software implementation) for computing this number for any polyhedral homotopy. We show how to construct a smooth projective toric variety equipped with a globally generated vector bundle such that the polyhedral homotopy arises as a section. This yields the main theoretical contribution in Theorem 3. Since this result depends upon determining a suitable nef divisor, we formulate an explicit linear program to determine a suitable ample divisor. An implementation in `Macaulay2` and `Python` demonstrates the results.

In Example 4, we say that some ramification points may lie on divisors at infinity. As such, it is natural to ask if the ramification points can be stratified by the closures of torus orbits. That is, given a face $\sigma \in \Sigma$ where $V(\sigma)$ is the closure of the corresponding orbit, can one determine the number of ramification points of the polyhedral homotopy which lie on $V(\sigma)$?

Acknowledgment

The authors thank Martin Helmer, Eric Riedl, and Andrew Sommese for helpful conversations. JDH was partially supported by NSF CCF-2331400 and the Robert and Sara Lumpkins Collegiate Professorship.

References

1. D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. *Numerically Solving Polynomial Systems with Bertini*, volume 25 of *Software, Environments, and Tools*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2013.
2. D. N. Bernstein. The number of roots of a system of equations. *Funkcional. Anal. i Priložen.*, 9(3):1–4, 1975.

3. T. Chen, T.-L. Lee, and T.-Y. Li. Mixed volume computation in parallel. *Taiwanese J. Math.*, 18(1):93–114, 2014.
4. D. A. Cox, J. B. Little, and H. K. Schenck. *Toric Varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
5. T. Gao, T. Y. Li, and M. Wu. Algorithm 846: MixedVol: a software package for mixed-volume computation. *ACM Trans. Math. Softw.*, 31(4):555–560, 2005.
6. D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.macaulay2.com.
7. C. R. Harris, K. J. Millman, S. J. van der Walt, R. Gommers, P. Virtanen, D. Cournapeau, E. Wieser, J. Taylor, S. Berg, N. J. Smith, R. Kern, M. Picus, S. Hoyer, M. H. van Kerkwijk, M. Brett, A. Haldane, J. F. del Río, M. Wiebe, P. Peterson, P. Gérard-Marchant, K. Sheppard, T. Reddy, W. Weckesser, H. Abbasi, C. Gohlke, and T. E. Oliphant. Array programming with NumPy. *Nature*, 585(7825):357–362, 2020.
8. J. D. Hauenstein, C. Hills, A. J. Sommese, and C. W. Wampler. Branch points of homotopies: Distribution and probability of failure. *Applied Mathematics and Computation*, 493:129273, 2025.
9. J. D. Hauenstein, A. J. Sommese, and C. W. Wampler. Regeneration homotopies for solving systems of polynomials. *Math. Comp.*, 80(273):345–377, 2011.
10. B. Huber and B. Sturmfels. A polyhedral method for solving sparse polynomial systems. *Math. Comp.*, 64(212):1541–1555, 1995.
11. A. N. Jensen. An implementation of exact mixed volume computation. In G.-M. Greuel, T. Koch, P. Paule, and A. Sommese, editors, *Mathematical Software – ICMS 2016*, pages 198–205, Cham, 2016. Springer International Publishing.
12. K. Kaveh and A. G. Khovanskii. Mixed volume and an extension of intersection theory of divisors. *Mosc. Math. J.*, 10(2):343–375, 479, 2010.
13. T. Mizutani and A. Takeda. *DEMiCs: A Software Package for Computing the Mixed Volume Via Dynamic Enumeration of all Mixed Cells*, pages 59–79. Springer New York, New York, NY, 2008.
14. G. G. Smith. NormalToricVarieties: routines for working with normal toric varieties and related objects. Version 1.9. A *Macaulay2* package available at https://github.com/Macaulay2/M2/tree/stable/M2/Macaulay2/packages.
15. A. J. Sommese, J. D. Hauenstein, C. Hills, and C. W. Wampler. Ramification points of homotopies: Enumeration and general theory. *Advances in Geometry*, 25(4):505–527, 2025.
16. A. J. Sommese and C. W. Wampler. *The Numerical Solution of Systems of Polynomials Arising in Engineering and Science*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
17. J. Verschelde. Algorithm 795: PHCpack: a general-purpose solver for polynomial systems by homotopy continuation. *ACM Trans. Math. Softw.*, 25(2):251–276, 1999.
18. P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S. J. van der Walt, M. Brett, J. Wilson, K. J. Millman, N. Mayorov, A. R. J. Nelson, E. Jones, R. Kern, E. Larson, C. J. Carey, İ. Polat, Y. Feng, E. W. Moore, J. VanderPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E. A. Quintero, C. R. Harris, A. M. Archibald, A. H. Ribeiro, F. Pedregosa, P. van Mulbregt, and SciPy 1.0 Contributors. SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. *Nature Methods*, 17:261–272, 2020.